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Determination of control parameter in an inverse time fractional diffusion equation using a linearized fourth-order finite difference scheme

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Abstract. The problem of finding the space- or time-dependent control parameter in partial differential equations has increasingly appeared in physical phenomena, for example, in the study of control theory, heat conduction process, and chemical diffusion. This study aims to construct an efficient numerical method to determine a time-dependent source term in a time fractional diffusion equation subject to over-specification at a point in the spatial domain. We use a second order scheme to discretize the equation in the time direction, then we replace the space derivative with a fourth-order compact finite difference approximation. We construct a linearized difference scheme and prove the solvability, and unconditional stability of the proposed method. Due to the ill-posed nature of inverse problems, we examine the stability of the method with respect to perturbations of the data. We show that the proposed method achieves stable and accurate numerical approximations without using any regularization techniques. Numerical experiments show satisfactory results for problems with smooth, non-smooth, and discontinuous initial conditions.

Keywords: Control parameter, inverse problem, compact finite difference, stability, perturbation.
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1 Introduction

Fractional partial differential equations are very important and interesting due to their extraordinary capability to model scientific phenomena, such as medicine, material engineering, control theory, and

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signal processing. The fractional diffusion equation describes the phenomena of anomalous diffusion in transport processes through complex and disordered systems, including fractal media, and fractional kinetic equations [25]. There have been several numerical methods proposed for solving the fractional diffusion equation, for example, some finite difference schemes are proposed in [17, 19, 26, 37], finite element methods in [14, 24], and meshfree method in [18].

In some practical situations, part of the initial data, boundary data, diffusion coefficients, or source term may not be given, and we want to find them using additional measured data, which will lead to some diffusion inverse problems. The parameter determination in a partial differential equation from the over-specified data plays a crucial role in physics and applied mathematics. Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of parabolic partial differential equations with source control parameters [2, 3].

Murio provided the early paper on fractional inverse problems [28]. An inverse problem for determining the order of the fractional derivative and the diffusion coefficient in a fractional diffusion equation is given in [5]. Authors of [20] solved a backward problem for the time-fractional diffusion equation by a quasi reversibility regularization method. The Cauchy problems for the time fractional diffusion equation on a strip domain by a Fourier regularization and a convolution method are considered in [38], and for the time fractional diffusion equation on a bounded domain is solved by a kernel-based method in [10]. Identifying a space-dependent source in a time-fractional diffusion-wave equation by using the final time data is shown in [32]. The authors of [4] studied a subdiffusion equation with a single boundary time trace as observation, and the uniqueness of the fractional order, along with a piecewise constant diffusion coefficient from a boundary trace, is shown. A semilinear subdiffusion equation and the identification of a time- dependent potential from integral observations over a time interval are studied in [15]. Yamamoto [33] investigated the identification of the space dependent part of a space-time fractional diffusion or diffusion wave equation.

The main contribution of this paper is to propose a higher order method to approximate the solution of the following inverse problem, i.e., to find the unknown solution $u(x,t)$ and control parameter $p(t)$ in the semi-linear time-dependent fractional diffusion equation

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx} + p(t)(\gamma u(x,t) + \kappa f(x)) + \phi(x,t), \quad 0 < x < 1,\ 0 < t \le T, \tag{1} $$

with initial condition

$$ u(x,0) = v(x), \quad 0 < x < 1, \tag{2} $$

and boundary conditions

$$ u(0,t) = g_0(t)\ ,\ u(1,t) = g_1(t), \quad 0 \le t \le T, \tag{3} $$

subject to the over-specification at a point in the spatial domain

$$ u(x^*,t) = E(t), \quad 0 \le t \le T, \tag{4} $$

where $f(x), v(x), g_0(t), g_1(t), \phi(x,t)$, and $E(t)$ are known functions, $\gamma, \kappa \ge 0$, $x^* \in (0,1)$, $u(x,t)$ and $p(t)$ are unknown functions, and ${}_0^C D_t^\alpha$ is the fractional derivative operator in Caputo sense

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, $$

in which $\alpha \in (0,1)$ and $\Gamma(.)$ is the Gamma function. Problem (1)-(4) with the time derivative of integer order has been considered by many researchers. For example, Cannon et al [3] formulated a backward Euler finite difference scheme via a transformation method. Dehghan in [8, 9] presented four difference schemes, the approximation of moving least-square (MLS) is proposed in [6], the reproducing kernel Hilbert space in [30], a method based on combination of compact finite difference scheme with boundary value methods is proposed in [27], and a linearized compact difference scheme is constructed in [35]. For problem (1)-(4) with $\gamma = 0$, authors of [31], based on the separation of variables and Duhamel's principle, transformed the inverse source problem into a first kind Volterra integral equation with the source term as unknown. Further, they used a boundary element method combined with a generalized Tikhonov regularization to solve the Volterra integral equation of the first kind. The generalized cross-validation choice rule is applied to find a suitable regularization parameter. Also, the authors show the ill-posedness of the problem. A local discontinuous Galerkin (LDG) method is also proposed in [36] for problem (1)-(4) with $\gamma = 0$. The proposed method is based on a finite difference scheme in the time and a LDG method in the space without using any regularization technique. Lyu and Cheng [23] investigated the inverse problem of identifying fractional order and time-dependent source term for fractional pseudo-parabolic equation. Altybay in [1] proposed a fully implicit finite-difference scheme and rigorously analysed the stability and convergence of the applied method in solving inverse problem of identifying a time-dependent coefficient in the time-fractional diffusion equation. An adaptive fractional physics-informed neural networks for numerical reconstructions of the fractional order and the space-dependent potential coefficient simultaneously is given in [21]. Authors of [29] determined a time-dependent coefficient using an efficient finite difference scheme and the existence and uniqueness theorem for small time is proved. Yan and Wei [34] proved the uniqueness of determining the fractional order and the time-dependent reaction coefficient of problem (1)-(4) with $\kappa = 0$ when $\alpha \in (1,2)$. Moreover, they adopted an iterative regularizing the ensemble Kalman method to solve the Bayesian inverse problem and present some numerical examples to confirm the effectiveness of the proposed method. Authors of [12] for problem (1)-(4) with $\gamma = 0$ and $\alpha \in (1,2)$, discussed the continuity of the weak solution for the direct problem. Then, they transformed the inverse problem into a first kind of Volterra integral equation and used a generalized Tikhonov regularization to solve the Volterra integral equation.

