Journal of Mathematical Modeling
Vol. 14, No. 2, 2026, pp. 787-801. Research Article

JMM

Modeling log-volatility with zero returns: empirical evidence for asymmetric SV and log-GARCH models

Abdeljalil Settar[†], Sara Chegdal[†], Mustapha Kabil[†], Ghassane Benrhmach[‡*]

[†]*Laboratory of mathematics, computer science and applications (LMCSA), FST Mohammedia, Hassan II university, 20650 Casablanca, Morocco*
[‡]*Department of Mathematics and Statistics, College of Engineering, Abu Dhabi University, 59911 Abu Dhabi, UAE*

Email(s): abdeljalilsettar@gmail.com, chegdalsara@gmail.com, kabilfstm@gmail.com, ghassane.benrhmach@adu.ac.ae

Abstract. In this work, we address the challenges posed by zero returns in both stochastic volatility (SV) and log-GARCH models in their asymmetric form. Building upon Expectation-Maximization imputation for handling zero returns, we propose a unified approach that enhances parameter estimation robustness for both model classes. Specifically, we employ the Quasi-Maximum Likelihood estimation, incorporating the Kalman filter for both asymmetric SV and asymmetric log-GARCH models, to ensure robust parameter estimation even in the presence of zero returns. By comparing the performance of these models under our proposed framework, we provide new insights into their relative strengths in capturing the asymmetric volatility dynamics in the presence of zero returns. This contribution extends the existing literature by proposing a computational framework applicable to such models, based on a logarithmic specification of volatility.

Keywords: Stochastic volatility; log-GARCH; zero returns; quasi-maximum likelihood; Kalman filter.
AMS Subject Classification 2010: 62M10; 62F10; 62P20.

1 Introduction

In financial econometrics, volatility modeling plays a central role in understanding the dynamics of asset returns and in applications such as risk management, option pricing, and portfolio optimization. Among the most widely used approaches are the logarithmic specifications of volatility, which have proven highly effective in capturing the complex dynamics of financial returns especially their ability to ensure the non-negativity of volatility addressing one of the key limitations of traditional GARCH

*Corresponding author
Received: 15 October 2025/ Revised: 16 December 2025/ Accepted: 07 January 2026
DOI: 10.22124/jmm.2026.31979.2889

models introduced by Bollerslev [5]. Two prominent models that represent different classes of volatility modeling are the log-GARCH (log-Generalized Autoregressive Conditional Heteroscedasticity) model and the stochastic volatility (SV) model.

The first-order log-GARCH (or log-GARCH(1,1)) model first introduced by Geweke [11] and further developed by Pantula [18] belongs to the family of observation-driven models which assume that volatility σ_t is conditionally determined by past information and driven by observed returns $\varepsilon_t = \log(p_t) - \log(p_{t-1})$ for an asset price series (p_t) as follows:

$$\begin{aligned} \varepsilon_t &= \sigma_t \eta_t , \quad \eta_t \sim iid(0,1), \\ \log(\sigma_{t+1}^2) &= \alpha_0 + \alpha \log(\varepsilon_t^2) + \beta \log(\sigma_t^2), \end{aligned} \tag{1}$$

where α_0, α and β are real parameters.

This logarithmic transformation ensures that $\sigma_t > 0$ and provides a stability that makes the model more robust in capturing extreme volatility dynamics. Moreover, as an observation-driven model, the log-GARCH model conditions on past observable data, making it computationally efficient and suitable for real-time forecasting [7].

A look at the log-GARCH litterature points to a first power log-GARCH generalization provided by Sucarrat [24] using a power log-variance specification describing $\log(\sigma^d)$ for a fixed power d. Further, general framework for the estimation and inference in so-called univariate and multivariate log-GARCH-X models was proposed in [9, 26] when covariates or other conditioning variables are added to the volatility equation. The log-GARCH model has seen various modifications to accommodate asymmetric volatility responses to positive and negative shocks [8, 13]. In the former, the log-moment structure, consistency and asymptotic normality of the Quasi-Maximum Likelihood estimation was established.

On the other hand, the first-order SV (or SV(1)) model introduced by Melino and Turnbull [17] and Taylor [29] is considered in contrast as a parameter-driven approach that treats volatility h_t as a latent process taking an AR(1) form evolving independently from the observable return process. The canonical SV model is defined by:

$$\begin{aligned} \varepsilon_t &= exp(h_t/2)\eta_t , \quad \eta_t \sim iid(0,1), \\ h_{t+1} &= \omega + \beta h_t + \delta_t , \quad \delta_t \sim iidN(0,\delta^2), \end{aligned} \tag{2}$$

where (η_t) and (δ_t) are stochastically independent, ω, β and δ are parameters which are beforehand free of any constraints.

The SV framework offers greater flexibility by allowing volatility to follow its own stochastic process, which often captures persistence and asymmetric responses to market shocks more effectively than GARCH-type models [12, 28]. However, one of the challenges of SV models lies in the complexity of parameter estimation due to the unobserved nature of volatility, often requiring advanced estimation techniques such as Bayesian inference or Markov Chain Monte Carlo (MCMC) methods [20]. The embedding of the Kalman filter in the QMLE is widely used in the literature for SV model estimation [15, 19] as an approach that is not computationally intensive. The SV model has been extended to handle features such as leverage effects giving rise to asymmetric versions of SV models as in Harvey [16] and Chirico [6].

It is important to note that log-GARCH model and SV model, though often compared, are not equivalent. As noted by Sucarrat [26], the log-GARCH model is more general than the SV model, as it can admit a log-GARCH representation, but not necessarily vice versa [3,9]. Despite their differences, both models encounter significant challenges in the presence of zero returns which can distort parameter estimates and hinder the performance of volatility forecasts [25]. For log-GARCH model, the volatility equation in (1) does not hold and requires to assume $\mathbb{P}(\eta_t = 0) = 0$ [8]. On the other hand, even if the SV model is well defined eventually in presence of zero returns, the direct use of the Kalman filter is impossible since the log-square of returns is ill-defined. One way to perform a QMLE of log-GARCH model in presence of zero returns was proposed by Francq [8] by multiplying the ARCH coefficient α by $\mathbb{I}_{\eta_{t-1}\neq 0}$, so that the zero returns do not occur in the volatility recursion equation. In Sucarrat [25], zeros are treated as missing values using an Expectation-Maximization algorithm, whereby log-zero returns are replaced by the conditional expectation $\mathbb{E}(\log \varepsilon_t^2 | \varepsilon_1, \ldots, \varepsilon_{t-1})$ estimated by an ARMA representation. In the same vein, we approached in [21, 22] an imputation algorithm of zero returns using the information filter allowing a QMLE of parameters with no assumptions on the existence of the log-moment orders greater than one.

