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Numerical solution of the time fractional nonlinear burgers equation using the quintic B-Spline method

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Abstract. This paper introduced a novel approach for resolving fractional partial differential equations. The time fractional nonlinear Burgers equation of order κ was solved to illustrate the efficacy of the technique, where $\kappa \in (0,1]$. The quintic B-spline method facilitated spatial partitioning, while the finite difference method addressed the fractional Caputo derivative, which simulates anomalous diffusion processes influenced by memory effects. The proposed methods stability is demonstrated utilizing the von Neumann technique; it has been shown to be unconditionally stable. Additionally, a convergence study is shown, demonstrating that the approach exhibits uniform convergence of $(\gamma h^4 + \sigma(\Delta\eta^2))$. We validated the methods correctness through numerical tests by comparing it with the exact solution and alternative numerical methods. Based on L^2 and L^∞ error norms, the quintic B-spline approach exhibits improved convergence rates and reduced computing costs.

Keywords: Quintic B-spline method, finite difference techniques, Caputo time-fractional derivative, Burgers equation.
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1 Introduction

Due to the fact that fractional differential operators have non-local properties while classical differential operators have local properties, the fractional calculus has grown significantly in recent years and is better able to describe real-life phenomena [31]. The significance of differential equations of fractional order has been demonstrated in recent years by researchers modeling scientific and engineering problems in a variety of demanding phenomena, such as the predator-prey food chain system [2], the unsteady fluid flow in a rotating annulus region, the non-linear oscillation of an earthquake, the unsteady rotational flow of a second-grade fluid, neutral differential systems with state-dependent delay [30], seepage flow in porous media [11], fluid dynamic traffic models [10], the spatial diffusion of biological populations [32],

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dynamical models of happiness [34], the magnetohydrodynamic flow and heat transfer model [5]. Recent years have seen a greater interest in fractional differential equations due to its use in many scientific and engineering fields [21, 22, 25, 38, 39]. In actuality, we may more precisely and accurately simulate a wide range of phenomena, such as fluid mechanics, viscoelasticity, chemistry, physics, economics, and other sciences, applying a range of methods from fractional calculus, which is the study of fractional order integrals and derivatives [7]. One of the main advantages of fractional derivatives is their ability to encapsulate the genetic properties of a phenomenon or memory, and to replicate a not small set of physical and geometric phenomena, which means that they enjoy a higher degree of freedom compared to classical derivatives [14, 18, 19, 23]. Many scholars are interested in creating or putting into practice a rigorous strategy for the analytical and numerical analysis of the behavior of fractional order models, given the increasing attention being paid to problems based on these models that arise in science and engineering. Numerous analytical and numerical methods including the new integral transform approach for homotopy perturbation are available in the literature to solve the fractional order model [29]. In fluid dynamics for diffusive waves, Burgers equation is a nonlinear equation. There are many issues that can be addressed by the burgers equation such as, in a material with limited electrical conductivity, magnetohydrodynamic waves, shock waves in a viscous medium, one-dimensional sound waves in a viscous medium, turbulence, etc [6, 8]. We examine the time fractional nonlinear Burgers equation (TFNBE) in this study [22] as follows:

$$\begin{aligned}
&\frac{\partial^\kappa z(\zeta,\eta)}{\partial\eta^\kappa} + z(\zeta,\eta)\frac{\partial z(\zeta,\eta)}{\partial\zeta} - v\frac{\partial^2 z(\zeta,\eta)}{\partial\zeta^2} = g(\zeta,\eta), \quad (\zeta,\eta)\in[c,d]\times(0,T],\\
&z(\zeta,0)=\xi(\zeta), \qquad \zeta\in[c,d]\\
&z(c,\eta)=\gamma_1(\eta) \quad and \quad z(d,\eta)=\gamma_2(\eta),
\end{aligned} \tag{1}$$

where v is a viscosity parameter, $\zeta\in[c,d]$, $\eta\in[0,T]$, $g(\zeta,\eta):[c,d]\times[0,T]\longrightarrow\mathbb{R}$, and $\xi:[c,d]\longrightarrow\mathbb{R}$. We need to define the fractional derivative [15]

$$\frac{\partial^\kappa z(\zeta,\eta)}{\partial\eta^\kappa} = \frac{1}{\Gamma(n-\kappa)}\int_c^\eta \frac{\partial^n z(\zeta,s)}{\partial s^n}(\eta-s)^{n-\kappa-1}ds, \tag{2}$$

where the Gamma function [26] is

$$\Gamma(n-\kappa) = \int_0^\infty \zeta^{n-\kappa-1}e^{-\zeta}d\zeta, \quad (\mathbb{R}(\kappa)>0). \tag{3}$$

One of the key equations in physics and engineering is the Burgers equation. Due to its superior ability to describe many phenomena within a frame of reference such as, turbulence problems, nonlinear acoustic waves, plane waves, shock waves, lattice gas issues, and traffic flow [3, 30, 36]. This equation plays an effective role in the field of oceanography, which falls under fluid mechanics [24]. Therefore, Burgers equation has been of interest to many researchers. Numerous analytical and numerical techniques have been proposed to solve Burgers equation [22,30]. The TFNBE is produced by substituting a real number κ, where $\kappa\in(0,1]$, for the exponent of the one derivative of time in the one-order partial differential equation [37]. To solve fractional time differential equations, some researchers use the B-spline method [23]. Its high flexibility, enables us to identify the solution at each node. Thus, we obtain a system written in the form of matrices, which makes it easier for us to find the solution using the computer. Using finite differences the fractional derivative of time is approximated which is defined using Caputo definition and the aim is to solve TFNBE and approximate the solution and its derivatives with respect to ζ by the quintic B-spline ($QNBS$) method. In Section 2 an outline of how the solution for ζ is approximated using numerical method is given. The suggested methodology's numerical implementation is explained in Section 3. Section 4 deals with the stability analysis. In Section 5, convergence is studied, while Section 6 analyzes numerical data using context-giving examples. Finally in Section 7, we conclude with our findings and future directions.