In this paper, we propose an efficient algorithm for solving problem (1)-(4), which is of order $\mathscr{O}(\tau^2 + h^4)$. The proposed linearized method is based on the compact finite difference scheme in space direction and a second order scheme for the time fractional derivative. We theoretically prove the solvability and unconditional stability of the presented method. It is well-known that inverse heat conduction problems are highly ill-posed, i.e., a small change in the input data can result in an enormous change in the resulting numerical solution. Thus, the stability of the numerical method with respect to the noise in the data is of vital importance for obtaining physically meaningful results. Since $E(t)$ is measured, there will be a measurement error in it. We will investigate the effect of this error on the approximate solutions.
The outline of this paper is as follows. In Section 2, we construct the proposed numerical algorithm using a second-order finite difference approximation in the time direction and a fourth-order compact scheme in the space variable. Section 3 gives the solvability of the proposed method. In Section 4, we investigate the stability of the presented method and prove that it is unconditionally stable. The numerical results obtained by applying the new method to several test problems, along with comparisons with analytical and other existing methods, are presented in Section 5. We conclude the article with a brief discussion in Section 6.

2 Derivation of proposed scheme

For positive integer numbers M and N, let $h = \frac{L}{M}$ denotes the step size of spatial variable, x, and $\tau = \frac{T}{N}$ denotes the step size of time variable, t, where $L = 1$ and T is the final time. So we define

$$x_j = ih, \;\; j = 0,1,2,...,M,$$
$$t_k = k\tau, \; k = 0,1,2,...,N.$$

Without the loss of generality, we can choose the mesh such that for some j_0, $1 \le j_0 \le M-1$, we have $x^* = j_0 h$. The exact and approximate solutions at the point (x_j, t_k) are denoted by u_j^k and U_j^k respectively. We define the following operators

$$\delta_x U_j = \frac{U_j - U_{j-1}}{h}, \quad \delta_x^2 U_j = \frac{U_{j+1} - 2U_j - U_{j-1}}{h^2}, \quad j = 1,2,...,M-1,$$

$$\mathscr{H} U_j = \begin{cases} \frac{1}{12}(U_{j-1} + 10U_j + U_{j+1}) = \left(1 + \frac{h^2}{12}\delta_x^2\right) U_j, & 1 \le j \le M-1, \\ U_j, & j = 0, M. \end{cases}$$

First, we present the following lemmas, which are needed in constructing the proposed method.

Lemma 1. *([11]) Suppose $s(x) \in C^6[x_L, x_R]$, then we can write*

$$\left(1 + \frac{h^2}{12}\delta_x^2\right) s^{(2)}(x_j) - \delta_x^2 s(x_j) = \frac{h^4}{240} s^{(6)}(\xi_j),$$

$$\xi_j \in (x_{j-1}, x_{j+1}), \; 1 \le j \le M-1.$$

Lemma 2. *Using Taylor series expansion for $p(t)$ we have*

$$p(t_k) = p(t_{k-1}) + \mathscr{O}(\tau),$$
$$p(t_k) = 2p(t_{k-1}) - p(t_{k-2}) + \mathscr{O}(\tau^2).$$

Lemma 3. *([22]) Let $g(t) \in C^3[0,T]$. For any $0 < \alpha < 1$, we can write*

$$\left| {}_0^C D_t^\alpha g(t_1) - \tilde{\alpha}_0{}^{-1}(g(t_1) - g(t_0)) \right| \le C \max_{0 \le t \le T} |g''(t)| \tau^{2-\alpha},$$

$$\left| {}_0^C D_t^\alpha g(t_k) - \alpha_0^{-1}\beta_0 (g(t_k) - \sum_{i=1}^{k} d_{k-i}^k g(t_{k-i})) \right| \le C \max_{0 \le t \le T} |g'''(t)| \tau^{3-\alpha}, \quad 2 \le k \le N,$$

where C is a positive constant and coefficients are

$$\tilde{\alpha_0}=\Gamma(2-\alpha)\tau^{\alpha},\ \ \alpha_0=\Gamma(3-\alpha)\tau^{\alpha},\ \ \beta_0=(1+\frac{\alpha}{2})2^{1-\alpha},\ \ 0<\alpha_0\beta_0^{-1}<\tilde{\alpha_0},$$

$$a_j=-\frac{3}{2}(2-\alpha)(j+1)^{1-\alpha}+\frac{1}{2}(2-\alpha)j^{1-\alpha}+(j+1)^{2-\alpha}-j^{2-\alpha},$$

$$b_j=2(2-\alpha)(j+1)^{1-\alpha}-2(j+1)^{2-\alpha}+2j^{2-\alpha},$$

$$c_j=-\frac{1}{2}(2-\alpha)((j+1)^{1-\alpha}+j^{1-\alpha})+(j+1)^{2-\alpha}-j^{2-\alpha},$$

for $k=2,3$

$$d_0^2=(-a_1-\frac{\alpha}{2})\beta_0^{-1},\ \ d_1^2=-(b_1-2)\beta_0^{-1},$$

$$d_0^3=-a_2\beta_0^{-1},\ \ d_1^3=(-a_1-b_2-\frac{\alpha}{2})\beta_0^{-1},\ \ d_2^3=(-b_1+c_2-2)\beta_0^{-1},$$

and for $k\geq 4$

$$d_{k-i}^k=-\beta_0^{-1}\times\begin{cases} b_1+c_2-2, & i=1,\\ a_1+b_2+c_3+\dfrac{\alpha}{2}, & i=2,\\ a_{i-1}+b_i+c_{i+1}, & 3\leq i\leq k-2,\\ a_{k-2}+b_{k-1}, & i=k-1,\\ a_{k-1}, & i=k.\end{cases}$$

