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New non-radial models for merger and acquisition to achieve strong efficiency and MPSS without changing in the efficient frontier

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Abstract. In this study, new models for the merger of two or more inefficient units are presented. The discussed mergers in this study are horizontal and acquisition mergers. In the presented models, unlike the previously presented models, the mergers are done non-radially, indeed, managers can manage each of the indicators separately in the presented new merger process. In this regard, efficient frontier of the production possibilities set does not change, and merger of several units does not affect the efficiency score of other decision-making units. Also, by merging via the new models one can obtain a strong efficient unit with the most productive scale size. The presented models are applied to the horizontal and acquisition merger of Iranian banks.

Keywords: Merger, acquisition, most productive scale size, strongly efficient, efficient frontier, non-radial.
AMS Subject Classification 2010: 65K05, 90B50, 90C90.

1 Introduction

Due to financial crises, for various reasons, mostly in governments, organizations, and private institutions, policymakers and managers usually decide to reduce costs by merging units, while the efficiency and service delivered by units under their management are not decreased. There exist different types of mergers based on the relationship between the units and this study is interested in the horizontal and acquisition merger. In horizontal mergers, two or more units merge to form an integrated new unit, while an acquisition merger is the purchase of one unit, in whole or in part, by another unit [11, 15]. In any way, the merger of the units should be done in a way that the efficiency of the obtained new unit is not lower than the efficiency of the merged units, also, the portion of the indicators of each merged unit should be determined. It is important to mention that after the economic collapse of 1990, to support companies and prevent their bankruptcy, countries including Japan recommended that these institutions

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should reorganize and the merger was one of the important and applicable ways to restructure institutions and companies, hence most of the studies in the field of mergers have been done in banks and parametric and non-parametric methods (such as Data Envelopment Analysis (DEA) [13]) have been proposed in the field of the merger of units.

Formally, the mergers debate was initiated by Healy et al. [16], who showed, there is a strong positive relation between post-merger increases in operating cash flows and abnormal stock returns at pre-merger. Shaffer in [26] used simulation to examine mergers between pairs of US commercial banks with assets greater than 1 billion. He showed that efficiency increasing is possible if the best banks merge with the least efficient banks. Rhoades in [24] investigated the efficiency effects of nine banks obtained from merger in the United States using the economic efficiency technique, his findings suggest that the cost-efficiency effects of mergers may depend on the motivation of the merger, and the merger process always does not increase performance. Zhu was another researcher who reached the results of [24] by conducting research [33]. In this direction, Al-Sharkas et al. in [1] provided an extensive empirical investigation on the cost and efficiency effects of the merger of banks and analyzed it by using a parametric approach and a non-parametric approach (DEA), while previous studies focused mainly on the merger of large banks, they examined the merger of banks in all sizes. Using SFA (a method of stochastic frontier analysis based on econometric models and microeconomic theories), they provided a statistical evidence that the merger of banks, in general, leads to increase the cost and profit efficiency and also showed that the merger of small banks improves the cost efficiency in comparison with the merger of large banks.

In another study, Bogetoft and Wang performed the merger process under the level of existing inputs and outputs using the standard models of DEA and obtained the potential profit from the merger by maximizing the input reduction of the merged inventory [8]. However, Cooper et al. in [12] showed that the obtained unit from the merger under standard models of DEA, including the presented model by Banker et al. [6], are not always efficient. Two DEA models are presented by Lozano and Villa, one for minimizing the post-merger input cost and the other for maximizing the post-merger profit. The first model assumes that input prices are known, while the second model assumes that output prices are known. Both models are suitable for in-market horizontal mergers, though considerable overlap may exist among the branches of the merging firms [21]. Lozano used a cost-minimizing model in DEA to select a partner from among different partners to form a joint venture, which was the best model for the strategic objective of horizontal cooperation [19]. Gattoufi et al. in 2014, proposed a new application of Inverse Data Envelopment Analysis (InvDEA [2, 22, 29]) in strategic decision-making on mergers in banking [14]. In this proposal, a new approach based on InvDEA is developed to determine the required level of input and output for the obtained bank from the merger to achieve a predetermined efficiency value. They implemented their approach for 42 banks in the Persian Gulf Council countries. Also, Xiao et al. in [32] presented a two-stage network model in DEA to estimate and analyze the potential benefits of cost efficiency from mergers. In [3] by using InvDEA a model is presented to predict the impact of the merger on the market. Amin and Oukil used the potential of InvDEA to create a flexible target of inputs and outputs that allows the decision-maker to favor a specific input in the merger target setting [5]. In the sequence an InvDEA model based on cost efficiency in [4] applied to estimate the potential profit from the merger, it is shown there that the obtained profit from mergers in the presented model is higher than the technical efficiency model of InvDEA. Then Soltanifar et al. presented a model for merging units with negative data [27]. Recently, the merger problem has been used in network data envelopment analysis. For instance, [18] have applied the merger scheme for units with the two-stage series network using inverse DEA.