2 An formulation of the *QNBS* method

To determine a solution of TFNBE, we apply a $QNBS$ method where, we impose the partition $\Delta : c = \zeta_0 < \zeta_1 < \zeta_2 < \ldots < \zeta_{n-1} < \zeta_n = d$ on endowed interval $[c,d]$ such that $h = \frac{d-c}{n}$ and $\zeta_i = c+ih$, n is the mesh size of Δ and $i = 0,1,2,\ldots,n$. We now define $QNBS$ in the following format [1]

$$\mathbb{B}_i(\zeta) = \frac{1}{h^5}\begin{cases} (\zeta-\zeta_{i-3})^5, & \text{if } \zeta_{i-3} \le \zeta \le \zeta_{i-2},\\ h^5+5h^4(\zeta-\zeta_{i-2})+10h^3(\zeta-\zeta_{i-2})^2+10h^2(\zeta-\zeta_{i-2})^3 & \\ \qquad +5h(\zeta-\zeta_{i-2})^4-5(\zeta-\zeta_{i-2})^5, & \text{if } \zeta_{i-2} \le \zeta \le \zeta_{i-1},\\ 26h^5+50h^4(\zeta-\zeta_{i-1})+20h^3(\zeta-\zeta_{i-1})^2-20h^2(\zeta-\zeta_{i-1})^3 & \\ \qquad -20h(\zeta-\zeta_{i-1})^4+10(\zeta-\zeta_{i-1})^5, & \text{if } \zeta_{i-1} \le \zeta \le \zeta_{i},\\ 26h^5+50h^4(\zeta_{i+1}-\zeta)+20h^3(\zeta_{i+1}-\zeta)^2-20h^2(\zeta_{i+1}-\zeta)^3 & \\ \qquad -20h(\zeta_{i+1}-\zeta)^4+10(\zeta_{i+1}-\zeta)^5, & \text{if } \zeta_{i} \le \zeta \le \zeta_{i+1},\\ h^5+5h^4(\zeta_{i+2}-\zeta)+10h^3(\zeta_{i+2}-\zeta)^2+10h^2(\zeta_{i+2}-\zeta)^3 & \\ \qquad +5h(\zeta_{i+2}-\zeta)^4-5(\zeta_{i+2}-\zeta)^5, & \text{if } \zeta_{i+1} \le \zeta \le \zeta_{i+2},\\ (\zeta_{i+3}-\zeta)^5, & \text{if } \zeta_{i+2} \le \zeta \le \zeta_{i+3},\\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We first begin by writing the approximate solution as follows [1]:

$$\hat{z}(\zeta,\eta) = \sum_{i=-2}^{n+2} c_i(\eta)\mathbb{B}_i(\zeta), \tag{5}$$

where $\mathbb{B}_i(\zeta)$ is a $QNBS$ functions, $\hat{z}_i^j = \hat{z}(\zeta_i,\eta_j)$ is approximate solution, $j = 0,1,\ldots,m$. So $\hat{z}_i^j, (\hat{z}_\zeta)_i^j, (\hat{z}_{\zeta\zeta})_i^j$ and $(\hat{z}_{\zeta\zeta\zeta})_i^j$ represented by the following formulas:

$$\begin{cases} \hat{z}_i^j = c_{i-2}^j + 26c_{i-1}^j + 66c_i^j + 26c_{i+1}^j + c_{i+2}^j,\\ (\hat{z}_\zeta)_i^j = -5\frac{c_{i-2}^j}{h} - 50\frac{c_{i-1}^j}{h} + 50\frac{c_{i+1}^j}{h} + 5\frac{c_{i+2}^j}{h},\\ (\hat{z}_{\zeta\zeta})_i^j = 20\frac{c_{i-2}^j}{h^2} + 40\frac{c_{i-1}^j}{h^2} - 120\frac{c_i^j}{h^2} + 40\frac{c_{i+1}^j}{h^2} + 20\frac{c_{i+2}^j}{h^2}. \end{cases} \tag{6}$$

Table 1 shows the solution values and their derivatives at the nodes using the $QNBS$ method. These values will be used to solve TFNBE.

Table 1: The values of $\mathbb{B}_i(\zeta)$ and their derivative.

	ζ_{i-2}	ζ_{i-1}	ζ_i	ζ_{i+1}	ζ_{i+2}	ζ_{i+3}
$\mathbb{B}_i$	1	26	66	26	1	0
$\mathbb{B}'_i$	$\frac{-5}{h}$	$\frac{-50}{h}$	0	$\frac{50}{h}$	$\frac{5}{h}$	0
$\mathbb{B}''_i$	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$-\frac{120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0

3 Time discretization formulation

In this section, we will approximate the fractional derivative using the Caputo's definition, and using forward differences to approximate the first derivative of η [33]. Accordingly, the term $\frac{\partial^\kappa z(\zeta,\eta)}{\partial \eta^\kappa}$ will be approximated as follows:

$$\frac{\partial^\kappa \hat{z}(\zeta,\eta_{j+1})}{\partial \eta^\kappa} = \frac{1}{\Gamma(2-\kappa)} \sum_{\nu=0}^{j} b_\nu [\frac{\hat{z}(\zeta,\eta_{j+1-\nu}) - \hat{z}(\zeta,\eta_{j-\nu}))}{\Delta\eta^\kappa}] + e_{\Delta\eta}^{j+1}. \tag{7}$$

In (7) $e_{\Delta\eta}^{j+1}$ is the truncation error, so

$$e_{\Delta\eta}^{j+1} \leq \sigma(\Delta\eta^2), \tag{8}$$

where, $\Delta\eta = \frac{\eta}{m}, \eta_j = j\Delta\eta$, $j = 0(1)m$ and $b_\nu = (\nu+1)^{2-\kappa} - \nu^{2-\kappa}$, such that

$$\begin{cases} b_0 = 1 \ and \ b_\nu > 0 \ , \ \nu = 0,1,2,....,j, \\ b_0 \gneqq b_1 \gneqq ...... \gneqq b_\nu, \ b_\nu \to 0 \ as \ \nu \to \infty, \\ \sum_{\nu=0}^{j}(b_\nu - b_{\nu+1}) + b_{j+1} = (1-b_1) + \sum_{\nu=1}^{j-1}(b_\nu - b_{\nu+1}) + b_j = 1. \end{cases} \tag{9}$$

Now, substituting (7) in (1) we obtain

$$\frac{1}{\Gamma(2-\kappa)} \sum_{\nu=0}^{j} b_\nu [\frac{\hat{z}(\zeta,\eta_{j+1-\nu}) - \hat{z}(\zeta,\eta_{j-\nu}))}{\Delta\eta^\kappa}] + z(\zeta,\eta_{j+1}) \frac{\partial \hat{z}(\zeta,\eta_{j+1})}{\partial \zeta} - \nu \frac{\partial^2 \hat{z}(\zeta,\eta_{j+1})}{\partial \zeta^2} = g(\zeta,\eta_{j+1}). \tag{10}$$

Suppose that $\hat{z}^{j+1} = \hat{z}(\zeta,\eta_{j+1})$, $\beta = \frac{1}{\Gamma(2-\kappa)\Delta\eta^\kappa}$ and $g^{j+1} = g(\zeta,\eta_{j+1})$. The equation (10) can be expressed in this way:

$$\beta\hat{z}^{j+1} - \beta\hat{z}^j + \hat{z}^{j+1}\frac{\partial \hat{z}^{j+1}}{\partial \zeta} - \nu\frac{\partial^2 \hat{z}^{j+1}}{\partial \zeta^2} = -\beta \sum_{\nu=0}^{j} b_\nu [\hat{z}^{j+1-\nu} - \hat{z}^{j-\nu}] + g^{j+1}. \tag{11}$$

