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Implementation of a meshless method for optimal control of elliptic variational inequality

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Abstract. In this paper, a mesh-free method is presented for the numerical solution of an optimal control problem constrained by an elliptic variational inequality. The proposed method is indirect and based on the element-free Galerkin method to solve the considered nonlinear optimal control problem. First, the optimality conditions of the problem are derived via the Lagrangian technique. The obtained conditions are mixed complementarity conditions which can be solved by specific efficient algorithms. Here, the moving least squares approximation is utilized within the element-free Galerkin approach to numerically solve the obtained optimality conditions. The proposed method is mesh-free and can be used with irregular meshes and even in irregular domains. Finally, the convergence of the proposed method is numerically investigated and results confirm high-order accuracy.

Keywords: Optimal control problem, elliptic variational inequality, optimality conditions, meshfree methods, element free Galerkin method.
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1 Introduction

An elliptic variational inequality is an inequality involving an elliptic function that holds for all possible values of a given variable, usually belonging to a convex set [24]. However, the mathematical theory of these types of problems was initially developed to deal with equilibrium problems, but it can appear in the description of many scientific phenomena. Mostly, contact problems such as the obstacle problem [19, 22] and the Signorini problem [4] can be formulated as an elliptic variational inequality.

As elliptic variational inequalities have become more prevalent in mathematical modeling, the optimal control of such systems has emerged as a subject of significant research attention. [14]. The optimal obstacle control problem is one of the important problems of this kind, in which the applied force is considered as the control of the problem to bring an elastic shell to a certain shape [21]. In [3], the authors

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have studied an optimal control problem of a three-dimensional elastic body in frictional contact with normal compliance with a deformable foundation. Optimal control problem for a nonlinear antiplane of an elastic cylindrical body infrictional contact with a rigid foundation has been studied in [2]. Also, the optimal control for a bilateral contact between an nonlinear elastic body and a rigid foundation has been studied in [25].

Since most of these problems do not have analytical solutions, the study of suitable numerical methods for solving these problems is of great interest among researchers. Most studies in this regard are traditionally based on the two popular methods of finite difference and finite element methods [21]. However, with the emergence of meshfree numerical methods in recent decades, a new field has opened for research in numerical methods suitable for solving such problems. Meshless methods are numerical methods that, unlike traditional numerical methods, are not based on a meshing of the solution domain and can be implemented on various types of regular or irregular domains [12, 13]. It would be more accurate to state that these methods operate on a set of scattered nodes, entirely avoiding a mesh for approximation, though a separate background grid may be needed for integration. Most of these methods are simpler to implement and can be easily extended to higher dimensions.

In this paper, we focus on presenting a mesh-free numerical method for solving the optimal control problem constrained by an elliptic variational inequality. The method we have considered is called the element-free Galerkin method (EFGM). This method is based on using the moving least squares shape functions in the Galerkin scheme as the test and trail functions. This technique employs a background mesh for integration which makes it different from the truly mesh procedures. The EFGM was initially proposed for solving boundary value problems by B Belytschko et al. in 1994 [5]. In this technique, the weak form integral equation of the problem is first extracted. The test and trial functions of Galerkin scheme in this method are the shape functions of the moving least squares method [1]. Then, by applying weak form to subdomains that are not necessarily regular and using numerical integration procedure, a system of algebraic equations is extracted, by solving which the solutions are obtained [9].

1.1 Problem statement

An optimal control of the elliptic variational inequality has the following general form:

$$\begin{array}{ll}\text{Minimize} & \mathscr{F}[y,u] \\ \text{over} & (y,u)\in \mathscr{W}\times\mathscr{U}_{ad} \\ \text{subject to} & b(y,y-w)\le (g+u,y-w)_{\mathscr{W}}, \quad \forall w\in S\subset\mathscr{W}\end{array} \tag{1}$$

where $\mathscr{F}$ is objective functional, $\mathscr{W}$ is a real Hilbert space with dual space $\mathscr{W}^*$ and inner product $(\cdot,\cdot)_{\mathscr{W}}$. $b(\cdot,\cdot)$ is a bounded and continuous bilinear form on $\mathscr{W}\times\mathscr{W}$. The S is a closed convex subset of $\mathscr{W}$, $\mathscr{U}_{ad}$ is a set of admissible control functions where the known function g and the unknown control function u are belong to it and y is the state function belongs to $\mathscr{W}$. If b be coercive then there exists a linear operator associated with the bilinear form b, i.e. $b(y,u)=(\mathscr{B}y,u)_{\mathscr{W}}$.