In this work, we employ the QML estimation incorporating the Kalman filter for both the asymmetric log-GARCH and asymmetric SV models to ensure robust parameter estimation even in the presence of zero returns building upon previous imputation techniques for handling zeros [8, 21, 22, 25] and using the so-called Stochastic Adaptive Stochastic Approximation (SASA) algorithm as a stochastic optimization routine [23]. By comparing thought real data the performance of these models under our proposed framework, we provide new insights into their relative strengths in enhancing parameter estimation accuracy and capturing volatility dynamics in the presence of zero returns. Standard notations are used through this work. By *iid* we denote any stochastic process which is independent and identically distributed. Especially, a white noise process is denoted WN. The $\mathbb{I}_A$ is the indicator function over a set A such that $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise.

The remainder of the paper is organized as follows. Section 2 introduces the asymmetric stochastic volatility model and its estimation framework. The same estimation procedure is applied to the asymmetric log-GARCH specification to be presented in Section 3. Section 4 reports the empirical results and comparative analysis. Section 5 concludes with key findings.

2 Asymmetric stochastic volatility model

2.1 State space representation

Let us start by deriving a state space representation of the SV model (2) with measurement $y_t = \log(\varepsilon_t^2)$ and log-volatility $h_t = \log(\sigma_t^2)$ in the sense that:

$$y_t = h_t + \mu + u_t, \tag{3}$$

where $u_t = \log(\eta_t^2) - \mu \sim WN(0, \sigma^2)$ and $\mu = \mathbb{E}\left(\log(\eta_t^2)\right)$.

The restricted estimation of the SV model involves in approximating the auxiliary parameters μ and σ^2 according to the distribution of η_t. Especially, if $\eta_t \sim iidN(0,1)$, μ is known to be -1.27 and σ^2 is thought to be 4.93 [1].

The asymmetric volatility is obtained according to the specification recently proposed by Chirico [6] assuming a negative correlation between error terms of stock returns ε_t and the volatility h_t namely between η_t and δ_t such that $\mathbb{E}(\delta_t|\eta_t) = \gamma\eta_t$, with $\gamma < 0$ as follows:

$$h_{t+1} = \omega + \phi h_t + \gamma\eta_t + \nu_t. \tag{4}$$

It is worth noting that equations (3) and (4) form a nonlinear state-space representation of the asymmetric SV model since η_t is endogenous as $u_t = \log(\eta_t^2)$. To address this nonlinearity, Chirico [6] proposed an iterative procedure in which η_t is estimated a priori using the Kalman smoother before predicting the latent log-volatility h_t. The resulting estimate, denoted $\widetilde{\eta}_t$, is treated as a deterministic proxy for the return innovation η_t. Following this approach, the state equation (4) is replaced by

$$h_{t+1} = \omega + \phi h_t + \gamma \widetilde{\eta}_t + \nu_t, \tag{5}$$

where ν_t represents the exogenous innovation on h_t derived from $\delta_t = \gamma \widetilde{\eta}_t + \nu_t$ with

$$\begin{pmatrix} u_t \\ \nu_t \end{pmatrix} \sim iidN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \nu^2 \end{pmatrix} \right).$$

Thus, the asymmetric SV model, hereafter denoted ASV, can be rewritten now using the following auxiliary state-space representation :

$$\begin{aligned} y_t &= \mu + h_t + u_t, \\ h_{t+1} &= \omega + \phi h_t + \gamma \widetilde{\eta}_t + \nu_t. \end{aligned} \tag{6}$$

2.2 QML estimation of ASV parameters

In this work, ASV (resp. ALGARCH) model will be estimated by (pseudo) QML (resp. QML) estimation combined with the Kalman filter from observed returns containing zeros. The Expectation-Maximization algorithm is then used to replace each measurement y_t corresponding to $\varepsilon_t = 0$ by its predicted estimate provided by the Kalman filter. The new measurement $\widetilde{y}_t$ generated after this imputation constitutes the information set:

$$\mathscr{I}_n = \{\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_n\}, \tag{7}$$

where $\widetilde{y}_t = \mathbb{E}(y_t | \mathscr{I}_{t-1})$ if $\varepsilon_t = 0$, and $\widetilde{y}_t = \log(\varepsilon_t^2)$, otherwise.

In this section for ASV model, we start by estimating h_t using the Kalman filter applied to model (6) given a parameter vector $\theta = (\theta^1, \theta^2, \theta^3, \theta^4)' = (\omega, \phi, \gamma, \nu)' \in \Theta$, with $\Theta = (-\infty, \infty) \times (-1, 1) \times (-\infty, 0] \times (0, \infty)$. Let $h_{t|t-1}(\theta) = \mathbb{E}(h_t | \mathscr{I}_{t-1}; \theta)$ and $h_{t|t}(\theta) = \mathbb{E}(h_t | \mathscr{I}_t; \theta)$ stand for the predicted and filtered estimates of h_t. $p_{t|t-1}(\theta) = \mathbb{V}(h_t | \mathscr{I}_{t-1}; \theta)$ and $p_{t|t}(\theta) = \mathbb{V}(h_t | \mathscr{I}_t; \theta)$ are their respective predicted and filtered estimate variances. Furthermore, $f_{t|t-1}(\theta) = \mathbb{V}(y_t | \mathscr{I}_{t-1}; \theta)$ and $k_t(\theta)$ is the kalman gain. To simplify the notation, let $h_{.|.}(\theta) = h_{.|.}$, $p_{.|.}(\theta) = p_{.|.}$, and $f_{.|.}(\theta) = f_{.|.}$. Next, we derive the Kalman filter equations under the covariance stationary assumption $|\phi| < 1$ as described by the KF-ASV Algorithm 1 below.