Since

$$\hat{z}^{j+1}\frac{\partial \hat{z}^{j+1}}{\partial \zeta} = \hat{z}^j \frac{\partial \hat{z}^{j+1}}{\partial \zeta} + \hat{z}^{j+1}\frac{\partial \hat{z}^j}{\partial \zeta} - \hat{z}^j \frac{\partial \hat{z}^j}{\partial \zeta}, \tag{12}$$

(11) can be written as follows:

$$\beta\hat{z}^{j+1}-\beta\hat{z}^{j}+\hat{z}^{j}\frac{\partial\hat{z}^{j+1}}{\partial\zeta}+\hat{z}^{j+1}\frac{\partial\hat{z}^{j}}{\partial\zeta}-\nu\frac{\partial^{2}\hat{z}^{j+1}}{\partial\zeta^{2}}=\hat{z}^{j}\frac{\partial\hat{z}^{j}}{\partial\zeta}-\beta\sum_{\nu=0}^{j}b_{\nu}[\hat{z}^{j+1-\nu}-\hat{z}^{j-\nu}]+g^{j+1}, \tag{13}$$

where, $j=0(1)m$. By substituting (6) into (13), the following system is obtained:

$$\begin{aligned}
&[\beta-\frac{5l_1}{h}+l_2-\nu\frac{20}{h^2}]c_{i-2}^{j+1}+[26\beta-\frac{50l_1}{h}+26l_2-\nu\frac{40}{h^2}]c_{i-1}^{j+1}+[66\beta+66l_2+\nu\frac{120}{h^2}]c_{i}^{j+1}\\
&\qquad+[26\beta+\frac{50l_1}{h}+26l_2-\nu\frac{40}{h^2}]c_{i+1}^{j+1}+[\beta+\frac{5l_1}{h}+l_2-\nu\frac{20}{h^2}]c_{i+2}^{j+1}\\
&=\beta l_1+l_1l_2-\beta\sum_{\nu=1}^{j}b_{\nu}(l_3-l_4)+g^{j+1},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
l_1&=c_{i-2}^{j}+26c_{i-1}^{j}+66c_{i}^{j}+26c_{i+1}^{j}+c_{i+2}^{j},\\
l_2&=-5\frac{c_{i-2}^{j}}{h}-50\frac{c_{i-1}^{j}}{h}+50\frac{c_{i+1}^{j}}{h}+5\frac{c_{i+2}^{j}}{h},\\
l_3&=c_{i-2}^{j-\nu+1}+26c_{i-1}^{j-\nu+1}+66c_{i}^{j-\nu+1}+26c_{i+1}^{j-\nu+1}+c_{i+2}^{j-\nu+1},\\
l_4&=c_{i-2}^{j-\nu}+26c_{i-1}^{j-\nu}+66c_{i}^{j-\nu}+26c_{i+1}^{j-\nu}+c_{i+2}^{j-\nu}.
\end{aligned}$$

There are $(n+1)$ equations and $(n+5)$ unknowns in the above system above. Therefore, we need to add two equations using the boundary conditions (1) as follows:

$$\begin{cases}c_{-2}^{j}+26c_{-1}^{j}+66c_{0}^{j}+26c_{1}^{j}+c_{2}^{j}=0\\ c_{n-2}^{j}+26c_{n-1}^{j}+66c_{n}^{j}+26c_{n+1}^{j}+c_{n+2}^{j}=0.\end{cases} \tag{15}$$

The Pseudoinverse approach [9] will be used to solve the problem in this instance since there will be one more unknown than equations. We must determine the values of c_i^j when $j=0$ before we can solve (14), and we do this by applying the initial condition (1)as follows:

$$\begin{cases}(\hat{z}_0^{j})_{\zeta}=\frac{d}{d\zeta}\xi(\zeta_i), i=0,\\ \hat{z}_i^{0}=u(\zeta_i,0)=\xi(\zeta_i), i=0,1,2,\ldots.n\\ (\hat{z}_i^{0})_{\zeta}=\frac{d}{d\zeta}\xi(\zeta_i), i=n.\end{cases} \tag{16}$$

The final result is a system that can be written as follows:

$$\mathbb{H}C^{0}=\delta, \tag{17}$$

where $\mathbb{H}$, C^0, and δ are follows:

$$\mathbb{H}=\begin{bmatrix} \frac{-5}{h} & \frac{-50}{h} & 0 & \frac{50}{h} & \frac{5}{h} & \dots & 0 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & 0 & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & \dots & \dots & 0 & 0 & \frac{-5}{h} & \frac{50}{h} & 0 & \frac{50}{h} & \frac{5}{h} \end{bmatrix},$$

$$\delta^T=\left[\frac{d}{d\zeta}\xi(\zeta_0) \quad \xi(\zeta_0) \quad \xi(\zeta_1) \quad \dots \quad \xi(\zeta_n) \quad \frac{d}{d\zeta}\xi(\zeta_n)\right]^T,$$

$$(C^0)^T=\left[c^0_{-2} \quad c^0_{-1} \quad c^0_0 \quad \dots \quad c^0_n \quad c^0_{n+1} \quad c^0_{n+2}\right]^T,$$

system (17) has $(n+5)$ of unknowns and $(n+3)$ equations.

4 Stability

The stability of suggested technique is carried out using Von Neumann stability analysis [33, 35]. First linearize the non-linear term in equation (11) by supposing

$$\hat{z}^{j+1}\frac{\partial \hat{z}^{j+1}}{\partial \zeta}=a\frac{\partial \hat{z}^{j+1}}{\partial \zeta},$$

we obtain

$$\beta\hat{z}^{j+1}-\beta\hat{z}^{j}+a\frac{\partial \hat{z}^{j+1}}{\partial \zeta}-\nu\frac{\partial^2 \hat{z}^{j+1}}{\partial \zeta^2}=-\beta\sum_{\nu=0}^{j}b_\nu[\hat{z}^{j+1-\nu}-\hat{z}^{j-\nu}]+g^{j+1}. \tag{18}$$