Problem (1) is the general form of the optimal control of the elliptic variational inequality. In more practical cases, the objective functional $\mathscr{F}$ is usually considered as follows:

$$\mathscr{F}[y,u]:=\mathscr{T}[y]+\omega\mathscr{S}[u]=\frac{1}{2}\int_\Omega (y-y_d)^2 d\mathbf{x}+\frac{\omega}{2}\int_\Omega u^2 d\mathbf{x},$$

where Ω is the computational domain of the problem with $\mathbf{x}$ as its variable. The state function belongs to $H_0^1(\Omega)$ and the set of admissible control considered as $L^2(\Omega)$. Also, $y_d \in H_0^1(\Omega)$ is the goal function to which the state function should be as close as possible. From now on, we will consider the elliptic form $b(\cdot,\cdot) = (\nabla\cdot,\nabla\cdot)$. A common choice for the admissible state function set S in practical problems is as follows:

$$S_z := \{y \in H_0^1(\Omega) \mid y(\mathbf{x}) \leq z(\mathbf{x}) \text{ a.e. in } \Omega\},$$

where $z \in H_0^1(\Omega)$ is a known function. So the optimal control problem we are considering in this paper is as follows, which we will refer to as (P) from now on:

$$\begin{aligned} &\text{Minimize} && \mathscr{F}[y,u] = \mathscr{T}[y] + \omega\mathscr{S}[u] \\ &\text{over} && (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \\ &\text{subject to} && (\nabla y, \nabla(y-w)) \leq (g+u, y-w), \quad \forall w \in S_z. \end{aligned} \tag{P}$$

Here, $y \in H_0^1$ means that the homogeneous Dirichlet boundary conditions hold for the state function, and given the condition of minimizing the norm of the control function in objective functional, the control function will also hold under the homogeneous Dirichlet boundary condition.

2 Optimality conditions

In this section, we focus on derivation of optimality conditions for optimal control problem (P) by the method described in [11]. The method is based on definition of the appropriate Lagrange multipliers for the problem.

First, we note that any solution of the following elliptic variational inequality

$$(\nabla y, \nabla(y-w)) \leq (g+u, y-w), \quad \forall w \in S_z, \tag{2}$$

$$S_z = \{y \in H_0^1(\Omega) \mid y(\mathbf{x}) \leq z(\mathbf{x}) \text{ a.e. in } \Omega\}, \tag{3}$$

satisfies the following complementarity problem [20]:

$$-\Delta y \leq g+u, \tag{4}$$

$$y \leq z, \tag{5}$$

$$(\Delta y + g + u)(y-z) = 0. \tag{6}$$

In fact, the later problem is the strong form of the problem (2)-(3). These conditions indicates that the analytical solution of variational inequality can divide the domain Ω into two parts:

$$\mathscr{A} := \{\mathbf{x} \in \Omega \mid y(\mathbf{x}) = z(\mathbf{x}) \ \wedge\ -\Delta y(\mathbf{x}) < g(\mathbf{x}) + u(\mathbf{x})\}, \tag{7}$$

$$\mathscr{N} := \{\mathbf{x} \in \Omega \mid -\Delta y(\mathbf{x}) = g(\mathbf{x}) + u(\mathbf{x}) \ \wedge\ y(\mathbf{x}) < z(\mathbf{x})\}, \tag{8}$$

that are called as active and non-active sets, respectively [8]. The important point is that the boundary between the two parts is unknown and will be determined by solving the problem and obtaining the solution. The non-smoothness along this unknown free boundary, is expected in both solution functions.

By definition of the residual function $\lambda \in H^{-1}$, as the Lagrange function, it is proven in Theorem 3.1 in [15] that the complementarity problem (4)-(6) is equivalent to the following equations:

$$-\Delta y + \lambda = u + g, \tag{9}$$
$$\lambda = \max\{0, \lambda + \eta(y - z)\}, \tag{10}$$

where $\eta > 0$ is an arbitrary real constant.

According to [11], for the purpose of proving the existence of Lagrange multipliers it may be useful to take the limit $\eta \to \infty$. However, for numerical purposes it is important to stress that at a solution y to the variational inequality, (9) holds for every $\eta > 0$ and there is no need to consider the limit as $\eta \to \infty$. Choosing $\eta \gg 1$ will only be useful when we want to use the preconditioner for the coefficient matrix, which is described in [15]. At the same time, choosing a very small η close to zero can also greatly increase the sensitivity of the numerical algorithm.