The -2 log-likelihood resulting from the prediction-error decomposition associated with the Kalman filter (1) applied to the auxiliary state-space representation (6) ignoring the constant term $-\frac{n}{2}\log(2\pi)$ writes:

$$l_n^{ASV}(\theta; \mathscr{I}_n) = n^{-1} \sum_{t=1}^{n} \left[\log(f_{t|t-1}(\theta)) + \left(\widetilde{y}_t - h_{t|t-1}(\theta) - \mu\right)^2 f_{t|t-1}^{-1}(\theta) \right]. \tag{8}$$

In our approach, the innovation proxy series $\{\widetilde{\eta}_{t,0}\}_t$ is initialized in the l_n^{ASV} optimization algorithm by $\{\widetilde{\varepsilon}_t / s\}_t$, being s the sample standard deviation of ε_t, and then updated at each optimization iteration

Algorithm 1: KF-ASV

Input: $\{\varepsilon_1, \dots, \varepsilon_n\}$, $\{\widetilde{\eta}_{1,k-1}, \dots, \widetilde{\eta}_{n,k-1}\}$, θ_{k-1}, $k \geq 1$
Output: $\{h_{t|t-1}\}$, $\{f_{t|t-1}\}$, $\{k_t\}$, $\{\widetilde{y}_t\}$
Initialize:
$\widetilde{\eta}_{0,k-1} = \mathbb{E}(\widetilde{\eta}_{t,k-1})$
$h_{1|0} = (\omega + \gamma\widetilde{\eta}_{0,k-1})(1-\phi)^{-1}$
$p_{1|0} = (\gamma^2 + \nu^2)(1-\phi^2)^{-1}$
$f_{1|0} = p_{1|0} + \sigma^2$
$y_{1|0} = h_{1|0} + \mu$
for $t = 1$ to n **do**
  if $\varepsilon_t = 0$ **then**
    $\widetilde{y}_t = y_{t|t-1}$
  else
    $\widetilde{y}_t = \log(\varepsilon_t^2)$
  end if
  $h_{t|t} = h_{t|t-1} + p_{t|t-1} f_{t|t-1}^{-1} (\widetilde{y}_t - h_{t|t-1} - \mu)$
  $p_{t|t} = p_{t|t-1} - p_{t|t-1}^2 f_{t|t-1}^{-1}$
  $k_t = \phi p_{t|t-1} f_{t|t-1}^{-1}$
  $h_{t+1|t} = \phi h_{t|t} + \gamma\widetilde{\eta}_{t,k-1} + \omega$
  $p_{t+1|t} = \phi^2 p_{t|t} + \nu^2$
  $f_{t+1|t} = p_{t+1|t} + \sigma^2$
  $y_{t+1|t} = h_{t+1|t} + \mu$
end for
return $\{h_{t|t-1}\}$, $\{f_{t|t-1}\}$, $\{k_t\}$, $\{\widetilde{y}_t\}$

$k \geq 1$ by $\{\widetilde{\eta}_{t,k} = exp((\widetilde{y}_t - \widetilde{h}_t)/2)\}_t$, where $\widetilde{h}_t$ is the smoothed estimate of h_t obtained by the RTS Algorithm 2, named SKF-ASV. In this way, to ensure the linearity of the auxiliary state-space model (6) at each iteration $k \geq 1$, $\widetilde{\eta}_{t,k-1}$ is treated as fixed when evaluating the likelihood criterion (8) which should therefore be interpreted as a pseudo-QML criterion conditional on $\{\widetilde{\eta}_{t,k-1}\}_t$.

The optimization method employed for (quasi) maximum likelihood estimation significantly impacts both convergence speed and estimation quality. We utilize algorithm to estimate ASV and ALGARCH model parameters, leveraging its adaptive learning rate mechanism and robust regularization approach. Here, the SASA method is employed to minimize l_n^{ASV} in (8) as given by Algorithm 3, called ASV-SASA. A projection operator Π ensuring the stability of the parameters within bounded regions [4] is defined by $\Pi : \mathbb{R}^4 \to \overline{\Theta}$ such that for all $\theta = (\theta^1, \theta^2, \theta^3, \theta^4)'$, $\Pi(\theta) = \left[\Pi^1(\theta^1), \Pi^2(\theta^2), \Pi^3(\theta^3), \Pi^4(\theta^4)\right]$ where $\overline{\Theta} = \prod_{i=1}^4 \left[\theta^i_{min}, \theta^i_{max}\right] \subset \Theta$ represents the constraint set for parameters within limits θ^i_{min} and θ^i_{max}. Each projection component $\Pi^i(\theta^i)$ for parameter θ^i $(i = 1,2,3,4)$ is defined, as given in Vijayan and Prashanth [30], by $\Pi^i(\theta^i) = min\left(max(\theta^i_{min}, \theta^i), \theta^i_{max}\right)$.

The Rauch-Tung-Striebel (RTS) smoother is a widely used approach in state-space modeling, enhancing Kalman filter estimates by integrating both forward and backward passes through the time series by incorporating future data, achieving refined state estimates for each time step.

Algorithm 2: SKF-ASV

Input: $\theta \in \Theta$
Output: $\{\widetilde{h}_t\}$
Call KF-ASV(θ)
Initialize $r_n = 0$
for $t = n$ down to 1 **do**
  $r_{t-1} = f_{t|t-1}^{-1}(\widetilde{y}_t - h_{t|t-1}) + (\phi - k_t) r_t$
end for
Initialize $\widetilde{h}_1 = h_{1|0} + p_{1|0} r_0$
for $t = 1$ to n **do**
  $\widetilde{h}_{t+1} = \omega + \phi \widetilde{h}_t + \nu^2 r_t$
end for
return $\{\widetilde{h}_t\}$