It calls for imposing the error in the following way:

$$\lambda^j_k=\varepsilon^j_k-\tilde{\varepsilon}^j_k,\ \ k=0,1,2......n,\ \ j=0,1,2.....,m, \tag{19}$$

where ε^j_k is the Fourier mode's growth factor and its approximation is $\tilde{\varepsilon}^j_k$. When the error disappear as the computation progresses, the numerical scheme is stable. Let $g(\zeta,\eta)=0$, and hence from (18) and (19), we derive the roundoff error equation:

$$\begin{aligned}&\beta(\lambda^{j+1}_{i-2}+26\lambda^{j+1}_{i-1}+66\lambda^{j+1}_{i}+26\lambda^{j+1}_{i+1}+\lambda^{j+1}_{i+2})+a(-5\frac{\lambda^{j+1}_{i-2}}{h}-50\frac{\lambda^{j+1}_{i-1}}{h}+50\frac{\lambda^{j+1}_{i+1}}{h}+5\frac{\lambda^{j+1}_{i+2}}{h})\\&-\nu(20\frac{\lambda^{j}_{i-2}}{h^2}+40\frac{\lambda^{j}_{i-1}}{h^2}-120\frac{\lambda^{j}_{i}}{h^2}+40\frac{\lambda^{j}_{i+1}}{h^2}+20\frac{\lambda^{j}_{i+2}}{h^2})\\=&\beta(\lambda^{j}_{i-2}+26\lambda^{j}_{i-1}+66\lambda^{j}_{i}+26\lambda^{j}_{i+1}+\lambda^{j}_{i+2})-\beta\sum_{\nu=0}^{j}b_\nu[(\lambda^{j-\nu+1}_{i-2}+26\lambda^{j-\nu+1}_{i-1}+66\lambda^{j-\nu+1}_{i}\\&+26\lambda^{j-\nu+1}_{i+1}+\lambda^{j-\nu+1}_{i+2})-(\lambda^{j-\nu}_{i-2}+26\lambda^{j-\nu}_{i-1}+66\lambda^{j-\nu}_{i}+26\lambda^{j-\nu}_{i+1}+\lambda^{j-\nu}_{i+2})].\end{aligned} \tag{20}$$

The boundary conditions of the equation (20) are

$$\lambda_0^j = h_1(\eta),\ \ \lambda_n^j = h_2(\eta),\ \ j = 0,1,2,...,m, \tag{21}$$

and the initial conditions are

$$\lambda_k^0 = \alpha(\zeta_k), \quad k = 0,1,2,...,n. \tag{22}$$

Define the grid function

$$\lambda^j(\zeta) = \begin{cases} \lambda_k^j, & \zeta_k - \frac{h}{2} < \zeta \leq \zeta_k + \frac{h}{2},\ k = 0(1)n, \\ 0, & c < \zeta < \frac{h}{2}\ or\ d - \frac{h}{2} < \zeta < d. \end{cases} \tag{23}$$

The Fourier series of $\lambda^j(\zeta)$ is

$$\lambda^j(\zeta) = \sum_{\mu=-\infty}^{\infty} \zeta^j(\mu) e^{\frac{i2\pi\mu\zeta}{(d-c)}},\ j = 0(1)m, \tag{24}$$

where

$$\zeta^j(\mu) = \frac{1}{(d-c)} \int_c^d \lambda^j(\zeta) e^{\frac{-i2\pi\mu\zeta}{(d-c)}} d\zeta. \tag{25}$$

Let $\lambda^j = [\lambda_1^j, \lambda_2^j, ..., \lambda_{n-1}^j]^T$, and introduce the norm [13]

$$\|\lambda^j\|_2 = (\sum_{j=1}^{m-1} h|\lambda_k^j|^2)^{\frac{1}{2}} = [\int_c^d |\lambda_k^j|^2 d\zeta]^{\frac{1}{2}}. \tag{26}$$

By using Parseval's equality [4], it is clear that $\int_c^d |\lambda_k^j|^2 d\zeta = \sum_{\mu=-\infty}^{\infty} |\zeta^j(\mu)|^2$.

Consequently, the following relationship is obtained:

$$\|\lambda^j\|_2^2 = \sum_{\mu=-\infty}^{\infty} |\zeta^j(\mu)|^2. \tag{27}$$

Now,we assume that $\lambda_k^j = \tau^j e^{ipkh}$ is the solution of equations (19)-(22),where $i \in \mathbb{C}$ and $p \in \mathbb{R}$. Thus, we can write equation (20) as follows:

$$\begin{aligned} &\beta\tau^{j+1}\big[e^{ip(k-2)h} + 26e^{ip(k-1)h} + 66e^{ipkh} + 26e^{ip(k+1)h} + e^{ip(k+2)h}\big] \\ &\quad + \frac{a}{h}\tau^{j+1}\big[-5e^{ip(k-2)h} - 50e^{ip(k-1)h} + 50e^{ip(k+1)h} + 5e^{ip(k+2)h}\big] \\ &\quad - \frac{v}{h^2}\tau^{j+1}\big[20e^{ip(k-2)h} + 40e^{ip(k-1)h} - 120e^{ipkh} + 40e^{ip(k+1)h} + 20e^{ip(k+2)h}\big] \\ &= \beta\tau^{j}\big[e^{ip(k-2)h} + 26e^{ip(k-1)h} + 66e^{ipkh} + 26e^{ip(k+1)h} + e^{ip(k+2)h}\big] \\ &\quad - \beta b_1\tau^{j}\big[e^{ip(k-2)h} + 26e^{ip(k-1)h} + 66e^{ipkh} + 26e^{ip(k+1)h} + e^{ip(k+2)h}\big] \\ &\quad - \beta\sum_{v=0}^{j-1}(-b_v + b_{v+1})\tau^{j-v}\big[e^{ip(k-2)h} + 26e^{ip(k-1)h} + 66e^{ipkh} + 26e^{ip(k+1)h} + e^{ip(k+2)h}\big]. \end{aligned} \tag{28}$$

We divide system (28) on e^{ipkh}, so we obtain

$$\tau^{j+1} = \frac{(1-b_1)p_1}{p_1+p_2-p_3}\tau^j + \frac{p_1}{p_1+p_2-p_3}\sum_{\nu=1}^{j-1}(b_\nu - b_{\nu+1})\tau^{j-\nu}. \tag{29}$$

where

$$p_1 = e^{-2iph} + 26e^{-iph} + 66 + 26e^{iph} + e^{2iph},$$
$$p_2 = -\frac{-5a}{h}e^{-2iph} - \frac{50a}{h}e^{-iph} + \frac{50a}{h}e^{iph} + \frac{5a}{h}e^{2iph},$$
$$p_3 = \frac{20}{h^2}e^{-2iph} + \frac{40}{h^2}e^{-iph} - \frac{120}{h^2} + \frac{40}{h^2}e^{iph} + \frac{20}{h^2}e^{2iph}.$$

Lemma 1. *If τ^j is a solution of (29), then $|\tau^j| \le |\tau^0|,\ j = 0(1)m$.*