Although substituting (9)-(10) for the constraints of problem (P) can remove inequality constraints, it should be noted that the new constraints also consist of a non-differentiable function. Thus, the optimal control problem (P) is not a regular problem and obtaining the optimality conditions needs regularization issue. By using the following approximation function [20]

$$\max_c\{0, x\} = \begin{cases} 0, & x \leq \frac{-1}{2c}, \\ \frac{c}{2}x^2 + \frac{1}{2}x + \frac{1}{8c}, & |x| \leq \frac{1}{2c}, \\ x, & x \geq \frac{1}{2c}, \end{cases}$$

for the function $\max\{0, x\}$, we can introduce a regularized form of the problem (P) as follows:

$$\begin{aligned} \text{Min} \quad & \mathscr{F}[y, u] \\ \text{s.t.} \ -\Delta y + \lambda &= u + g, \\ \lambda &= \max_c\{0, \lambda + \eta(y - z)\}. \end{aligned} \tag{Pc}$$

Since the function $\max_c$ is a smooth function and its derivative is defined as follows:

$$\delta_c(x) = \begin{cases} 0, & x \leq \frac{-1}{2c}, \\ cx + \frac{1}{2}, & |x| \leq \frac{1}{2c}, \\ 1, & x \geq \frac{1}{2c}, \end{cases}$$

the optimality conditions for the approximation problem (Pc) can be obtained easily. On the other hand, when $c \to \infty$, the sequence of the problems (Pc) for any c have the unique solutions (y_c, u_c) [11] whose weak cluster point (y^*, u^*) solve the original problem (P).

Now, to obtain the optimality conditions for the approximation problem (Pc), we define the Lagrangian functional as follows:

$$\mathscr{L}[y, u, \lambda, \alpha, \beta] = \mathscr{F}[y, u] - (\Delta y - \lambda + u + g, \alpha) + (\max_c\{0, \lambda + \eta(y - z)\} - \lambda, \beta)$$

with α and β as the slack variables. By defining $F(y, u) = -\Delta y + \max_c\{0, \lambda + \eta(y - z)\} - u - g$, since $x \to \max_c\{0, x\}$ is a C^1-mapping, the Fréchet derivative $F \in \mathscr{L}\left(H_0^1 \times L^2, H^{-1}\right)$ exists and is given by [11]

$$F(h, v) = -\Delta h + \eta \delta_c h - v \quad \text{for} \quad (h, v) \in H_0^1 \times L^2,$$

where

$$\delta_c = \delta_c\left(\lambda + \eta\left(y - z\right)\right).$$

Moreover, since

$$(-\Delta\phi + \eta\delta_c\phi, \phi) \geq \int_\Omega (\nabla\phi \cdot \nabla\phi)dx,$$

it follows from the Lax-Milgram theorem that F is surjective. Thus, the necessary optimality conditions for problem (Pc) is given by taking the derivative of a functional $\mathscr{L}$ with respect to all its main and slack variables, so we have [11]

$$\begin{aligned}
&-\Delta y + \lambda - u - g = 0,\\
&\lambda - \max_c\{0, \lambda + \eta(y - z)\} = 0,\\
&y - y_d - \Delta\alpha + \eta\delta_c(\lambda + \eta(y - z))\beta = 0,\\
&\alpha + (\delta_c(\lambda + \eta(y - z)) - 1)\beta = 0,\\
&\omega u - \alpha = 0.
\end{aligned}$$

If we define α^* and β^* as the cluster points of $\{\alpha_c\}$ and $\{\beta_c\}$, respectively, they satisfy in

$$\begin{aligned}
&-\Delta y^* + \lambda^* - u^* - g = 0,\\
&\lambda^* - \max\{0, \lambda^* + \eta(y^* - z)\} = 0,\\
&y^* - y_d - \Delta\alpha^* + \eta\chi_{\mathscr{A}}(\lambda^* + \eta(y^* - z))\beta^* = 0,\\
&\alpha^* + (\chi_{\mathscr{A}}(\lambda^* + \eta(y^* - z)) - 1)\beta^* = 0,\\
&\omega u^* - \alpha^* = 0,
\end{aligned} \tag{11}$$

where $\chi_{\mathscr{A}}$ represents the identity function of the set $\mathscr{A}$. Finally, by using the property of the identity function and eliminating of the slack variables from (11), one can summarize the optimality conditions of the original problem (P) as follows [11]:

$$-\Delta y^* + \lambda^* - u^* - g = 0, \tag{12a}$$

$$\lambda^* - \max\{0, \lambda^* + \eta(y^* - z))\} = 0, \tag{12b}$$

$$y^* - y_d - \omega\Delta u^* = 0, \quad \text{in } \mathscr{N}, \tag{12c}$$

$$u^* = 0, \quad \text{in } \mathscr{A}. \tag{12d}$$

In the next section, we use the optimality system (12) for the numerical purposes.

3 Element free Galerkin method

In this section, we will introduce the element-free Galerkin method. To do this, it is necessary to first introduce the moving least squares (MLS) shape functions. Let $\Xi_I = \{\mathbf{x}_l\}_{l=1}^{N_I}$ and $\Xi_b = \{\mathbf{x}_l\}_{l=1}^{N_b}$ be the set of the selected nodes in the computational domain Ω and its boundary Γ, respectively. Set their union as $\Xi = \{\mathbf{x}_l\}_{l=1}^{N}$ where $N = N_I + N_b$.