3 Asymmetric log-GARCH model

3.1 State-space representation

The first-order asymmetric log-GARCH model (denoted simply ALGARCH) has been suggested by Francq et al. [8] where the log-volatility is specified as:

$$h_{t+1} = \alpha_0 + \left(\alpha_1 \mathbb{I}_{\{\varepsilon_t > 0\}} + \alpha_2 \mathbb{I}_{\{\varepsilon_t < 0\}}\right) y_t + \beta h_t \tag{9}$$

with y_t is the same measurement described by (3), α_0, α_1, α_2 et β are parameters a priori without any constraints. Noting that the usual symmetric log-GARCH (1) corresponds to the case $\alpha_1 = \alpha_2$. To obtain the state-space representation of ALGARCH model, let substitute y_t in (3) into (9) and rewrite h_t as:

$$h_{t+1} = \alpha_0^* + \beta^* h_t + \alpha^* u_t, \tag{10}$$

where $\alpha^* = \alpha_1$ if $\varepsilon_t > 0$ and $\alpha^* = \alpha_2$ if $\varepsilon_t < 0$, $\beta^* = \alpha^* + \beta$ and $\alpha_0^* = \alpha_0 + \alpha^* \mu$.
Finally, the state space equations in the innovation form [2, 31] corresponding to ALGARCH model write:

$$\begin{aligned} y_t &= \mu + h_t + u_t, \\ h_{t+1} &= \alpha_0^* + \beta^* h_t + \alpha^* u_t. \end{aligned} \tag{11}$$

3.2 The QML estimation of ALGARCH parameters

The state-space model (11) is given in innovation form, then the uncorrelated noise assumption is no longer fulfilled. The Kalman filter equations must be changed in equations of k_t, $h_{t+1|t}$ and $p_{t+1|t}$ [14, p. 113].

Given parameter vector $\psi = \left(\psi^1, \psi^2, \psi^3, \psi^4\right)' = (\alpha_0^*, \beta^*, \alpha_1, \alpha_2)'$ belongs to $\Psi = (-\infty, \infty) \times (-1, 1) \times (-\infty, \infty)^2$, Algorithm 4 outlines the kalman filter of ALGARCH model applied to stationary state space representation (11) with $|\beta^*| < 1$. Now, a QML estimate of ψ is obtained as any solution minimizing -2

Algorithm 3: ASV-SASA

Input: $\{\varepsilon_1,\dots,\varepsilon_n\}$, $\theta_0 \in \Theta$
Output: θ_{k+1}
Calibration: $(a_{init},\xi,r,\lambda) = (0.005, 0.05, 0.6, 0.001)$
Initialize $H_0 = I_4$ (4×4 identity matrix)
for $t = 1$ to n **do**
  if $\varepsilon_t = 0$ **then**
    $\widetilde{\varepsilon}_t = y_{1|0}$
  else
    $\widetilde{\varepsilon}_t = \varepsilon_t$
  end if
  $\widetilde{\eta}_{t,0} = \widetilde{\varepsilon}_t / s$
end for
for $k = 1$ to I **do**
  Call KF-ASV($\theta_{k-1}, \widetilde{\eta}_{t,k-1}$) 1
  Call SKF-ASV(θ_{k-1}) 2
  Compute $\widetilde{\eta}_{t,k} = \exp((\widetilde{y}_t - \widetilde{h}_t)/2)$ for $t = 1,\dots,n$
  Generate $\Delta_k \sim U[-1,1]^{4\times 2}$
  Compute $c_k = 0.1k^{-0.101}$
  Compute $f_k^{\pm} = l_n^{ASV}(\theta_k \pm c_k\Delta_k)$
  Estimate gradient $g_k = (f_k^{+} - f_k^{-})(2c_k)^{-1}\Delta_k$
  Update gradient average $\overline{g}_k = rg_{k-1} + (1-r)g_k$
  Compute step size $a_k = a_{init}(1+\xi k)^{-0.6}$
  Update Hessian $H_k = H_k + \lambda I_4$
  Update parameters $\theta_{k+1} = \Pi(\theta_k - a_k H_k^{-1}\overline{g}_k)$
  if $\|\theta_{k+1} - \theta_k\| < \varepsilon$ **then**
    return θ_{k+1}
  end if
end for

log-likelihood ignoring the constant term - $\frac{n}{2}\log(2\pi)$:

$$l_n^{ALGARCH}(\psi;\mathscr{I}_n) = n^{-1}\sum_{t=1}^{n}\left(\frac{\exp(\tilde{y}_t(\psi))}{\exp(h_{t|t-1}(\psi))} + h_{t|t-1}(\psi)\right). \tag{12}$$

In the following algorithm called ALGARCH-SASA (5), SASA is combined with the Algorithm KF-ALGARCH 4 in order to minimize $l_n^{ALGARCH}$ (12) with respect to ψ, following the same setup as in Algorithm 3 for ASV model.

4 Empirical study

To provide an empirical comparison of the performance of ASV and ALGARCH specifications in the presence of zero returns, we consider five daily financial return series: The FTSE100 index (source:

Algorithm 4: KF-ALGARCH

Input: $\{\varepsilon_1, \ldots, \varepsilon_n\}$, $\psi_0 \in \Psi$
Output: $\{h_{t+1|t}\}$, $\{\widetilde{y}_t\}$
Initialize:
$h_{1|0} = \alpha_0^* / (1 - \beta^*)$
$p_{1|0} = \alpha^{*2} \sigma^2 / (1 - \beta^{*2})$
$f_{1|0} = p_{1|0} + \sigma^2$
$y_{1|0} = h_{1|0} + \mu$
for $t = 1$ to n **do**
  if $\varepsilon_t = 0$ **then**
    $\widetilde{y}_t = y_{t|t-1}$
  else
    $\widetilde{y}_t = \log(\varepsilon_t^2)$
  end if
  $k_t = (\beta^* p_{t|t-1} + \alpha_1^* \sigma^2) f_{t|t-1}^{-1}$
  $h_{t+1|t} = (\beta^* - k_t) h_{t|t-1} + k_t \widetilde{y}_t + (\alpha_0^* - \mu k_t)$
  $p_{t+1|t} = \beta^{*2} p_{t|t-1} - (\beta^* p_{t|t-1} + \alpha_1^* \sigma^2)^2 f_t^{-1} + (\alpha_1^*)^2 \sigma^2$
  $f_{t+1|t} = p_{t+1|t} + \sigma^2$
  $y_{t+1|t} = h_{t+1|t} + \mu$
end for
return $\{h_{t+1|t}\}$, $\{\widetilde{y}_t\}$