Proof. We prove the result by induction, for $j = 0$, equation (29) implies

$$|\tau^1| = \frac{(1-b_1)p_1}{p_1+p_2-p_3}|\tau^0|.$$

Since $p_2 = -\frac{-5a}{h}e^{-2iph} - \frac{50a}{h}e^{-iph} + \frac{50a}{h}e^{iph} + \frac{5a}{h}e^{2iph}$ and by the appropriate choice of the value of a, where a is arbitrary constant we have

$$\frac{p_1}{p_1+p_2-p_3} \le 1. \tag{30}$$

From (9) we obtain

$$(1-b_1) < 1. \tag{31}$$

Hence from (30) and (31) one get

$$|\tau^1| \le |\tau^0|.$$

Now, suppose $|\tau^j| \le |\tau^0|,\ j = 0(1)m-1$. Using equation (29) one has

$$|\tau^{j+1}| = \frac{(1-b_1)p_1}{p_1+p_2-p_3}|\tau^j| + \frac{p_1}{p_1+p_2-p_3}\sum_{\nu=1}^{j-1}(b_\nu - b_{\nu+1})|\tau^{j-\nu}|.$$

This implies

$$|\tau^{j+1}| \le (1-b_1)|\tau^j| + \sum_{\nu=1}^{j-1}(b_\nu - b_{\nu+1})|\tau^{j-\nu}|.$$

From (9) we get

$$|\tau^{j+1}| \le |\tau^0|.$$ □

Theorem 1. *A system (14) is unconditionally stable .*

Proof. Lemma 1 and (27) allow us to proceed to $\|\lambda^j\|_2 \le \|\lambda^0\|_2,\ j = 0, 1 \ldots, m$. This suggests unconditionally stability of system (14). □

5 Convergence of the *QNBS* method

The convergence of the *QNBS* schemes is explained in this section. We start by outlining some of the key conditions for the proof.

Theorem 2 ([12, 27]). *Let $z(\zeta,\eta) \in C^6([c,d])$, $g(\zeta,\eta) \in C^2([c,d])$, and $\Delta : c = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{n-1} < \zeta_n = d$ be the uniform division of $[c,d]$. If $S(\zeta,\eta)$, the Quintic spline function, is used to interpolate the function's values of $z(\zeta,\eta)$ at the knots $\zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{n-1} < \zeta_n \in \Delta$, then $\exists$ R_s independent of the step size h, so we have for each $t > 0$ and $\zeta \in [c,d]$*

$$\|D^s(z(\zeta,\eta) - S(\zeta,\eta)\|_\infty \leq R_s h^{6-s}, s = 0,1,2,3,4.$$

Lemma 2 ([17, 20]). *The Quintic B-spline $\{\mathbb{B}_{-2},\mathbb{B}_{-1},\mathbb{B}_0,\mathbb{B}_1,...,\mathbb{B}_n,\mathbb{B}_{n+1},\mathbb{B}_{n+2}\}$ defined in equation (4) satisfy the following inequality*

$$\sum_{i=-2}^{n+2} |\mathbb{B}_i(\zeta)| \leq 186, \quad c \leq \zeta \leq d.$$

Theorem 3. *If $g(\zeta,\eta) \in C^2([c,d])$, $\hat{z}(\zeta,\eta)$ and $z(\zeta,\eta)$ are approximate and exact solutions of (1), respectively, then for all $\eta > 0$ we have*

$$\|z(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty \leq \gamma h^4 + \sigma(\Delta\eta^2),$$

where γ is independent of the step size h, and h is sufficiently small.

Proof. Let

$$S(\zeta,\eta) = \sum_{i=-2}^{n+2} u_i(\eta)\mathbb{B}_i(\zeta),$$

be the spline that is computed for both $\hat{z}(\zeta,\eta)$ and $z(\zeta,\eta)$. When triangular inequality is used, one has

$$\|z(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty \leq \|z(\zeta,\eta) - S(\zeta,\eta)\|_\infty - \|S(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty.$$

From Theorem 2 one gets

$$\|z(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty \leq R_0 h^6 - \|S(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty. \tag{32}$$

Linearizing the nonlinear term $\hat{z}^{j+1}\frac{\partial \hat{z}^{j+1}}{\partial \zeta}$ in (11) by taking $\hat{z}^{j+1}$ as a constant a , we obtain

$$\beta\hat{z}^{j+1} - \beta\hat{z}^j + a\frac{\partial \hat{z}^{j+1}}{\partial \zeta} - \nu\frac{\partial^2 \hat{z}^{j+1}}{\partial \zeta^2} = -\beta\sum_{\nu=0}^{j} b_\nu[\hat{z}^{j+1-\nu} - \hat{z}^{j-\nu}] + g^{j+1}. \tag{33}$$

Theorize collocation requirement as:

$$\begin{aligned} L(z(\zeta_i,\eta)) &= L(\hat{z}(\zeta_i,\eta)) \\ &= g(\zeta_i,\eta), \; i = 0,1,....,n. \end{aligned}$$

Assume that

$$L(S(\zeta_i,\eta)) = \hat{g}(\zeta_i,\eta), \; i = 0,1,....,n.$$

The TFNBE (1), conserving (2) and (3) in the form of $L(S(\zeta,\eta)-\hat{z}(\zeta,\eta))$, can therefore be stated as follows at time level:

$$\begin{aligned}&[\beta-\frac{5}{h}-v\frac{20}{h^2}]f_{i-2}^{j+1}+[26\beta-\frac{50}{h}-v\frac{40}{h^2}]f_{i-1}^{j+1}\\&+[66\beta+v\frac{120}{h^2}]f_i^{j+1}+[26\beta+\frac{50}{h}-v\frac{40}{h^2}]f_{i+1}^{j+1}+[\beta+\frac{5}{h}-v\frac{20}{h^2}]f_{i+2}^{j+1}\\=&\beta(f_{i-2}^j+26f_{i-1}^j+66f_i^j+26f_{i+1}^j+f_{i+2}^j)-\beta\sum_{v=1}^{j}b_v(f_{i-2}^{j-v+1}+26f_{i-1}^{j-v+1}+66f_i^{j-v+1}\\&+26f_{i+1}^{j-v+1}+f_{i+2}^{j-v+1})-(f_{i-2}^{j-v}+26f_{i-1}^{j-v}+66f_i^{j-v}+26f_{i+1}^{j-v}+f_{i+2}^{j-v})+\frac{1}{h^2}t_i^{j+1},\quad\forall j.\end{aligned}\tag{34}$$