If we have the value of the function f on Ξ as the known data $\mathbf{f} = \{f_l = f(\mathbf{x}_l)\}_{l=1}^{N}$, we can define the local approximating function $\hat{f}$ at the collocation point $\mathbf{x}$ as

$$\hat{f}(\mathbf{x},\bar{\mathbf{x}}) := \mathbf{\Phi}^t(\bar{\mathbf{x}}) \cdot \mathbf{a}(\mathbf{x}) = \sum_{l=1}^{M} a_l(\mathbf{x})\phi_l(\bar{\mathbf{x}}), \tag{13}$$

where ϕ_ls are the basis function of the spatial coordinates which are usually chosen as the monomial multivariate polynomials. Here, we choose quadratic basis $\boldsymbol{p}^T = \left(1, x_1, x_2, x_1^2, x_1x_2, x_2^2\right)$ and $\bar{\mathbf{x}} \in \Xi$ is the point in the local influenced domain of $\mathbf{x}_i$ which is determined by a compact support radial weight function $w_i(\mathbf{x})$. In the MLS approximation the unknown coefficients $a_l(\mathbf{x})$ are obtained by minimizing the following residual functional

$$\sum_{l=1}^{N}\left[\sum_{k=1}^{M} a_k(\mathbf{x})\phi_k(\mathbf{x}_l) - f_l\right]^2 w_l(\mathbf{x}). \tag{14}$$

The Gaussian weight function can be chosen as a good option for the weight function $w_i(\mathbf{x})$ as follows [7]:

$$w_l(\mathbf{x}) = \begin{cases} \frac{\exp(-(\|\mathbf{x}-\mathbf{x}_l\|/c)^2)-\exp(-(r/c)^2)}{1-\exp(-(r/c)^2)}, & 0 \le \|\mathbf{x}-\mathbf{x}_l\| \le r \\ 0, & o.w. \end{cases} \tag{15}$$

The necessary condition of the minimizing functional (14) with respect to $a(\mathbf{x})$ leads to the following condition [18]:

$$\left(\mathbf{A}^t\mathbf{Q}(\mathbf{x})\mathbf{A}\right)\mathbf{a}(\mathbf{x}) = \mathbf{A}^t\mathbf{Q}(\mathbf{x})\mathbf{f}, \tag{16}$$

where $\mathbf{A}$ is a $N \times M$ matrix with elements $\mathbf{A}_{m,n} = \phi_m(\mathbf{x}_n)$ when $n = 1,\ldots,N$ and $m = 1,\ldots,M$ and $\mathbf{Q}$ is a diagonal $N \times N$ weight matrix with the non-zero diagonal elements $\mathbf{Q}_{i,i} = w_i(\mathbf{x})$. Now, by substituting the solution of (16) for $\mathbf{a}(\mathbf{x})$ in (13), we have

$$\hat{f}(\mathbf{x},\bar{\mathbf{x}}) := \mathbf{\Phi}^t(\bar{\mathbf{x}}) \cdot \left(\mathbf{A}^t\mathbf{Q}(\mathbf{x})\mathbf{A}\right)^{-1}\mathbf{A}^t\mathbf{Q}(\mathbf{x})\mathbf{f}. \tag{17}$$

By definition of *the MLS shape functions* as

$$\mathbf{S}(\mathbf{x}) = (S_1(\mathbf{x}),\cdots,S_N(\mathbf{x})) = \mathbf{\Phi}^t(\mathbf{x})\left(\mathbf{A}^t\mathbf{Q}(\mathbf{x})\mathbf{A}\right)^{-1}\mathbf{A}^t\mathbf{Q}(\mathbf{x}), \tag{18}$$

the MLS approximation (17) of the function f can be rewritten as

$$\hat{f}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \cdot \mathbf{f} = \sum_{l=1}^{N} S_l(\mathbf{x}) f_l.$$

Now, if we use the MLS shape functions as the both test and trail functions in standard Galerkin scheme, the obtained method is called as *the element-free Galerkin method*. It should be noted that the MLS shape functions do not satisfy the Kronecker delta property and the essential boundary conditions cannot be applied directly in the MLS-based methods.

4 Implementation

In order to achieve a suitable numerical method for the optimal control problem (P), we use the optimality conditions extracted in (12). Our numerical method is based on selecting active and non-active set points, applying optimality conditions to these two sets, and then improving this selection.