Considering Eq. (1) at point (x_j,t_k) we have

$$ {}_0^CD_t^{\alpha}u(x_j,t_k)=u_{xx}(x_jt_k)+p(t_k)(\gamma u(x_j,t_k)+\kappa f(x_j))+\phi(x_j,t_k). \tag{5}$$

Using Lemma 3, for $k=1$ we can write

$$u_j^1-u_j^0=\tilde{\alpha}_0(u_{xx})_j^1+\tilde{\alpha}_0p^1(\gamma u_j^1+\kappa f_j)+\tilde{\alpha}_0\phi_j^1+R_0, \tag{6}$$

where

$$|R_0|\leq C\tau^2.$$

By applying the operator $\mathscr{H}$ to both sides of Eq. (6) and invoking Lemmas 1 and 2, we obtain

$$\mathscr{H}u_j^1-\mathscr{H}u_j^0=\tilde{\alpha}_0\delta_x^2u_j^1+\tilde{\alpha}_0p^0\mathscr{H}(\gamma u_j^1+\kappa f_j)+\tilde{\alpha}_0\mathscr{H}\phi_j^1+R_j^1,$$

or

$$\left(\mathscr{H}-\tilde{\alpha}_0\delta_x^2-\tilde{\alpha}_0\gamma p^0\mathscr{H}\right)u_j^1=\mathscr{H}u_j^0+\tilde{\alpha}_0p^0\kappa\mathscr{H}f_j+\tilde{\alpha}_0\mathscr{H}\phi_j^1+R_j^1, \tag{7}$$

where

$$|R_j^1|\leq C(\tau^{1+\alpha}+h^4).$$

Note that we can compute p^0 as follows:

$$p^0=\frac{1}{\gamma v(x^*)+\kappa f(x^*)}\left({}_0^CD_t^{\alpha}E(t)|_{t=0}-v''(x^*)-\phi_{j_0}^0\right).$$

For $2 \leq k \leq N$, using Lemma 3 we have

$$\alpha_0^{-1}\beta_0 u_j^k - \alpha_0^{-1}\beta_0 \sum_{i=1}^{k} d_{k-i}^k u_j^{k-i} = (u_{xx})_j^k + p^k(\gamma u_j^k + \kappa f_j) + \phi_j^k + R_j^k, \tag{8}$$

in which

$$|R_j^k| \leq C\tau^2.$$

Acting operator $\mathscr{H}$ to both sides of Eq. (8) and using Lemmas 2.1 and 2.2, we can write

$$\left((\alpha_0^{-1}\beta_0 - 2\gamma p^{k-1} + \gamma p^{k-2})\mathscr{H} - \delta_x^2\right) u_j^k =$$

$$\alpha_0^{-1}\beta_0 \mathscr{H} \sum_{i=1}^{k} d_{k-i}^k u_j^{k-i} + \kappa\left(2p^{k-1} - p^{k-2}\right)\mathscr{H} f_j + \mathscr{H}\phi_j^k + R_j^k, \tag{9}$$

in which

$$|R_j^k| \leq C(\tau^2 + h^4).$$

Regarding to the relation

$$\frac{\partial^2 u}{\partial x^2}(x_{j_0}, t_k) = \frac{1}{12h^2}\left(-u_{j_0-2}^k + 16u_{j_0-1}^k - 30u_{j_0}^k + 16u_{j_0-1}^k - u_{j_0+2}^k\right) + \mathscr{O}(h^4),$$

we can compute p^k, $k = 1, 2, \ldots, N$ from the following fourth-order scheme

$$\begin{aligned} p^k &= \frac{1}{\gamma E^k + \kappa f(x^*)}\left\{ {}_0^C D_t^\alpha E(t)|_{t=t_k} \right. \\ &- \left. \frac{1}{12h^2}\left(-U_{j_0-2}^k + 16U_{j_0-1}^k - 30U_{j_0}^k + 16U_{j_0-1}^k - U_{j_0+2}^k\right) - \phi_{j_0}^k \right\}. \end{aligned} \tag{10}$$

Omitting the small terms in (7), (9) and (10), we construct the following linearized difference scheme for the problem (1)-(4):

$$\left(\mathscr{H} - \tilde{\alpha}_0\delta_x^2 - \tilde{\alpha}_0\gamma p^0\mathscr{H}\right)U_j^1 = \mathscr{H}U_j^0 + \tilde{\alpha}_0 p^0\kappa\mathscr{H} f_j + \tilde{\alpha}_0\mathscr{H}\phi_j^1, \tag{11}$$

$$\begin{gathered} \left((\alpha_0^{-1}\beta_0 - 2\gamma p^{k-1} + \gamma p^{k-2})\mathscr{H} - \delta_x^2\right) U_j^k = \alpha_0^{-1}\beta_0\mathscr{H}\sum_{i=1}^{k} d_{k-i}^k U_j^{k-i} + \kappa\left(2p^{k-1} - p^{k-2}\right)\mathscr{H} f_j + \mathscr{H}\phi_j^k, \\ k = 2, 3, \ldots, N, \end{gathered} \tag{12}$$

$$\begin{aligned} p^k &= \frac{1}{\gamma E^k + \kappa f(x^*)}\left\{ {}_0^C D_t^\alpha E(t)|_{t=t_k} \right. \\ &- \left. \frac{1}{12h^2}\left(-U_{j_0-2}^k + 16U_{j_0-1}^k - 30U_{j_0}^k + 16U_{j_0-1}^k - U_{j_0+2}^k\right) - \phi_{j_0}^k \right\}, k = 1, 2, \ldots, N. \end{aligned} \tag{13}$$