Algorithm 5: ALGARCH-SASA

Input: $\{\varepsilon_1, \ldots, \varepsilon_n\}$, $\psi_0 \in \Psi$
Output: ψ_{k+1}
Calibration: $(a_{init}, \xi, r, \lambda) = (0.005, 0.05, 0.6, 0.001)$
Initialize Hessian: $H_0 = I_4$ (4×4 identity matrix)
for $k = 1$ to I **do**
  Call KF-ALGARCH(ψ_{k-1}) 4
  Generate $\Delta_k \sim U[-1,1]^{4\times 2}$
  Compute $c_k = 0.1\, k^{-0.101}$
  Compute perturbed function values: $f_k^{\pm} = l_n^{ALGARCH}(\psi_k \pm c_k \Delta_k)$
  Estimate gradient: $g_k = (f_k^+ - f_k^-)(2c_k)^{-1} \Delta_k$
  Update gradient average: $\overline{g}_k = r g_{k-1} + (1-r) g_k$
  Compute step size: $a_k = a_{init}(1 + \xi k)^{-0.6}$
  Update Hessian: $H_k = H_k + \lambda I_4$
  Update parameters: $\psi_{k+1} = \Pi(\psi_k - a_k H_k^{-1} \overline{g}_k)$
  if $\|\psi_{k+1} - \psi_k\| < \varepsilon$ **then**
    return ψ_{k+1}
  end if
end for

Table 1: Sample characteristics of daily return series.

Series	Sample period	n	$\widehat{\pi}$ (%)	s^2	s^4	ARCH	$cor(r_t^+, (r_{t+1}^+)^2)$	$cor(r_t^-, (r_{t+1}^-)^2)$
FTSE100	5/1/1998–2/6/2015	4397	0.14	1.51	8.59	97.9 (0.00)	0.8444 (0.00)	-0.8769 (0.00)
Apple	10/9/1984–12/10/2011	6835	4.60	9.63	53.85	5.90 (0.02)	0.8118 (0.00)	-0.7081 (0.00)
USD/EUR	5/1/1999–12/10/2011	3274	0.80	0.45	5.45	205.9 (0.00)	0.8598 (0.00)	-0.8653 (0.00)
Oil price	21/5/1987–4/10/2011	6190	2.47	5.59	17.47	78.6 (0.00)	0.8452 (0.00)	-0.7630 (0.00)
Gold	4/1/2006–12/10/2011	1448	0.55	1.96	6.18	6.76 (0.01)	0.8556 (0.00)	-0.9013 (0.00)

Bloomberg), the Apple stock price (source: Yahoo Finance, https://yahoo.finance.com), the USD/EUR exchange rate (source: The European Central Bank, http://www.ecb.int/), the Brent blend oil price (source: The US Energy Information Agency, http://www.eia.gov/) and the Gold price (source: Kitco, http://www.kitco.com/) studied also in [27]. Each series denoted r_t is centered and thus represented by ε_t and then fitted by ASV model (6) and ALGARCH model (11) using Algorithms 3 and 5, respectively.

4.1 Descriptive preliminary

Table 1 reports the descriptive statistics of the selected series (as can also be found in [27]. We also compute the correlation between positive (resp. negative) returns denoted r_t^+ (resp. r_t^-) and their squares $(r_t^+)^2$ (resp. $(r_t^-)^2$) used as proxy of series volatility to detect beforehand, the presence of a leverage effect. For all series, the findings reveal the presence of the main stylized facts of excess kurtosis that exceeds 3, evidence of ARCH effect at the 5% significance level measured by the first-order serial correlation in the squared return, and the presence of a leverage effect according to a significant correlation between r_t^- and $(r_t^-)^2$ at the 5% significance level exceeding in magnitude that between r_t^+ and $(r_t^+)^2$ for the FTSE100 and Gold series and, to a lesser extent, for the USD/EUR exchange rate. The percentage of observed zeros is noted $\widehat{\pi}$ and it ranges from 0.14% for FTSE100 to 4.6% for Apple.

4.2 Comparison in model performance

Table 2 contains the estimates of ASV model. The asymmetry parameter estimate $\widehat{\gamma}$ is negative across all series, except for Gold. The strength of this asymmetry, as captured by the correlation parameter estimate $\widehat{\rho} = \widehat{\gamma}/\sqrt{\widehat{\gamma}^2 + \widehat{\nu}^2}$, is relatively pronounced for FTSE100 ($\widehat{\rho} = -0.7$) and moderate for Apple ($\widehat{\rho} = -0.2$). In contrast, for Gold, the model yields a positive estimate of γ, suggesting the absence of a leverage effect in this series. Volatility persistence is generally captured by the model across all series, as reflected by $\widehat{\phi}$ being close to unity. However, the further $\widehat{\phi}$ deviates from 1, the higher the proportion of zero returns. For example, ϕ is estimated at 0.98 for FTSE100 and 0.99 for Gold, which contain 0.14% and 0.55% of zeros, respectively, while ϕ is estimated at 0.91 for Apple, which contains 4.6% of zeros. This pattern is in accordance with the fact that volatility filtering tends to smooth out extreme spikes, thereby reducing the persistence estimate via $\widehat{\phi}$ when the frequency of zero returns increases. As a result, the persistence estimates remain high but slightly biased downward as the proportion of zeros grows.

Looking now at Table 3, where the estimates of ALGARCH specification are displayed, the leverage effect, characterized by $|\widehat{\alpha}_1| < |\widehat{\alpha}_2|$, is detected only for FTSE100. Moreover, volatility persistence is also highly captured by ALGARCH for all series as confirmed by the estimated $\widehat{\beta} + (\widehat{\alpha}_1 + \widehat{\alpha}_2)/2$ being close to 1. Similarly to ASV specification, persistence estimates approach 1 as the percentage of zero returns

Table 2: Parameter estimates of ASV model with goodness-of-fit criteria for daily financial return series (standard errors in parentheses). The most relevant values are displayed in boldface.