Suppose that $\left(y^{(k)}, u^{(k)}, \lambda^{(k)}\right)$ are the solution functions obtained at k-th iteration of our algorithm. Based on (7)-(8) and Lagrangian definition in (10), the new choice for active and non-active sets is as follows:

$$\mathscr{A}^{(k+1)} = \{\mathbf{x} \in \Omega |\ \lambda^{(k)}(\mathbf{x}) + \eta(y^{(k)}(\mathbf{x}) - z(\mathbf{x})) > 0\}, \tag{19}$$

$$\mathscr{N}^{(k+1)} = \{\mathbf{x} \in \Omega |\ \lambda^{(k)}(\mathbf{x}) + \eta(y^{(k)}(\mathbf{x}) - z(\mathbf{x})) \leq 0\}. \tag{20}$$

On the other hand, obtaining λ and substituting it in (12b) yields that

$$\Delta y^* + u^* + g - \max\{0, \Delta y^* + u^* + g + \eta(y^* - z)\} = 0,$$

which suggests using the following iterative method to solve equations (12a)-(12b):

$$\Delta y^{(k+1)} + u^{(k+1)} + g = \max\{0, \Delta y^{(k)} + u^{(k)} + g + \eta(y^{(k+1)} - z)\}.$$

Here, we can write

$$\begin{aligned}
&\|\Delta y^{(k+1)} + u^{(k+1)} - \Delta y - u\|^2 \leq \\
&\|\Delta y^{(k)} + u^{(k)} - \Delta y - u + \eta(y^{(k+1)} - z)\|^2 - \|\Delta y^{(k+1)} + u^{(k+1)} - \Delta y^{(k)} - u^{(k)} - \eta(y^{(k+1)} - z)\|^2 + \\
&2\int_\Omega \left[\Delta y^{(k+1)} + u^{(k+1)} - \Delta y^{(k)} - u^{(k)} - \eta(y^{(k+1)} - z)\right] \cdot \left[\Delta y^{(k+1)} + u^{(k+1)} - \Delta y - u\right] dx \leq \\
&\|\Delta y^{(k)} + u^{(k)} - \Delta y - u\|^2 - \|\Delta y^{(k+1)} + u^{(k+1)} - \Delta y^{(k)} - u^{(k)}\|^2 + \\
&2\eta \int_\Omega \left[\Delta y^{(k+1)} + u^{(k+1)} - \Delta y - u\right] \cdot \left[y^{(k+1)} - z\right] dx.
\end{aligned}$$

Now, if

$$\int_\Omega \left[\Delta y^{(k+1)} + u^{(k+1)} - \Delta y - u\right] \cdot \left[y^{(k+1)} - z\right] dx \leq 0,$$

then

$$\begin{aligned}
\|\Delta y^{(k+1)} + u^{(k+1)} - \Delta y - u\|^2 &\leq \|\Delta y^{(k)} + u^{(k)} - \Delta y - u\|^2 - \|\Delta y^{(k+1)} + u^{(k+1)} - \Delta y^{(k)} - u^{(k)}\|^2 \\
&\leq \|\Delta y^{(k)} + u^{(k)} - \Delta y - u\|^2.
\end{aligned}$$

By considering equation (12d) and (19), for any $\mathbf{x} \in \mathscr{A}^{(k+1)}$ we can use the following equation for updating:

$$\Delta y^{(k+1)}(\mathbf{x}) = \Delta y^{(k)}(\mathbf{x}) + u^{(k)}(\mathbf{x}) + \eta\left(y^{(k+1)}(\mathbf{x}) - z(\mathbf{x})\right), \tag{21}$$

and according to (12c) and considering $\lambda^{(k+1)} = 0$, for updating in non-active regions, we have

$$-\Delta y^{(k+1)}(\mathbf{x}) - u^{(k+1)}(\mathbf{x}) = g(\mathbf{x}), \tag{22}$$
$$y^{(k+1)}(\mathbf{x}) - y_d(\mathbf{x}) - \omega\Delta u^{(k+1)}(\mathbf{x}) = 0. \tag{23}$$

Now, by using local MLS approximation for approximating state and control functions, we have

$$y^{(k+1)}(\mathbf{x}) \simeq \hat{y}^{(k+1)}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \cdot \mathbf{y}^{(k+1)} = \sum_{l=1}^{N} S_l(\mathbf{x}) y_l^{(k+1)}, \tag{24}$$
$$u^{(k+1)}(\mathbf{x}) \simeq \hat{u}^{(k+1)}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \cdot \mathbf{u}^{(k+1)} = \sum_{l=1}^{N} S_l(\mathbf{x}) u_l^{(k+1)}.$$

Applying Gauss formula and obtaining weak form of (21)-(23) yeilds

$$\int_\Omega \nabla y^{(k+1)} \nabla v d\mathbf{x} + \eta \int_{\mathscr{A}^{(k+1)}} y^{(k+1)} v d\mathbf{x} - \int_{\mathscr{N}^{(k+1)}} u^{(k+1)} v d\mathbf{x} = \int_\Omega f v d\mathbf{x}, \tag{25}$$
$$\omega \int_{\mathscr{N}^{(k+1)}} \nabla u^{(k+1)} \nabla v d\mathbf{x} + \int_{\mathscr{N}^{(k+1)}} y^{(k+1)} v d\mathbf{x} = \int_{\mathscr{N}^{(k+1)}} y_d v d\mathbf{x}, \tag{26}$$

where

$$f(\mathbf{x}) := \begin{cases} -\Delta y^{(k)}(\mathbf{x}) - u^{(k)}(\mathbf{x}) + \eta z(\mathbf{x}) & \mathbf{x} \in \mathscr{A}^{(k+1)}, \\ g(\mathbf{x}) & \mathbf{x} \in \mathscr{N}^{(k+1)}. \end{cases} \tag{27}$$