3 Solvability

This section is devoted to the investigation of the solvability of the applied method. Let

$$U^k=[U_1^k,U_2^k,...,U_{M-1}^k]^T, F=[f_1,f_2,...,f_{M-1}]^T, \phi^k=[\phi_1^k,\phi_2^k,...,\phi_{M-1}^k]^T,$$

$$H=\frac{1}{12}\begin{bmatrix} 10 & 1 & 0 & \dots & 0 \\ 1 & 10 & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & 10 & 1 \\ 0 & \dots & 0 & 1 & 10 \end{bmatrix}_{(M-1)\times(M-1)}, \tag{14}$$

$$J=\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}_{(M-1)\times(M-1)}, \tag{15}$$

$\kappa_1=(\tilde{\alpha}_0)^{-1}-\gamma p^0$ and $\kappa_2=\alpha_0^{-1}\beta_0-2\gamma p^{k-1}+\gamma p^{k-2}$. For homogeneous boundary conditions, we can write Eq. (11) in the matrix-vector form

$$AU^1=b^1,$$

and Eq. (12) as

$$BU^k=b^k, \quad k=2,3,...,N,$$

in which $A=\kappa_1 H-J$, $B=\kappa_2 H-J$,

$$b^1=(\tilde{\alpha}_0)^{-1}HU^0+p^0\kappa HF+H\phi^1,$$

$$b^k=\alpha_0^{-1}\beta_0 H\sum_{i=1}^{k} d_{k-i}^k U^k+\kappa\left(2p^{k-1}-p^{k-2}\right)HF+H\phi^k.$$

For sufficiently small τ, the constants κ_1 and κ_2 are positive and it can be shown that the tridiagonal matrices A and B are diagonally dominant. For instance

$$A=\kappa_1 H-J=\begin{bmatrix} \frac{10\kappa_1}{12}+\frac{2}{h^2} & \frac{\kappa_1}{12}-\frac{1}{h^2} & 0 & \dots & 0 \\ \frac{\kappa_1}{12}-\frac{1}{h^2} & \frac{10\kappa_1}{12}+\frac{2}{h^2} & \frac{\kappa_1}{12}-\frac{1}{h^2} & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & \frac{\kappa_1}{12}-\frac{1}{h^2} & \frac{10\kappa_1}{12}+\frac{2}{h^2} & \frac{\kappa_1}{12}-\frac{1}{h^2} \\ 0 & \dots & 0 & \frac{\kappa_1}{12}-\frac{1}{h^2} & \frac{10\kappa_1}{12}+\frac{2}{h^2} \end{bmatrix},$$

and it is clear that

$$\left|\frac{10\kappa_1}{12}+\frac{2}{h^2}\right|>\left|\frac{\kappa_1}{12}-\frac{1}{h^2}\right|$$

and

$$\left|\frac{10\kappa_1}{12}+\frac{2}{h^2}\right|>\left|\frac{2\kappa_1}{12}-\frac{2}{h^2}\right|.$$

So the difference scheme (11)-(12) has a unique solution.

4 Stability analysis

In this section we investigate the stability of the proposed method. To this end, we assume that the boundary conditions (3) are homogeneous. We first give some lemmas and notations. Let

$$V_h = \{V \mid V = (V_0, V_1, ..., V_M),\ V_0 = V_M = 0\}$$

be the space of grid functions. For any grid function $U, W \in V_h$ we define l^2 inner product and norm as follows:

$$(U,W) = \sum_{j=1}^{M-1} U_j W_j, \qquad \|U\|^2 = (U,U).$$

Lemma 4. *([11]) Suppose $U, W \in V_h$, then*

$$a)\ (\delta_x^2 U, W) = -(\delta_x U, \delta_x W),$$

$$b)\ \|HU\| \leqslant \|U\|.$$

Lemma 5. *([16]) For any symmetric matrix* A*, the property of Rayleigh-Ritz ratio is*

$$\lambda_{min}(\mathrm{A}) \le \frac{(\mathrm{AX},\mathrm{X})}{(\mathrm{X},\mathrm{X})} \le \lambda_{max}(\mathrm{A}),$$

where $\lambda_{min}, \lambda_{max}$ denote the smallest and largest eigenvalue of A*, respectively.*

Lemma 6. *([13]) For $0 < \alpha < 1$, and $d_{k-i}^k, i = 1,2,...,k,\ k = 2,3,...,M$ defined in Lemma 3, we have*
i) $\sum_{i=1}^{k} | d_{k-i}^k | \le 2,$
ii) $| d_0^k | < 1.$

Lemma 7. *(Gronwall's inequality, [7]) Let $\{y_n\}$ and $\{g_n\}$ are non-negative sequences and C is a non-negative constant such that*

$$y_n \le C + \sum_{k=0}^{n-1} g_k y_k, \quad n \ge 0,$$

then

$$y_n \le C \prod_{j=0}^{n-1} (1 + g_j) \le C\ \exp\left(\sum_{j=0}^{n-1} g_j\right), \quad n \ge 0.$$

Note that the eigenvalues of matrix H in (14) are as

$$\lambda_j(H) = \frac{1}{12}\left(10 + 2\cos\left(\frac{j\pi}{M}\right)\right), \quad j = 1,2,...,M-1,$$

so

$$\frac{2}{3} \le \lambda_j(H) \le 1. \tag{16}$$

Theorem 1. *The compact difference scheme (11) -(13) is unconditionally stable.*

Proof. Multiplying both sides of (11) by U_j^1 and summing up j from 1 to $M-1$ we get

$$\begin{aligned}\left(HU^1,U^1\right) &- \tilde{\alpha}_0\left(\delta_x^2U^1,U^1\right)-\tilde{\alpha}_0\gamma p^0\left(HU^1,U^1\right)\\ &= \left(HU^0,U^1\right)+\tilde{\alpha}_0 p^0\kappa\left(HF,U^1\right)+\tilde{\alpha}_0\left(H\phi^1,U^1\right). \end{aligned}\tag{17}$$