The formation is in fact occupied by the boundary conditions:

$$f_{i-2}^j+26f_{i-1}^j+66f_i^j+26f_{i+1}^j+f_{i+2}^j=0,\ \ i=0,1,....,n,$$

where

$$\begin{aligned}f_i^j&=c_i^j-u_i^j,\ \ i=0,1,....,n,\\t_i^j&=h^2(g_i^j-\hat{g}_i^j),\ \ i=0,1,....,n.\end{aligned}$$

From Theorem 2 we obtain

$$|t_i^j|=h^2|g_i^j-\hat{g}_i^j|\le R_0h^6,$$

consequently, we define

$$\begin{cases}|t^j|=max\{|t_i^j|,0\le i\le n\},\\|\lambda_i^j|=|f_i^j|\\|\lambda^j|=max\{|\lambda_i^j|,0\le i\le n\}.\end{cases}$$

For $j=0$ in (34), one gets

$$\begin{aligned}&[\beta-\frac{5}{h}-v\frac{20}{h^2}]f_{i-2}^1+[26\beta-\frac{50}{h}-v\frac{40}{h^2}]f_{i-1}^1+[66\beta+v\frac{120}{h^2}]f_i^1\\&+[26\beta+\frac{50}{h}-v\frac{40}{h^2}]f_{i+1}^1+[\beta+\frac{5}{h}-v\frac{20}{h^2}]f_{i+2}^1\\&=\beta(f_{i-2}^0+26f_{i-1}^0+66f_i^0+26f_{i+1}^0+f_{i+2}^0)+\frac{1}{h^2}t_i^1,\end{aligned}\tag{35}$$

The initial conditions will yield the following outcome $\lambda^0=0$:

$$\begin{aligned}&[\beta-\frac{5}{h}-v\frac{20}{h^2}]f_{i-2}^1+[26\beta-\frac{50}{h}-v\frac{40}{h^2}]f_{i-1}^1+[66\beta+v\frac{120}{h^2}]f_i^1\\&+[26\beta+\frac{50}{h}-v\frac{40}{h^2}]f_{i+1}^1+[\beta+\frac{5}{h}-v\frac{20}{h^2}]f_{i+2}^1\\&=\frac{1}{h^2}t_i^1.\end{aligned}\tag{36}$$

With a small enough h, and taking the absolute values of f_i^1 and t_i^1, we obtain

$$\lambda^1\le\frac{R_0h^4}{120\beta}\le n_1h^4,\tag{37}$$

where n_1 is unaffected by h. Assuming $\lambda^x \le n_x h^4$ is true for $x = 1,2,....,j$, process and apply the induction approach. After setting $n = max n_x, 0 \le x \le j$, (34) turns into

$$\begin{aligned}[\beta - \frac{5}{h} - v\frac{20}{h^2}]f_{i-2}^{j+1} + [26\beta - \frac{50}{h} - v\frac{40}{h^2}]f_{i-1}^{j+1} + [66\beta + v\frac{120}{h^2}]f_i^{j+1} \\ + [26\beta + \frac{50}{h} - v\frac{40}{h^2}]f_{i+1}^{j+1} + [\beta + \frac{5}{h} - v\frac{20}{h^2}]f_{i+2}^{j+1} \\ =(\beta - \beta b_1)(f_{i-2}^j + 26f_{i-1}^j + 66f_i^j + 26f_{i+1}^j + f_{i+2}^j) - \beta\sum_{v=0}^{j-1}(-b_v + b_{v+1}) \\ \times (f_{i-2}^{j-v} + 26f_{i-1}^{j-v} + 66f_i^{j-v} + 26f_{i+1}^{j-v} + f_{i+2}^{j-v}) + \frac{1}{h^2}t_i^{j+1}.\end{aligned} \tag{38}$$

Henceforth,

$$120\beta\lambda^{j+1} \le 120\beta(1-b_1)\lambda^j - 120\beta\sum_{v=0}^{j-1}(-b_v + b_{v+1})\lambda^{j-v}. \tag{39}$$

From (9) we conclude

$$\lambda^{j+1} \le nh^4. \tag{40}$$

It is now possible to create that

$$S(\zeta,\eta) - \hat{z}(\zeta,\eta) = \sum_{i=-2}^{n+2}(c_i(\eta) - u_i(\eta))\mathbb{B}(\zeta). \tag{41}$$

Using Lemma 2, we obtain

$$\|S(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty \le 186nh^4. \tag{42}$$

From (8) , (32) and (42) we prove

$$\|z(\zeta,\eta) - \hat{z}(\zeta,\eta)\|_\infty \le \gamma h^4 + \sigma(\Delta\eta^2),$$

where $\gamma = R_0 + 186n$. □

6 Numerical application

To ensure that the required process is performed as effectively as possible, two instances are conducted in this section. All computations are carried out using Maple 17 to demonstrate the precision and efficacy of this technique. The error norms L_2, and L_∞ are used to test the accuracy of the method that is being described which are computed in [28] as follows:

$$L_2 = \|z - \hat{z}_n\|_2 \simeq \sqrt{h\sum_{j=0}^{n}|z_j - (\hat{z}_n)_j|^2}$$

and

$$L_\infty = \|z - \hat{z}_n\|_\infty \simeq max_j|z_j - (\hat{z}_n)_j|,$$

where $\hat{z}$ and z stand for the approximate and exact solutions at the ith knot, respectively.

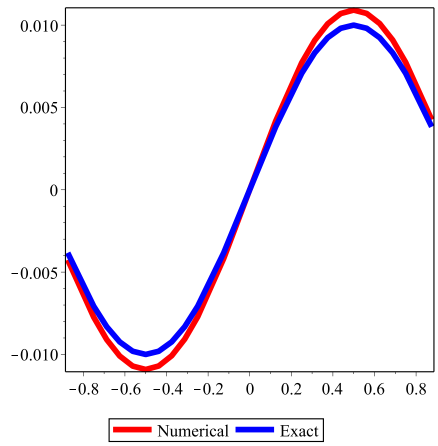

Figure 1: Comparison of the numerical and exact solutions at $\kappa = 0.5$

Table 2: The error norms L_∞ comparison of our method with method of [22] and method of [16] for Example 1

$\Delta\eta$	Proposed method	Method of [22]	Method of [16]
$\frac{1}{10}$	0.0009140	0.0020697	0.0052339
$\frac{1}{40}$	0.0000888	0.0001412	0.0013443
$\frac{1}{160}$	0.0000077	0.0000090	0.0003438

Example 1. We consider the following TFNBE [22]:

$$\begin{aligned}&\frac{\partial^\kappa z(\zeta,\eta)}{\partial \eta^\kappa} + z(\zeta,\eta)\frac{\partial z(\zeta,\eta)}{\partial \zeta} - \frac{\partial^2 z(\zeta,\eta)}{\partial \zeta^2} = g(\zeta,\eta), \quad (\zeta,\eta) \in [-1,1]\times(0,T],\\ &z(\zeta,0) = 0, \qquad \zeta \in [-1,1]\\ &z(-1,\eta) = 0 \quad and \quad z(1,\eta) = 0.\end{aligned} \tag{43}$$

The exact solution of (42) is

$$z(\zeta,\eta) = \eta^2 sin(\pi\zeta)$$

and

$$g(\zeta,\eta) = \frac{\eta^{2-\kappa}\Gamma(3)}{\Gamma(3-\kappa)} sin(\pi\zeta) + \pi\eta^4 sin(\pi\zeta)cos(\pi\zeta) + \pi^2\eta^2 sin(\pi\zeta).$$

Figure 1's precise and approximative solutions have a strong and verified agreement. While the Figures 2 and 3 show the error level for different values of v. After comparing the findings from the suggested approach with those taken from the [16, 22] in Table 2 , it is found that the results are superior.