If we use the MLS shape functions (18) as the test functions, we can discretize (25)-(26) as

$$(H + \eta L)\mathbf{y} - M\mathbf{u} = F, \tag{28}$$
$$M\mathbf{y} + \omega H\mathbf{u} = G, \tag{29}$$

where for $i, j \in \{1, \ldots, N\}$

$$H(i,j) = \int_\Omega \nabla S_i(\mathbf{x}) \nabla S_j(\mathbf{x}) d\mathbf{x}, \tag{30}$$
$$L(i,j) = \int_{\mathscr{A}^{(k+1)}} S_i(\mathbf{x}) S_j(\mathbf{x}) d\mathbf{x}, \tag{31}$$
$$M(i,j) = \int_{\mathscr{N}^{(k+1)}} S_i(\mathbf{x}) S_j(\mathbf{x}) d\mathbf{x}, \tag{32}$$
$$F(i) = \int_\Omega f(\mathbf{x}) S_i(\mathbf{x}) d\mathbf{x}, \tag{33}$$
$$G(i) = \int_{\mathscr{N}^{(k+1)}} y_d(\mathbf{x}) S_i(\mathbf{x}) d\mathbf{x}. \tag{34}$$

To start the algorithm, we can obtain $(\bar{y}, \bar{u})$ from solving the optimal control problem $\min \mathscr{F}$ constrained to $-\Delta y = u + g$ and then set $\bar{\lambda} = \max\{0, \Delta\bar{y} + \bar{u} + g\}$. We can now summarize the algorithm as follows.

Algorithm 1. EFG algorithm for optimal control of elliptic variational inequality (P):

1. Initialize with choosing $(\mathbf{y}^{(0)}, \mathbf{u}^{(0)}, \boldsymbol{\lambda}^{(0)})$ from MLS approximation of $(\bar{y}, \bar{u}, \bar{\lambda})$ and $\eta > 0$.
2. Divide the computational indices into $\mathscr{A}^{(k+1)} = \{i | \ \boldsymbol{\lambda}^{(k)}(\mathbf{x}_i) > 0\}$and $\mathscr{B}^{(k+1)} = \{i | \ \boldsymbol{\lambda}^{(k)}(\mathbf{x}_i) = 0\}$.
3. Define f from (27).
4. Compute matrices from (30)–(34).
5. Solve (28)–(29) for $\mathbf{y}^{(k+1)}$ and $\mathbf{u}^{(k+1)}$.
6. Update $\boldsymbol{\lambda}^{(k+1)} = \max\{0, \boldsymbol{\lambda}^{(k)} + \eta(\mathbf{y}^{(k+1)} - \mathbf{z})\}$.
7. Repeat steps 2-7 until the stopping condition is met.

$$\mathscr{B}^{(k)} = \mathscr{B}^{(k+1)}.$$

The Laplacian of the state variable at the previous iterate, $\Delta y^{(k)}(x)$, is needed in computing (27). It is computed by taking the *second-order spatial derivatives of the MLS shape function approximation*. Specifically, by considering (24) the Laplacian is evaluated as

$$\Delta y^{(k)}(x) = \sum_{l=1}^{N} \Delta S_l(x)\, y_l^{(k)},$$

where $\Delta S_l(x)$ denotes the Laplacian of the MLS shape functions. We note that, as commonly reported in the literature, second-order derivatives of MLS approximations are generally more sensitive to node distribution and numerical integration than first-order derivatives, and may introduce a higher level of numerical noise. In the present computations, numerical stability was preserved by using sufficiently smooth weight functions, adequate support sizes, and refined background integration grids. No spurious oscillations were observed in the reported numerical results.

5 Numerical illustrations

In this section, the element-free Galerkin method is used to solve two numerical examples of the optimal control problem constrained with elliptic variational inequality in the form of (P). The method is implemented in MATLAB software and on a PC with 4 Gigabytes RAM and 3.2 Giga-Hertz Core i5 Processor. In our examples, we choose the radius of the influence r in (15) as a ratio of fill distance h, which is defined as follows:

$$h = \sup_{\mathbf{x} \in \Omega} \min_{\mathbf{x}_i \in \Xi} \|\mathbf{x}_i - \mathbf{x}\|_2.$$

To be more precise, we have set $r = \beta h$, and experience has shown that choosing a β coefficient between 2 and 3 can yield better results [9].