Using Cauchy-Schwartz inequality, Lemmas 4, and 5, we can write

$$\frac{2}{3}\left\|U^1\right\|^2+\tilde{\alpha}_0\left\|\delta_xU^1\right\|^2-\tilde{\alpha}_0\gamma p^0\left\|U^1\right\|^2\le\left(HU^1,U^1\right)-\tilde{\alpha}_0\left(\delta_x^2U^1,U^1\right)-\tilde{\alpha}_0\gamma p^0\left(HU^1,U^1\right),\tag{18}$$

and

$$\begin{aligned}\left(HU^0,U^1\right)+\tilde{\alpha}_0p^0\kappa\left(HF,U^1\right) &+ \tilde{\alpha}_0\left(H\phi^1,U^1\right)\le\left\|U^0\right\|\left\|U^1\right\|\\ &+ \tilde{\alpha}_0p^0\kappa\left\|HF\right\|\left\|U^1\right\|+\tilde{\alpha}_0\left\|H\phi^1\right\|\left\|U^1\right\|. \end{aligned}\tag{19}$$

Using (18), and (19), Eq. (17) gives

$$\left(\frac{2}{3}-\tilde{\alpha}_0\gamma p^0\right)\left\|U^1\right\|^2+\tilde{\alpha}_0\left\|\delta_xU^1\right\|^2\le\left\|U^0\right\|\left\|U^1\right\|+\tilde{\alpha}_0p^0\kappa\left\|HF\right\|\left\|U^1\right\|+\tilde{\alpha}_0\left\|H\phi^1\right\|\left\|U^1\right\|.\tag{20}$$

For enough small values of τ, and $q_1=\frac{2}{3}-\tilde{\alpha}_0\gamma p^0>0$ we can write

$$\left\|U^1\right\|\le\frac{1}{q_1}\left\|U^0\right\|+\frac{\tilde{\alpha}_0p^0\kappa}{q_1}\left\|HF\right\|+\frac{\tilde{\alpha}_0}{q_1}\left\|H\phi^1\right\|.\tag{21}$$

Note that we assume that $p^0\ge0$, however for the case $p^0<0$ we can show that (21) is also hold. For $2\le k\le N$, multiplying both sides of (12) by U_j^k and summing up j from 1 to $M-1$ we get

$$\begin{aligned}(\alpha_0^{-1}\beta_0-2\gamma p^{k-1}+\gamma p^{k-2})\left(HU^k,U^k\right) &+ \left\|\delta_xU^k\right\|^2\\ =\alpha_0^{-1}\beta_0\sum_{i=1}^{k}d_{k-i}^k\left(HU^{k-i},U^k\right) &+ \kappa\left(2p^{k-1}-p^{k-2}\right)\left(HF,U^k\right)+\left(H\phi^k,U^k\right). \end{aligned}\tag{22}$$

Using Cauchy-Schwartz inequality, Lemmas 4, and 5, Eq. (22) results

$$\begin{aligned}\frac{2}{3}(\alpha_0^{-1}\beta_0-2\gamma p^{k-1} &+ \gamma p^{k-2})\left\|U^k\right\|^2+\left\|\delta_xU^k\right\|^2\le\alpha_0^{-1}\beta_0\sum_{i=1}^{k}d_{k-i}^k\left\|U^{k-i}\right\|\left\|U^k\right\|\\ &+ \kappa\left|2p^{k-1}-p^{k-2}\right|\left\|HF\right\|\left\|U^k\right\|+\left\|H\phi^k\right\|\left\|U^k\right\|. \end{aligned}\tag{23}$$

For enough small values of τ, and $q_2=\frac{2}{3}(\alpha_0^{-1}\beta_0-2\gamma p^{k-1}+\gamma p^{k-2})>0$ we have

$$\left\|U^k\right\|\le\frac{\alpha_0^{-1}\beta_0}{q_2}\sum_{i=1}^{k}d_{k-i}^k\left\|U^{k-i}\right\|+\frac{\kappa\left|2p^{k-1}-p^{k-2}\right|}{q_2}\left\|HF\right\|+\frac{1}{q_2}\left\|H\phi^k\right\|.\tag{24}$$

Using Lemmas 6 and 7, (24) gives

$$\begin{aligned}\left\|U^k\right\| &\leq \left(\frac{\kappa\left|2p^{k-1}-p^{k-2}\right|}{q_2}\|HF\|+\frac{1}{q_2}\left\|H\phi^k\right\|\right)\exp\left(\frac{\alpha_0^{-1}\beta_0}{q_2}\sum_{i=1}^{k}d_{k-i}^k\right)\\ &\leq C\left(\|HF\|+\left\|H\phi^k\right\|\right), \qquad (25)\end{aligned}$$

which completes the proof. □

5 Numerical results

In this section, we present the numerical results of the new method on some test problems. We tested the accuracy and stability of the method for different values of h and τ. We performed our computations using **MATLAB** 7 software. To illustrate the accuracy of the method and to compare with other methods in the literature, we compute the following error norms

$$E_\infty=\left\|U^N-u^N\right\|_\infty,\ F_\infty=\|P-p\|_\infty. \qquad (26)$$

Let E_1 and E_2 be the errors correspond to step sizes h_1 and h_2, respectively, then the computational order of the given method can be calculated as

$$Order=\frac{log\frac{E_1}{E_2}}{log\frac{h_1}{h_2}}.$$

Also we investigate a measurement error in $E(t)$. We will replace exact value of $E(t)$ by noisy data given as follows

$$E^\delta(t_i)=E(t_i)(1+\delta\,.\mathrm{rand(i)}), \qquad (27)$$

where rand(i) is a random number uniformly distributed in $[-1,1]$ and the magnitude δ indicates a relative noise level. In this case we compute the relative root mean square error by

$$\varepsilon(p)=\left(\sum_{i=1}^{n}\left(P_i^\delta-p(t_i)\right)^2/\sum_{i=1}^{n}p(t_i)^2\right),$$

where n is the total number of uniformly distributed points on time internal $[0,T]$ and P^δ is the control parameter corresponds to the noisy data E^δ. For all test problems we put $T=1$.