Series	$\widehat{\omega}$	$\widehat{\phi}$	$\widehat{\gamma}$	$\widehat{\nu}$	Log-Lik	AIC	BIC
FTSE100	-0.004	0.980	-0.121	0.124	**-9878.535**	**19765.070**	**19790.620**
	(0.002)	(0.015)	(0.007)	(0.002)			
Apple	0.160	0.910	-0.069	0.339	**-15179.820**	**30367.640**	**30394.950**
	(0.002)	(0.011)	(0.006)	(0.001)			
USD/EURO	-0.099	0.990	-0.002	0.057	-10831.840	21671.680	21696.010
	(0.003)	(0.017)	(0.008)	(0.002)			
Oil price	0.033	0.973	-0.027	0.193	-13889.530	27787.060	27813.990
	(0.002)	(0.012)	(0.005)	(0.001)			
Gold	-0.098	0.990	0.026	0.109	-3897.559	7803.118	7824.244
	(0.023)	(0.010)	(0.008)	(0.016)			

declines. However, ASV model generally captures volatility persistence more effectively, yielding $\widehat{\phi}$ more closer to one, except for Apple, where the persistence under ALGARCH specification appears less affected by the relatively high proportion of zeros (4.6%).

In terms of goodness-of-fit, the last three columns of Tables 2 and 3 indicate that ASV model provides a more adequate fit for FTSE100 and Apple, as evidenced by the higher log-likelihood values and the lower AIC and BIC compared to ALGARCH model. In contrast, ALGARCH specification achieves a better fit for the USD/EUR exchange rate and Gold, with a modest improvement over ASV model. This conclusion follows directly from the log-likelihood values reported, which exceed those of ASV model, and from the lower AIC and BIC criteria, as presented in Table 3.

At this stage of analysis, it is challenging to declare one model clearly better, unless one considers the $\widehat{\pi}$'s proportion in each series. Indeed, ASV specification demonstrates its robustness to the presence of zero returns, outperforming not only for FTSE100, where $\widehat{\pi} = 0.14\%$, but also for Apple, where $\widehat{\pi}$ is relatively higher at 4.6%, and even for Oil price, where its performance remains non-negligible, as indicated by the log-likelihood, AIC, and BIC values being close to those obtained from ALGARCH model. The latter tends to perform better when $\widehat{\pi}$ is low, as illustrated by the USD/EUR, where it fully exploits a log-likelihood representing approximately 0.3 of that obtained from ASV model. A key advantage of ASV specification lies in the relative comparison of fit criteria. For FTSE100 and Apple, ASV log-likelihoods represent approximately 0.14 and 1.2 of ALGARCH's, respectively; for Oil price and Gold, they reach 1.01 and 1.54 of ALGARCH's. These results suggest that ASV model is a viable alternative to ALGARCH, particularly when zero returns are substantial. These findings hold true when considering AIC and BIC criteria, reinforcing ASV's robustness.

4.3 Volatility estimation accuracy and zero returns

The accuracy of the filtered volatility provided by ASV and ALGARCH specifications is evaluated by the Quasi-likelihood loss function (Qlike) and the Root Mean Squared Error (RMSE) given respectively by

$$\text{Qlike} = \frac{1}{n}\sum_{t=1}^{n}\left(\frac{\varepsilon_t^2}{\widehat{\sigma}_t^2} - \log\frac{\varepsilon_t^2}{\widehat{\sigma}_t^2} - 1\right), \tag{13}$$

Table 3: Parameter estimates of ALGARCH model with goodness-of-fit criteria for daily financial return series (standard errors in parentheses). The most relevant values are displayed in boldface.

Series	$\widehat{\alpha}_0$	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}$	Log-Lik	AIC	BIC
FTSE100	-0.118	0.015	0.073	0.947	-70520.430	141048.860	141074.415
	(0.041)	(0.019)	(0.015)	(0.061)			
Apple	0.047	0.028	0.016	0.955	-18240.98	36489.960	36517.279
	(0.007)	(0.002)	(0.001)	(0.097)			
USD/EURO	0.018	0.026	0.005	0.974	**-3350.620**	**6709.240**	**6733.615**
	(0.001)	(0.002)	(0.001)	(0.098)			
Oil price	0.059	0.034	0.014	0.956	**-14180.25**	**28368.500**	**28395.423**
	(0.006)	(0.002)	(0.001)	(0.094)			
Gold	0.063	0.039	0,003	0.963	**-2685.910**	**5379.820**	**5400.931**
	(0.007)	(0.003)	(0.002)	(0.097)			

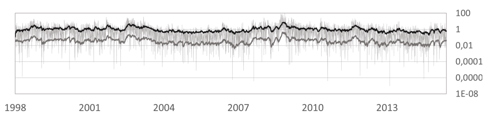
and

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\widehat{\sigma}_t^2 - \varepsilon_t^2\right)^2}, \tag{14}$$

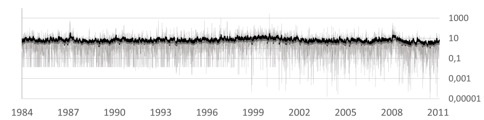
where the squared return ε_t^2 is used as proxy of the realized volatility σ_t^2 whereas $\widehat{\sigma}_t^2$ denotes the filtered volatility obtained from ASV and ALGARCH models as $exp(h_{t|t-1})$.

Both criteria are expressed as the ratio of ASV to ALGARCH, as reported in the second and third columns of Table 4. ASV specification exhibits the better predictive performance for FTSE100 with a low ASV's Qlike loss representing 0.036 of ALGARCH's, and somewhat less for Apple since the ratio is closer to 1 (0.862) and considerably higher for Oil price (0.987). The results change to ALGARCH for Gold (5.130) and less strongly for USD/EUR (3.115), showing ratios far exceeding 1. These results are consistent with the comparison analysis of the two specifications in terms of goodness of fit. This is notably true for Oil price, whose Qlike loss ratio is close to 1, confirming nearly equivalent performance between ASV and ALGARCH specifications. Furthermore, Figure 1 illustrates these trends by highlighting the gap between the volatility filtered by each specification and the proxy volatility. The two filtered volatility series overlap for Apple (Figure b) and Oil price (Figure d), but differ for FTSE100 (Figure a). For USD/EUR (Figure c) and Gold (Figure e), the gap is clearly noticeable in favor of ALGARCH specification. Overall, we can inspect from the sub-figures in Figure 1 that the two filtered volatilities have the same trend behavior, following the sharp upward and downward variations in proxy volatility. Besides, the RMSE criterion depicts surprisingly comparable accuracy of volatility filtering with ratio values close to 1, supporting however, the enhanced accuracy of ASV specification for FTSE100 and Apple while ALGARCH specification remains its outperformance for filtered volatility of USD/EUR and Gold.