In the numerical implementation, the integration domain is first partitioned into a collection of nonoverlapping subregions, and a Gaussian quadrature rule is then applied locally on each subdomain. In the present study, the computational domain is divided into 16 subdomains, and the 8–point Gauss–Lobatto–Legendre quadrature rule is employed for numerical integration in each part. It is worth noting that, in general, increasing the number of Gaussian quadrature points does not necessarily lead to higher accuracy and may even introduce instability or loss of convergence in the numerical scheme. More advanced numerical integration techniques for the EFG frameworks have been proposed in [16, 17].

It should be noted that due to the high computational cost required to calculate the H, M, and L matrices, this method cannot be compared with methods such as the finite difference method and the finite element method. The main goal here is to demonstrate the feasibility and accuracy of using the EFG method to solve the optimal control problem constrained by the variational inequality.

Example 1. As the first example, we consider the two dimensional optimal control problem (P) in $\Omega = [-1,1] \times [-1,1]$ with the following parameters:

$$\begin{aligned}
y_d(x_1,x_2) &= \begin{cases} 200x_1x_2(x_1-0.5)^2(1-x_2), & 0 < x_1 \leq 0.5, \\ 200x_2(x_1-1)(x_1-0.5)^2(1-x_2), & 0.5 < x_1 \leq 1, \end{cases} \\
g(x_1,x_2) &= \begin{cases} 400\left(x_1(x_1-0.5)^2 - x_2(3x_1-1)(1-x_2)\right), & 0 < x_1 \leq 0.5, \\ -200(x_1-0.5), & 0.5 < x_1 \leq 1, \end{cases} \\
z(x_1,x_2) &= 0,
\end{aligned}$$

and control parameter $\omega = 100$ in objective functional. This problem has an optimal state as follows, which is obtained for the optimal null control $u = 0$:

$$y = \begin{cases} y_d, & 0 < x_1 \leq 0.5, \\ 0, & 0.5 < x_1 \leq 1. \end{cases}$$

In addition, the optimal value of the objective functional is $\mathscr{F} = \frac{25}{504}$.

Table 1 contains L_2-error of the optimal state and control obtained with presented method for different values of h with different vlues for β. Also, absolute error of objective functional value is reported. Here, the computational order of error is defined as follows:

$$C_{order} = \frac{\log(E_1) - \log(E_2)}{\log(h_1) - \log(h_2)},$$

where E_1 and E_2 are the computed errors for h_1 and h_2, respectively.

Table 2 also compares the state function error in the L_2 norm between different meshless methods and the EFG method. In this table, the results of the EFG method with different h and $r = 2.5h$ are reported. Also, the results of the generalized finite difference methods (GFD) [6], the radial point interpolator method (RPIM) [19], the direct local meshless Petrov-Glerkin method(DMLPG) [10], and the radial basis function-finite difference method (RBF-FD) [23], considering the same radius and fill distance parameters, can be seen in this table.

Example 2. As the second example, we consider the two dimensional optimal control problem (P) in $\Omega = [-1,1] \times [-1,1]$ with the following parameters:

$$\begin{aligned}
y_d(x_1,x_2) &= 5x_1 + x_2 - 1, \\
z(x_1,x_2) &= 4(x_1(x_1-1) + x_2(x_2-1)) + 1.5, \\
g(x_1,x_2) &= 0.1,
\end{aligned}$$

Table 1: The computational order of error for Example 1

		state		*control*		*objective functional*	
β	h	$\\|y^* - \hat{y}\\|_2$	C_{order}	$\\|u^* - \hat{u}\\|_2$	C_{order}	$\|\mathscr{F}(y^*, u^*) - \mathscr{F}(\hat{y}, \hat{u})\|$	C_{order}
2.50	0.2	$1.8291e-02$	–	8.8291e-03	–	3.7148e-02	–
	0.1	$4.5877e-03$	1.9953	2.1879e-03	2.0127	9.8479e-03	1.9154
	0.05	$1.1419e-03$	2.0063	4.8272e-04	2.1803	2.4839e-03	1.9872
	0.02	$1.9080e-04$	1.9527	7.3534e-05	2.0536	4.1590e-04	1.9504
2.65	0.2	$1.1805e-02$	–	7.6825e-03	–	1.9628e-02	–
	0.1	$3.0687e-03$	1.9437	1.9137e-03	2.0052	5.2225e-03	1.9101
	0.05	$7.7638e-04$	1.9828	4.4110e-04	2.1172	1.3286e-03	1.9748
	0.02	$1.2570e-04$	1.9871	6.6185e-05	2.0701	2.1737e-04	1.9757
2.80	0.2	$1.1372e-02$	–	6.3091e-03	–	1.6510e-02	–
	0.1	$2.8745e-03$	1.9841	1.5841e-03	1.9938	4.3493e-03	1.9245
	0.05	$7.2157e-04$	1.9941	3.7140e-04	2.0926	1.0920e-03	1.9938
	0.02	$1.2020e-04$	1.9560	5.6332e-05	2.0583	1.7502e-04	1.9981