5.1 Test problem 1

We consider the semi-linear problem

$${}_0^CD_t^\alpha u(x,t)=u_{xx}+p(t)u(x,t)+\phi(x,t), \qquad (28)$$

with the following conditions

$$\begin{aligned}&u(x,0)=\sin(\pi x),\\ &E(t)=u(0.5,t)=1+t^{2.5},\\ &\phi(x,t)=\left(\tfrac{\Gamma(3.5+\alpha)}{\Gamma(3.5)}t^{2.5-\alpha}+\pi^2(1+t^{2.5})-(1+t^{2.5})^2\right)\sin(\pi x).\end{aligned}$$

Table 1: E_∞, CPU time and computational order for Test problem 1

		$\alpha=0.25$		$\alpha=0.5$		$\alpha=0.75$		
h	Δt	E_∞	Order	E_∞	Order	E_∞	Order	CPU(s)
$\frac{1}{40}$	$\frac{1}{80}$	7.8728×10^{-2}	–	4.3810×10^{-3}	–	1.5337×10^{-3}	–	0.028491
$\frac{1}{80}$	$\frac{1}{320}$	6.4100×10^{-3}	3.6148	2.9543×10^{-4}	3.8903	1.0259×10^{-4}	3.9021	0.084596
$\frac{1}{160}$	$\frac{1}{1280}$	4.7426×10^{-4}	3.7566	1.8680×10^{-5}	3.9832	6.6045×10^{-6}	3.9573	0.907856
$\frac{1}{320}$	$\frac{1}{5120}$	2.9225×10^{-5}	4.0204	1.2084×10^{-6}	3.9503	4.2408×10^{-7}	3.9610	14.73807

Table 2: F_∞ and computational order for Test problem 1

		$\alpha=0.25$		$\alpha=0.5$		$\alpha=0.75$	
h	Δt	F_∞	Order	F_∞	Order	F_∞	Order
$\frac{1}{40}$	$\frac{1}{80}$	3.9112×10^{-1}	–	2.1624×10^{-2}	–	7.5726×10^{-3}	–
$\frac{1}{80}$	$\frac{1}{320}$	3.1632×10^{-2}	3.6281	1.4581×10^{-3}	3.8904	5.0650×10^{-4}	3.9021
$\frac{1}{160}$	$\frac{1}{1280}$	2.3404×10^{-3}	3.7565	9.2201×10^{-5}	3.9832	3.2608×10^{-5}	3.9573
$\frac{1}{320}$	$\frac{1}{5120}$	1.4422×10^{-4}	4.0204	5.9643×10^{-6}	3.9504	2.0938×10^{-6}	3.9610

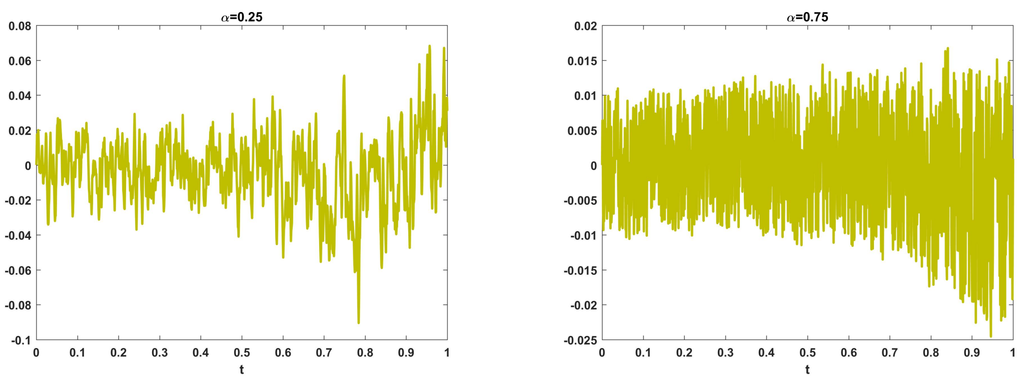

Figure 1: Resulting error of control parameter from noisy data for Test problem 1

The exact solutions of this problem are

$$u(x,t)=(1+t^{2.5})\sin(\pi x),\ \ p(t)=1+t^{2.5}.$$

Regarding to the second order of accuracy of method in the time variable and fourth-order of accuracy in the space component, we reduce the spatial mesh size by a factor of 2 and the temporal mesh size by a factor of 4 to show the fourth-order of accuracy of proposed method. Tables 1 and 2 show the errors, CPU time and computational order of the presented method. As we see the results of this tables confirm the fourth order of accuracy which is compatible with theoretical ones. We consider a noisy data and put $\delta=0.01$ in (27). Figure 1 displays the resulting error of control parameter with $h=1/160$ and $\Delta t=1/1280$ for $\alpha=0.25$ and $\alpha=0.75$.

Now we investigate problem (28) with the non-smooth exact solution

$$u(x,t)=t^{\alpha+\beta}\sin(\pi x),\ \ \beta<1,$$

Table 3: E_∞, CPU time and computational order with $\Delta t = 0.0005$

	$\alpha=0.75,\beta=0.25$		$\alpha=0.5,\beta=0.5$		$\alpha=0.3,\beta=0.6$		
h	E_∞	Order	E_∞	Order	E_∞	Order	CPU(s)
$\frac{1}{6}$	8.8865×10^{-3}	–	2.3411×10^{-2}	–	6.2830×10^{-2}	–	1.318004
$\frac{1}{12}$	6.0407×10^{-4}	3.8887	1.9546×10^{-3}	3.5822	2.1338×10^{-2}	1.5580	1.400598
$\frac{1}{24}$	3.8206×10^{-5}	3.9828	1.2574×10^{-4}	3.9583	2.2970×10^{-3}	3.2156	1.474718
$\frac{1}{48}$	2.3938×10^{-6}	3.9964	7.8910×10^{-6}	3.9941	2.3932×10^{-4}	3.3811	1.549540