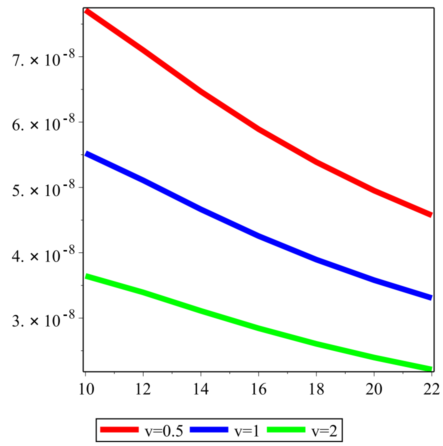

Figure 2: Error comparison for various n values when $\kappa = 0.25$ and $\Delta\eta = 0.001$

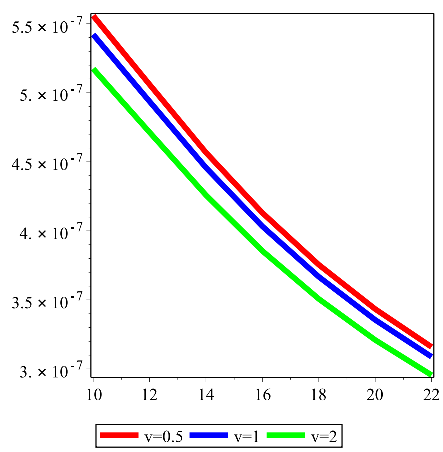

Figure 3: Error comparison for various n values when $\kappa = 0.75$ and $\Delta\eta = 0.001$

Example 2. We consider the following TFNBE [8]:

$$\begin{aligned} &\frac{\partial^\kappa z(\zeta,\eta)}{\partial \eta^\kappa} + z(\zeta,\eta)\frac{\partial z(\zeta,\eta)}{\partial \zeta} - \frac{\partial^2 z(\zeta,\eta)}{\partial \zeta^2} = g(\zeta,\eta), \qquad (\zeta,\eta) \in [0,1]\times(0,T], \\ &z(\zeta,0) = 0, \qquad \zeta \in [0,1] \\ &z(0,\eta) = \eta^2 \quad and \quad z(1,\eta) = \eta^2 e^1. \end{aligned} \tag{44}$$

The exact solution of problem (43) is $z(\zeta,\eta) = \eta^2 e^\zeta$, and $g(\zeta,\eta) = 2\frac{\eta^{2-\kappa}}{\Gamma(3-\kappa)}e^\zeta + \eta^4 e^{2\zeta} - \eta^2 e^\zeta$. The exact and approximate solutions in Figure 4 have a strong and verified agreement. The solutions for various values of κ, are compared in Figure 5, and the solutions for various values of $\Delta\eta$ are compared in Figure 6. Table 3 shows a comparison between L_2 and L_∞ for different values of n and Table 4 shows a comparison between L_2 and L_∞ for different values of $\Delta\eta$.

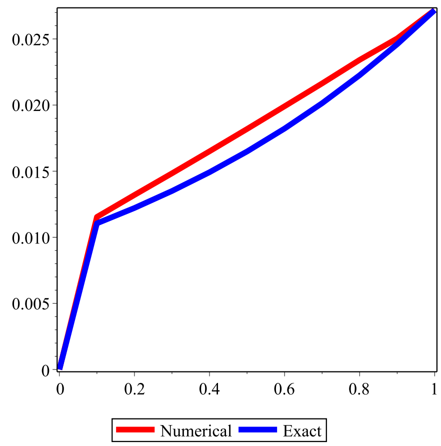

Figure 4: The numerical and precise solutions are compared at $\kappa = 0.5$.

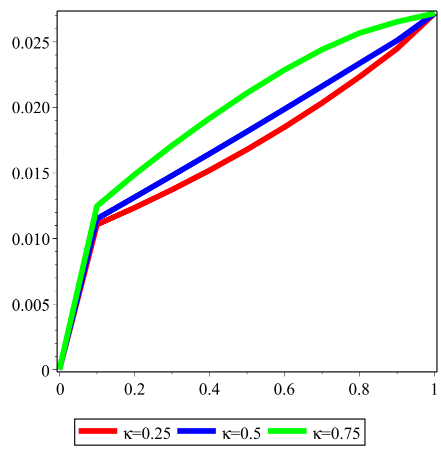

Figure 5: Comparison of solution for different values of κ when $\Delta\eta = 0.1$ and $n = 40$

Table 3: The error norms L_∞ and L_2 of Example 2 for $\kappa = 0.5$ and $\Delta\eta = 0.1$

	$n = 10$	$n = 20$	$n = 40$	$n = 80$
L_∞	0.0017012	0.0016922	0.0016742	0.0016383
L_2	0.0012250	0.0008628	0.0006007	0.0004125

Table 4: The error norms L_∞ and L_2 of Example 2 for $\kappa = 0.5$ and $n = 40$

	$\Delta\eta = 0.001$	$\Delta\eta = 0.003$	$\Delta\eta = 0.005$
L_∞	5.269×10^{-7}	4.100×10^{-6}	1.037×10^{-5}
L_2	2.001×10^{-7}	1.529×10^{-6}	3.850×10^{-6}

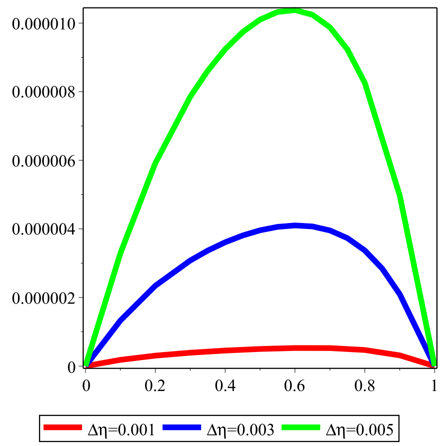

Figure 6: Comparison of error for different values of $\Delta\eta$ when $\kappa = 0.5$ and $n = 40$