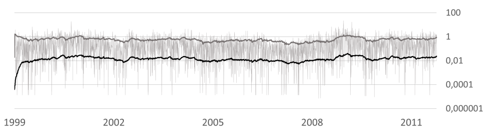
More precisely, the difference D_t and ratio R_t between ASV and ALGARCH filtered volatilities reveal on average systematic patterns. For FTSE100 and Apple, ASV specification generally overestimates volatility, with positive average of D_t and R_t above 1; for FTSE100, the average difference is 1.045 and the ratio 2.910, with relatively low standard deviation (0.872) indicating that this bias is stable over time (see also Figure a), whereas for Apple, the average difference is 3.120 and the ratio 1.865, with higher standard deviation (2.310) showing more variable overestimation (see also Figure b). For USD/EUR and Gold, ASV model underestimates volatility, with negative average difference and very



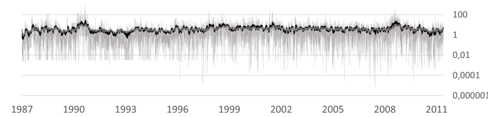
(a) FTSE100



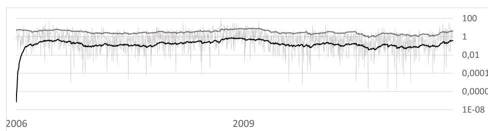
(b) Apple



(c) USD/EUR



(d) Oil price



(e) Gold

Figure 1: Realized volatility measured by squared returns (light gray) and estimated volatility from ASV (black) and ALGARCH (dark gray) for daily financial return series (y-axis on a logarithmic scale)

low ratio; USD/EUR has an average difference of -0.590 and ratio 0.026, Gold shows -3.080 and 0.058, both with low standard deviations (0.298 and 0.019), indicating that this underestimation is consistent over time (see also Figures c and e). Finally, as expected, Oil price displays a modest positive difference (0.395) and a ratio close to 1 (1.045), suggesting that both models generate broadly comparable volatility estimates, with ASV specification being slightly higher on average (see also Figure d).

Now, in the presence of zero returns, as shown in Figure 2, the volatility filtered by the ASV model that best fits the FTSE100 and Apple as seen before remains robust in terms of the Qlike loss ratio, with a slight decrease from 0.036 for the FTSE100 to 0.895 for Apple when $\widehat{\pi}$ increases from 0.14% to 4.6%, respectively. On the other hand, ALGARCH model is more impacted even by a slight increase in $\widehat{\pi}$ from 0.55% to 0.8%, resulting in a large decrease in Qlike loss from 5.130 for Gold to 3.115 for USD/EUR. Note that the RMSE values being overall close to 1, does not provide a clear indication of the impact of zero returns on the relative accuracy of volatility estimates between the two models (see Figure 2). Regarding the discrepancy between the volatilities estimated by ASV and ALGARCH in terms of difference D_t and ratio R_t, Figure 3 indicates that higher proportions of zero returns are linked to greater divergences between ASV and ALGARCH. For example, Apple, with 4.6% zeros, shows on average a strong overestimation by ASV, with a difference of 3.120 and a ratio of 1.865, while USD/EUR and Gold, with negligible zeros (0.8% and 0.55%, respectively), show a stable underestimation on average with a difference of -0.590, -3.080 and a ratio of -3.028, 0.058, respectively. The Oil price, with 2.47% of zeros, lies in between, with nearly equivalent estimates on average, with a difference of 0.371 and a ratio of 1.030.

5 Conclusion

This paper proposed and implemented an iterative quasi-maximum likelihood estimation method for asymmetric stochastic volatility (ASV) and asymmetric log-GARCH (ALGARCH) models, with an EM imputation using a Kalman filter-based framework for the treatment of zero returns. The method, com-

Table 4: Relative performance metrics of volatility forecasts from ASV and ALGARCH models for daily financial return series (Standard deviations in parentheses). The most relevant values are displayed in boldface.

Series	Qlike loss ratio	RMSE's ratio	Average of D_t	Average of R_t
FTSE100	**0.036**	**0.895**	1.045 (0.781)	2.910 (0.872)
Apple	**0.862**	**0.990**	3.120 (2.310)	1.865 (0.468)
USD/EUR	3.115	1.120	-0.590 (0.211)	0.026 (0.006)
Oil price	**0.987**	**0.989**	0.395 (1.481)	1.045 (0.298)
Gold	5.130	1.052	-3.080 (1.462)	0.058 (0.019)

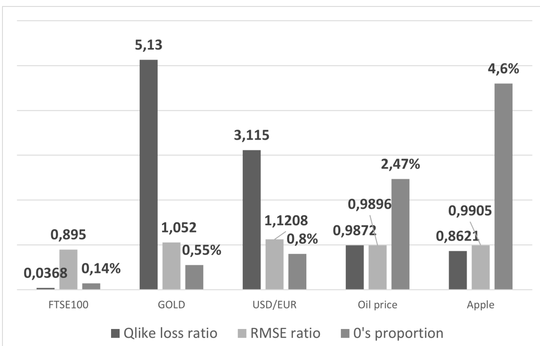


Figure 2: Comparison of volatility forecast performance from ASV and ALGARCH models based on Qlike loss ratio, RMSE ratio according to the proportion of zero returns

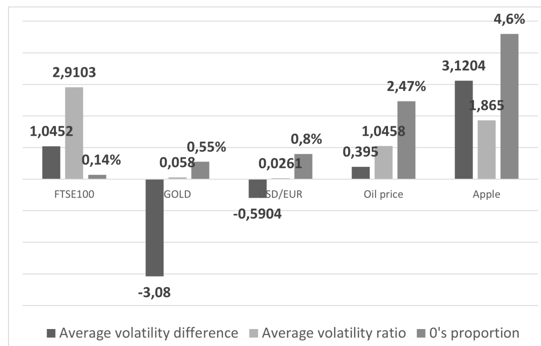


Figure 3: Comparison of volatility forecast performance from ASV and ALGARCH models based on the average volatility difference D_t and the average volatility ratio R_t according to the proportion of zero returns

bined with the SASA optimization routine, ensures stable estimation of latent volatilities even in the presence of zeros in the return series. Empirical results on daily financial data show that both models reproduce the main stylized facts of volatility persistence, clustering, and leverage effects, while ASV specification provides greater robustness to zero returns, thus, when the proportion of zeros increases, ASV model maintains stable parameter estimates and smoother volatility paths, whereas ALGARCH model becomes more sensitive and less stable.