Table 4: F_∞ and computational order with $\Delta t = 0.0005$

	$\alpha=0.75,\beta=0.25$		$\alpha=0.5,\beta=0.5$		$\alpha=0.3,\beta=0.6$	
h	F_∞	Order	F_∞	Order	F_∞	Order
$\frac{1}{6}$	9.5678×10^{-2}	–	2.3891×10^{-1}	–	6.2765×10^{-1}	–
$\frac{1}{12}$	6.4736×10^{-3}	3.8855	2.3891×10^{-2}	3.5927	2.1110×10^{-1}	1.5720
$\frac{1}{24}$	4.0923×10^{-4}	3.9836	1.2731×10^{-3}	3.9593	2.2651×10^{-2}	3.2203
$\frac{1}{48}$	4.0923×10^{-5}	3.9966	8.5058×10^{-5}	3.9037	2.4103×10^{-3}	3.2323

in which

$$p(t) = 1+t,$$
$$\phi(x,t) = \left(\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}t^\beta + \pi^2 t^{\alpha+\beta} - (1+t)t^{\alpha+\beta}\right)\sin(\pi x).$$

We present the E_∞, F_∞, CPU time and computational orders with $\Delta t = 0.0005$ in Tables 3 and 4. While the results in Tables 3 and 4 are satisfactory, in some cases the computational orders do not align with the theoretical ones. This mismatch is attributed to the lack of sufficient regularity in the solution, where convergence is dictated by the solution's smoothness rather than the formal accuracy of the discretization.

5.2 Test problem 2

We consider Eq. (1) with $\gamma = 0$, $\kappa = 1$ and $\phi(x,t) = 0$ with the following conditions

$$u(x,0) = 0,$$
$$E(t) = u(0.5,t) = t^2\sin(0.25\pi).$$

The exact solutions of this problem are

$$u(x,t) = t^2\sin(\frac{\pi x}{2}), \;\; p(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + \frac{\pi^2}{4}t^2.$$

Tables 5 and 6 show the errors, CPU time and computational order of presented method. As we see the results of this tables confirm the fourth order of accuracy which is compatible with theoretical ones.

In Table 7 we compare the relative root mean square error of the proposed method with the method presented in [31] for different values of α, $x^* = 0.5$, $h = 0.01$ and $\Delta t = 0.001$. In the method of [31] $\varepsilon_1(p)$ is the error for no regularization method and $\varepsilon_2(p)$ is the error for the first order Tikhonov regularization method. As we see, the proposed method can achieve better results without any regularization technique. Table 8 gives the comparisons of the relative error of the proposed method with the method of [31] ($\varepsilon_1(p)$ and $\varepsilon_2(p)$) for different values of x^*, $\delta = 0.10$, $\alpha = 0.8$, $h = 0.01$ and $\Delta t = 0.001$. Again we can see, the proposed method can achieve better results without any regularization technique. Also in spite of [31], the results are independent of x^*.

Table 5: E_∞, CPU time and computational order for Test problem 2

		$\alpha=0.15$		$\alpha=0.65$		$\alpha=0.95$		
h	Δt	E_∞	Order	E_∞	Order	E_∞	Order	CPU(s)
$\frac{1}{20}$	$\frac{1}{40}$	3.2812×10^{-3}	–	3.2941×10^{-3}	–	2.9195×10^{-3}	–	0.008556
$\frac{1}{40}$	$\frac{1}{160}$	2.0495×10^{-4}	4.0009	2.0886×10^{-4}	3.9793	1.8718×10^{-4}	3.9632	0.014286
$\frac{1}{80}$	$\frac{1}{640}$	1.2820×10^{-5}	3.9988	1.3115×10^{-5}	3.9932	1.1790×10^{-5}	3.9888	0.112437
$\frac{1}{160}$	$\frac{1}{2560}$	8.0188×10^{-7}	3.9989	8.2004×10^{-7}	3.9994	7.3990×10^{-7}	3.9941	1.888553
$\frac{1}{320}$	$\frac{1}{10240}$	5.1661×10^{-8}	3.9562	4.0161×10^{-8}	4.3518	1.5422×10^{-8}	5.5843	54.835182

Table 6: F_∞ and computational order for Test problem 2

		$\alpha=0.15$		$\alpha=0.65$		$\alpha=0.95$	
h	Δt	F_∞	Order	F_∞	Order	F_∞	Order
$\frac{1}{20}$	$\frac{1}{40}$	3.7870×10^{-2}	–	3.8109×10^{-2}	–	3.3779×10^{-2}	–
$\frac{1}{40}$	$\frac{1}{160}$	2.3641×10^{-3}	4.0017	2.4157×10^{-3}	3.9796	2.1654×10^{-3}	3.9634
$\frac{1}{80}$	$\frac{1}{640}$	1.4783×10^{-4}	3.9993	1.5163×10^{-4}	3.9938	1.3634×10^{-4}	3.9894
$\frac{1}{160}$	$\frac{1}{2560}$	9.2464×10^{-6}	3.9989	9.4817×10^{-6}	3.9993	8.5568×10^{-6}	3.9940
$\frac{1}{320}$	$\frac{1}{10240}$	6.0366×10^{-7}	3.9371	4.6982×10^{-7}	4.3350	3.6515×10^{-7}	4.5505

Table 7: Comparison of error with the method of [31] for Test problem 2 with noisy data