Table 2: Comparison of errors in different methods for Example 1

h	EFG	GFE	RPIM	DMLPG	RBF-FD
0.20	$1.8291e-02$	$1.1217e-02$	$6.8105e-02$	$9.183e-02$	$6.7162e-02$
0.10	$4.5877e-03$	$3.6329e-03$	$9.1547e-03$	$5.8350e-02$	$8.9176e-03$
0.05	$1.1419e-03$	$7.0911e-04$	$2.3549e-03$	$1.0915e-02$	$2.0228e-03$
0.02	$1.9080e-04$	$8.1285e-05$	$5.2850e-04$	$7.1185e-03$	$3.0473e-04$

and control parameter $\omega = 0.1$ in objective functional. The obtained optimal state and control by applying the element free Galerkin method with $h = 0.05$ are plotted in Figure 1. The obtained Lagrangian function λ is also plotted in Figure 2, showing that the element free Galerkin method has been able to distinguish the active and non-active regions well.

Table 3 reports the numerical value obtained for the objective functional along with the number of iterations of the algorithm to solve the problem for different values of h and $r = 2.50$. Since this problem has no analytical solution and it is not possible to measure the absolute error for it, the solution obtained from solving the problem with the spectral method with $N = 31$ reported in [21] is considered as the reference solution and the distance of the solution obtained from the EFG method with this solution is reported as the approximated error of the method. Also, in this table, the ratio of nodes that are in the active region to the total number of nodes is reported.

6 Conclusion

In this paper, moving least squares shape functions are employed to design an element-free Galerkin approach to solve the optimal control of an elliptic variational inequality. The optimality conditions of the problem are obtained by a regularization method, and a new iterative numerical method is introduced for solving this optimality system. The MLS shape functions are used in the element free Galerkin

Table 3: Comparison of errors in different methods for Example 2

h	$\mathscr{F}(y^*,u^*)$	approximated error	iterations	CPU time(s)	active nodes ratio
0.20	3.2743	$1.7429e-01$	6	3.15	0.198
0.10	3.3481	$4.1046e-02$	7	3.36	0.171
0.05	3.3516	$6.3260e-03$	11	6.98	0.153
0.02	3.3579	$2.2159e-03$	23	29.50	0.144

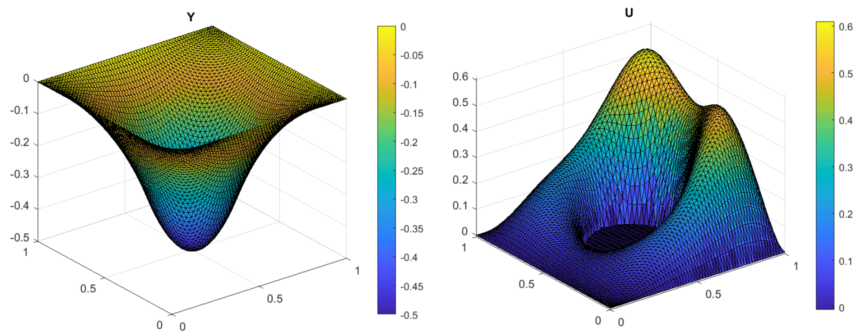

Figure 1: The state and control functions for Example 2

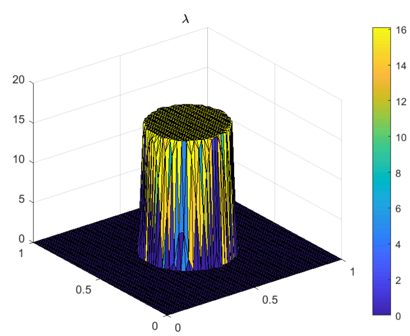

Figure 2: The obtained Lagrangian function for Example 2

approach for numerical solving of mixed complementarity conditions in each iteration of the method. Finally, two examples of the considered optimal control problem are solved by the presented method. The calculated results indicate numerical convergence of the second order for the method used. This show that the presented element-free Galerkin method is efficient and provides accurate solutions for this nonlinear optimal control problem.

However, the main problem in applying the EFG method is still the use of a suitable integration method, which may require a lateral meshing, and this somewhat affects the independence of the method from the choice of mesh. Research on a suitable numerical integration method for use in the EFG method is an open issue that needs to be addressed. The author's subsequent research is on the use of the EFG method to solve optimal control problems constrained by the inequalities of parabolic variations. Also, using the EFG method in numerically solving optimal control of contact problems, which are usually three-dimensional variational inequalities, may be considered for future research.

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