The forecast comparison based on Qlike loss and RMSE criteria reveals that the predictive performance of each model is strongly related to the proportion of zero returns observed in the data. For series with frequent zeros, such as Apple and Oil price, ASV model yields smoother and more reliable volatility forecasts. In contrast, ALGARCH specification performs slightly better for assets characterized by a low proportion of zero returns such as USD/EUR and Gold. In this case, both models can be considered suitable candidates. We conjecture that the superior performance of ASV model in the presence of zero returns stems from its parameter-driven structure, where volatility evolves as a latent state variable. In contrast, ALGARCH model as an observation-driven that depends directly on the logarithmic transformation of observed returns, it is more sensitive to zeros, leading to a less accurate volatility estimation.

6 Conflicts of interest

The authors declare that there are no conflicts of interest.

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Vol. 55, Courier Corp., North Chelmsford, MA, 1965.

[2] B.D.O. Anderson, J.B. Moore, *Optimal Filtering*, Prentice-Hall, Englewood Cliffs, NJ, 1979.

[3] M. Asai, A new method to estimate stochastic volatility models: A LOG-GARCH approach, *J. Japan Statist. Soc.* **28** (1998) 101–114.

[4] S. Bhatnagar, N. Hemachandra, V.K. Mishra, Stochastic approximation algorithms for constrained optimization via simulation, *ACM Trans. Model. Comput. Simul.* **21** (2011) 1–22.

[5] T. Bollerslev, Generalized autoregressive conditional heteroskedasticity, *J. Econometrics* **31** (1986) 307–327.

[6] P. Chirico, Iterative QML estimation for asymmetric stochastic volatility models, *Stat. Methods Appl.* (2024).

[7] P. Christoffersen, K. Jacobs, C. Ornthanalai, GARCH option valuation: Theory and evidence, *J. Derivatives* (2013).

[8] C. Francq, O. Wintenberger, J.-M. Zakoian, GARCH models without positivity constraints: Exponential or log GARCH?, *J. Econometrics* **177** (2013) 34–46.

[9] C. Francq, G. Sucarrat, An equation-by-equation estimator of a multivariate log-GARCH-X model of financial returns, *J. Multivariate Anal.* **153** (2017) 16–32.

[10] C. Francq, J.-M. Zakoian, *GARCH Models: Structure, Statistical Inference and Financial Applications*, Wiley, Hoboken, NJ, 2019.

[11] J. Geweke, Modelling the persistence of conditional variances: A comment, *Econometric Rev.* **5** (1986) 57–61.

[12] E. Ghysels, A.C. Harvey, E. Renault, Stochastic volatility, in: G.S. Maddala, C.R. Rao (Eds.), *Handbook of Statistics*, Vol. 14, Elsevier, Amsterdam, 1996, pp. 119–191.

[13] L.R. Glosten, R. Jagannathan, D.E. Runkle, On the relation between the expected value and the volatility of the nominal excess return on stocks, *J. Finance* **48** (1993) 1779–1801.

[14] A.C. Harvey, *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge Univ. Press, Cambridge, 1990.

[15] A.C. Harvey, E. Ruiz, N. Shephard, Multivariate stochastic variance models, *Rev. Econom. Stud.* **61** (1994) 247–264.

[16] A.C. Harvey, N. Shephard, Estimation of an asymmetric stochastic volatility model for asset returns, *J. Bus. Econom. Statist.* **14** (1996) 429–434.

[17] A. Melino, S.M. Turnbull, Pricing foreign currency options with stochastic volatility, *J. Econometrics* **45** (1990) 239–265.

[18] S.G. Pantula, Modeling the persistence of conditional variances: A comment, *Econometric Rev.* **5** (1986) 71–74.

[19] E. Ruiz, Quasi-maximum likelihood estimation of stochastic volatility models, *J. Econometrics* **63** (1994) 289–306.

[20] S. Kim, N. Shephard, S. Chib, Stochastic volatility: Likelihood inference and comparison with ARCH models, *Rev. Econom. Stud.* **65** (1998) 361–393.

[21] A. Settar, M. Badaoui, On the treatment of zero returns in the estimation of log-GARCH model: Empirical study, *IOP Conf. Ser. Mater. Sci. Eng.* **1099** (2021) 012029.

[22] A. Settar, N.I. Fatmi, M. Badaoui, Numerical estimation of log-GARCH model with relaxed moment assumptions, *J. Phys. Conf. Ser.* **1743** (2021) 012036.

[23] J.C. Spall, Adaptive stochastic approximation by the simultaneous perturbation method, *IEEE Trans. Automat. Control* **45** (2000) 1839–1853.

[24] G. Sucarrat, A. Escribano, The power log-GARCH model, Working Paper, Dept. of Economics, Univ. Carlos III Madrid, 2010.

[25] G. Sucarrat, A. Escribano, Unbiased QML estimation of log-GARCH models in the presence of zero returns, Working Paper, Dept. of Economics, Univ. Carlos III Madrid, 2013.

[26] G. Sucarrat, S. Grønneberg, A. Escribano, Estimation and inference in univariate and multivariate log-GARCH-X models when the conditional density is unknown, *Comput. Statist. Data Anal.* **100** (2016) 582–594.

[27] G. Sucarrat, The log-GARCH model via ARMA representations, in: *Financial Mathematics, Volatility and Covariance Modelling*, Routledge, London, 2019, pp. 336–359.

[28] S.J. Taylor, *Modelling Financial Time Series*, Wiley, Chichester, 1986.

[29] S.J. Taylor, Modeling stochastic volatility: A review and comparative study, *Math. Finance* **4** (1994) 183–204.

[30] N. Vijayan, L.A. Prashanth, Smoothed functional-based gradient algorithms for off-policy reinforcement learning: A non-asymptotic viewpoint, *Syst. Control Lett.* **155** (2021) 104988.

[31] A. Wills, T.B. Schön, B. Ninness, Estimating state-space models in innovations form using the expectation maximisation algorithm, in: *Proc. 49th IEEE Conf. Decis. Control (CDC)*, IEEE, 2010, pp. 5524–5529.