		$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
$\delta=0.05$	$\varepsilon_1(p)$	0.0773	0.0880	0.1230	0.2651	0.8079
	$\varepsilon_2(p)$	0.0331	0.0362	0.0393	0.0413	0.0431
	$\varepsilon(p)$	0.0017	7.3904×10^{-4}	6.1507×10^{-4}	6.8304×10^{-4}	7.2054×10^{-4}
$\delta=0.10$	$\varepsilon_1(p)$	0.1545	0.1738	0.2436	0.5296	1.6159
	$\varepsilon_2(p)$	0.0634	0.0633	0.0662	0.0721	0.0812
	$\varepsilon(p)$	0.0062	0.0030	0.0025	0.0027	0.0028
$\delta=0.15$	$\varepsilon_1(p)$	0.2322	0.2611	0.3658	0.7948	2.4241
	$\varepsilon_2(p)$	0.0952	0.0943	0.0984	0.1083	0.1248
	$\varepsilon(p)$	0.0148	0.0063	0.0056	0.0057	0.0066

Table 8: Comparison of error with the method of [31] for Test problem 2 with noisy data

	$x^*=0.1$	$x^*=0.2$	$x^*=0.3$	$x^*=0.4$	$x^*=0.5$	$x^*=0.6$	$x^*=0.7$	$x^*=0.8$
$\varepsilon_1(p)$	0.9047	0.9197	0.9323	0.9222	0.9071	0.9147	0.9256	1.0010
$\varepsilon_2(p)$	0.0661	0.0767	0.0694	0.0839	0.0761	0.1026	0.0974	0.1801
$\varepsilon(p)$	0.0029	0.0027	0.0028	0.0026	0.0029	0.0028	0.0028	0.0028

5.3 Test problem 3

We consider Eq. (1) with $\gamma=0$, $\kappa=1$ and $\phi(x,t)=0$ with the following conditions

$$\begin{aligned} &u(x,0)=0,\\ &u(0,t)=u(1,t)=0,\\ &u(x,0)=\sin(2\pi x),\\ &f(x)=x^2. \end{aligned}$$

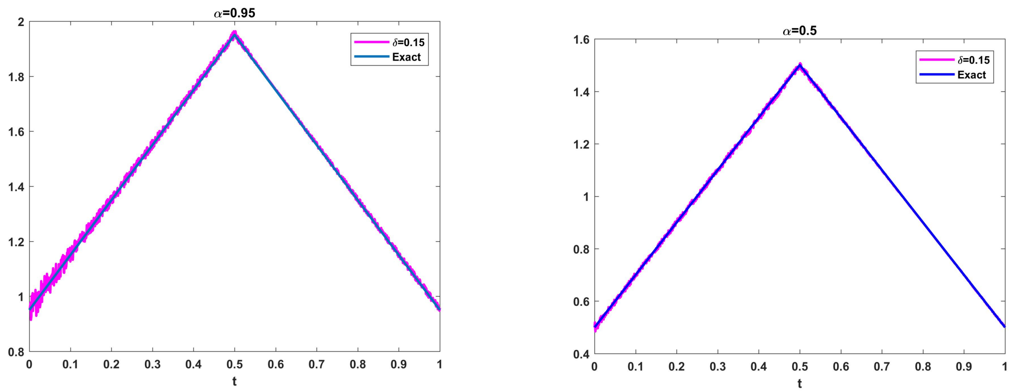

Figure 2: Numerical approximations to $p(t)$ for Test problem 3 with control parameter (29)

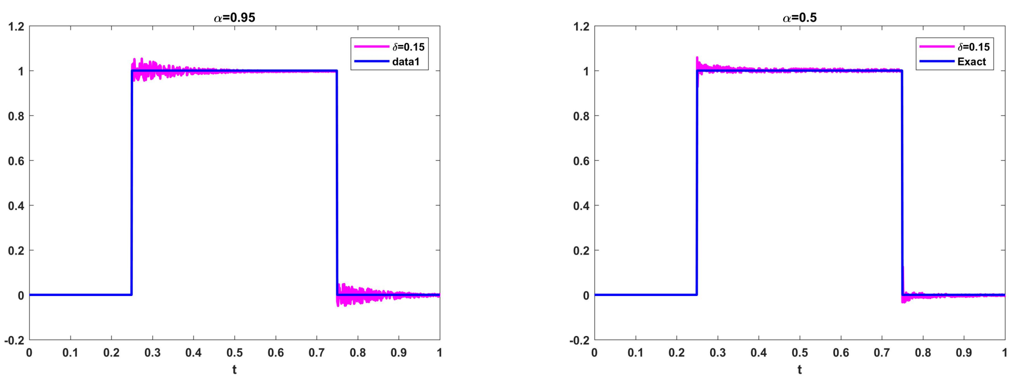

Figure 3: Numerical approximations to $p(t)$ for Test problem 3 with control parameter (30)

We investigate this problem with the following nonsmooth and discontinuous exact control parameters

$$p(t) = \begin{cases} 2t+\alpha, & t \in [0,0.5], \\ -2t+2+\alpha & t \in (0.5,1], \end{cases} \tag{29}$$

and

$$p(t) = \begin{cases} 1, & t \in [0.25,0.75], \\ 0 & t \in [0,0.25) \cup (0.75,1]. \end{cases} \tag{30}$$

The exact solution of this problem is not accessible, so we first solve the direct problem using a compact finite difference scheme and the input data $E(t)$ for each case can be obtained. Afterwards we solve the inverse problem using the proposed method. In all computations we put $x^* = 0.5$, $h = 0.005$ and $\Delta t = 0.001$. Numerical approximations of $p(t)$ with both control parameters (29) and (30) and noise amplitude $\delta = 0.15$ are presented in Figures 2 and 3. As we see the results are in good agreement with the results of [31].

6 Conclusion

In this paper, we introduce a linearized high-order numerical scheme for determining the control parameter in a time-fractional diffusion equation subject to overspecification at a point in the spatial domain. A second-order discretization is employed in the time direction, while the spatial derivative is approximated using a fourth-order compact finite difference scheme. We prove that the proposed method is solvable and unconditionally stable. Various numerical examples confirm the theoretical results and demonstrate the high accuracy of the scheme. Additionally, we investigate the effect of data noise and obtain satisfactory results for both smooth and non-smooth solutions.

Conflict of Interest

The authors declare that they have no conflict of interest.

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