Table 5: Comparison of the proposed method with [30] for certain values of n

		$n = 10$	$n = 20$	$n = 40$	$n = 80$
Proposed method	L_∞	1.9313×10^{-8}	1.1613×10^{-8}	1.1598×10^{-8}	1.1596×10^{-8}
	L_2	8.4900×10^{-9}	5.8999×10^{-9}	4.1706×10^{-9}	2.9487×10^{-9}
[30]	L_∞	1.9866×10^{-5}	1.9805×10^{-5}	1.9579×10^{-5}	1.9531×10^{-5}
	L_2	1.4626×10^{-5}	1.3963×10^{-5}	1.3799×10^{-5}	1.3759×10^{-5}

Example 3. We consider The following TFNBE [30]:

$$\begin{aligned}
&\frac{\partial^\kappa z(\zeta,\eta)}{\partial \eta^\kappa} + z(\zeta,\eta)\frac{\partial z(\zeta,\eta)}{\partial \zeta} - \frac{\partial^2 z(\zeta,\eta)}{\partial \zeta^2} = g(\zeta,\eta), \quad (\zeta,\eta) \in [0,1] \times (0,T], \\
&z(\zeta,0) = 0, \qquad \zeta \in [0,1] \\
&z(0,\eta) = \eta_2 \;\; and \;\; z(1,\eta) = -\eta^2.
\end{aligned} \tag{45}$$

The exact solution of problem (44) is $z(\zeta,\eta) = \eta^2 cos(\pi\zeta)$, for

$$g(\zeta,\eta) = \frac{2\eta^{2-\kappa} cos(\pi\zeta)}{\Gamma(3-\kappa)} - \pi\eta^4 sin(\pi\zeta)cos(\pi\zeta) + \pi^2\eta^2 cos(\pi\zeta).$$

Table 5 compares L_∞ and L_2 errors for various n values and parametric values, including $\kappa = 0.5$ $v = 1$, and $\Delta\eta = 0.00025$. Table 6 compares error norms with values for $\kappa = 0.5$, $n = 80$, and $v = 1$.

Table 6: Comparison of the proposed method with [30] for certain values of $\Delta\eta$

		$\Delta\eta = 0.002$	$\Delta\eta = 0.001$	$\Delta\eta = 0.0005$
Proposed method	L_∞	4.5777×10^{-7}	1.3800×10^{-7}	4.0546×10^{-8}
	L_2	1.1669×10^{-7}	3.5120×10^{-8}	1.0308×10^{-8}
[30]	L_∞	1.6442×10^{-4}	8.7080×10^{-5}	4.8293×10^{-5}
	L_2	1.1600×10^{-4}	6.1505×10^{-5}	3.4177×10^{-5}

Table 7: The error norms L_∞ and L_2 of Example 4

	$\Delta\eta = 0.1$	$\Delta\eta = 0.3$	$\Delta\eta = 0.5$
L_∞	3.0601×10^{-14}	1.0217×10^{-13}	1.7987×10^{-13}
L_2	1.9446×10^{-14}	6.7744×10^{-14}	1.2286×10^{-13}

Example 4. We consider the following TFNBE:

$$\begin{aligned}
&\frac{\partial^\kappa z(\zeta,\eta)}{\partial \eta^\kappa} + z(\zeta,\eta)\frac{\partial z(\zeta,\eta)}{\partial \zeta} - v\frac{\partial^2 z(\zeta,\eta)}{\partial \zeta^2} = g(\zeta,\eta), \quad (\zeta,\eta) \in [0,10] \times (0,T], \\
&z(\zeta,0) = 0, \qquad \zeta \in [0,10] \\
&z(0,\eta) = 0 \quad and \quad z(10,\eta) = 0,
\end{aligned} \tag{46}$$

where $g(\zeta,\eta) = \frac{\eta^{1-\kappa} sin(\pi\zeta)}{\Gamma(2-\kappa)} + \pi\eta^2 sin(\pi\zeta)cos(\pi\zeta) + v\pi^2\eta sin(\pi\zeta)$. The exact solution for this problem is $z(\zeta,\eta) = \eta sin(\pi\zeta)$. We apply the transformation $\zeta = 5x+5$ on the interval $[0,10]$ to get $[-1,1]$. Thus, we have the equation:

$$\frac{\partial^\kappa z(x,\eta)}{\partial \eta^\kappa} + 0.2z(x,\eta)\frac{\partial z(x,\eta)}{\partial x} - 0.04v\frac{\partial^2 z(x,\eta)}{\partial x^2} = g(x,\eta), \qquad (x,\eta) \in [-1,1] \times (0,T],$$

with the initial condition

$$z(\zeta,0) = 0, \qquad \zeta \in [-1,1],$$

and

$$g(\zeta,\eta) = \frac{\eta^{1-\kappa} sin(\pi(5x+5))}{\Gamma(2-\kappa)} + \pi\eta^2 sin(\pi(5x+5))cos(\pi(5x+5)) + v\pi^2\eta sin(\pi(5x+5)).$$

Table 7 shows a comparison between L_2 and L_∞ for different values of $\Delta\eta$ at $\kappa = 0.5$, $v = 5$ and $n = 10$.

7 Conclusions

The spline function is a recognized and effective instrument for approximating solutions of fractional partial differential equations, due to its piecewise polynomial framework and inherent smoothness characteristics. In this study, we have proposed an efficient numerical method, quintic B-splines (QNBS) functions, for solving TFNBE. The Caputo time-fractional derivative has been approximated by means of the usual finite difference scheme, and the quintic B-spline functions are used for spatial discretization. Additionally, four numerical cases have been examined utilizing the QNBS approach; graphs and tables show the accuracy and practicality of the approach. The theoretical findings are validated by the numerical outcomes of the QNBS method. Comparing the scheme suggested in this work to others already established in the literature, it offers sufficient precision, and it is innovative. The implementation of the recommended approach demonstrates that it is more efficient, simple, and palatable than [16,22,30]. A significant theoretical outcome is the demonstration of unconditional stability within the proposed QNBS Fourier framework. In contrast to numerous time-stepping methods that necessitate adherence to the Courant-Friedrichs-Lewy condition, our algorithm maintains stability for any selection of time and spatial discretization. This robustness is especially advantageous for addressing long-term integration issues or when utilizing adaptive meshes to capture localized features. It is no secret that proving convergence is of great importance in demonstrating the accuracy and validity of the method. In this work, the convergence of the method has been proven, which means that the method is characterized by accuracy.

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