In addition, most applications of mergers occur in financial institutions, especially banks, while, some researchers have investigated the application of mergers in other organizations as well. Pyra and Siedlecka by using a simulation and a DEA model, investigated the potential benefits of the merger from universities in Poland [23]. Also, Lozano and Contreras thoroughly analyzed the Spanish public universities system to maximize their efficiency and evaluated the achievements from the potential merger of universities in each Spanish region separately [20]. In addition to discussing how the units are merged, another issue that researchers follow is the selection of appropriate units in the merger process. Wu et al. in [31] proposed the first DEA- based algorithm to assist a bidder unit to choose a suitable unit to merge. Blancard et al. proposed a DEA-based mixed-integer linear programming model to identify the units to enter into a merger with a given unit [7]. Zhu et al. proposed a 0-1 integer linear programming model based on DEA for choosing the best partner for the merger [34]. Chang et al. proposed new models of Nash bargaining in DEA to help units obtain their most desired target units for the vertical merger [10].

In all the studies conducted in the field of the merger in DEA, technical or radial efficiency has been used. However, in this article, new models are presented that the merger of units is done non-radially, indeed, the input and output of the merged units decrease or increase non-radially. Also, in these models, unlike the previously presented models, the efficiency frontier does not change. Using these models can obtain a strongly efficient unit with the most productive scale size (MPSS). In this regard in Section 2 of this paper, an introduction of DEA and required concepts are presented. Then, after briefly introducing the merger process, DEA-related models are examined. Section 3, is devoted to the main discussions, where new models are presented. Then, a real example is presented to illustrate our approach in Section 4, and the final section is devoted conclusion.

2 Some concepts of DEA and Merger

As we know, all the decision-making units (DMUs) in an organization, indeed are located in a set called the production possibility set (PPS). In a PPS, if for all efficient DMUs, the increase ratio of production factors (input) is equal to the increase ratio of the production (output), PPS has the property of constant returns to scale (CRS), otherwise, it has the property of variable returns to scale (VRS). DMUs with the CRS feature, have the MPSS and they are efficient in both types of PPS [12].

Definition 1. *Let $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^t \geqslant 0$ and $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^t \geqslant 0$ be the input and output vectors of DMU_j for $j = 1, 2, \ldots, n$ respectively. The PPS introduced by Charnes et al. [9] and Banker et al. [6] are displayed as $PPS_{CRS} = \{(X,Y) \mid \sum_{j=1}^{n} \lambda_j X_j \leqslant X, \sum_{j=1}^{n} \lambda_j Y_j \geqslant Y, \lambda \geqslant 0\}$, $PPS_{VRS} = \{(X,Y) \mid \sum_{j=1}^{n} \lambda_j X_j \leqslant X, \sum_{j=1}^{n} \lambda_j Y_j \geqslant Y, 1\lambda = 1, \lambda \geqslant 0\}$.*

Since $PPS_{CRS} \subseteq PPS_{VRS}$, the obtained efficiency value of DMU_j for all j, in the BCC model is higher than or equal to the obtained efficiency value from the CCR model, and DMUs that are efficient in both sets PPS_{CRS} and PPS_{VRS}, they are MPSS units [12].

Definition 2. *DMU_A dominates DMU_B if $X_A \leqslant X_B$ and $Y_A \geqslant Y_B$.*

Definition 3. *A DMU is strongly efficient or Pareto-Koopmans efficient if and if it is not possible to improve any input or output without worsening some other input or output. In other words, a unit $(X,Y) \in PPS_{VRS}$ is strongly efficient, if there does not exist a unit $(\overline{X},\overline{Y}) \in PPS_{VRS}$, such that $\overline{X} \leqslant X$, $\overline{Y} \geqslant Y$ and $(\overline{X},\overline{Y}) \neq (X,Y)$.*

Table 1: Inputs and outputs of 5 units.

DMU	A	B	C	D	E
Input	1	2	3	2	4
Output	2	3	3	2	4

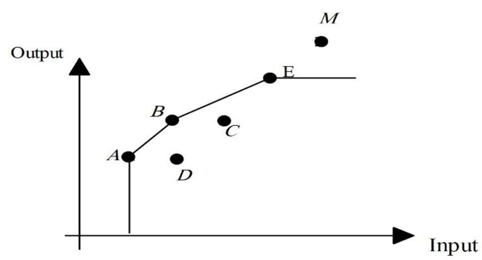

Figure 1: Production possibility set

Definition 4. *A unit* $(X,Y) \in PPS_{VRS}$ *is weakly efficient or Farrell efficient, if there does not exist a unit* $(\overline{X},\overline{Y}) \in PPS_{VRS}$*, such that* $\overline{X} < X$*,* $\overline{Y} > Y$*.*

Definition 5 ([12])**.** *Suppose* $\left(\xi^*, \psi^*, S^{-*}, S^{+*}, \lambda^*\right)$ *is the optimal solution of the following problem (1).*

$$\begin{array}{ll} Max & \frac{\psi}{\xi} \\ S.to: & \sum_{j=1}^{n} \lambda_j X_j - \xi X_P + S^- = 0, \\ & \sum_{j=1}^{n} \lambda_j Y_j - \psi Y_p - S^+ = 0, \\ & \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1,2,...,n, \\ & \psi \geq 1, \quad 0 < \xi \leq 1, \\ & \lambda_j \geq 0, \quad S^-, S^+ \geq 0. \end{array} \tag{1}$$

Then DMU_p *is MPSS if* $\frac{\psi^*}{\xi^*} = 1$ *and* $S^* = 0$*.*

To avoid losses and bankruptcy, organizations managers and decision makers sometimes try to merge inefficient units. Indeed, merging is one of the most effective ways to increase profits and productivity of companies. Generally, what is considered and comes to mind in merging, is that the input indicators are added together, and also the output indicators are added together as well. However, merging in this way is not profitable in many cases, since merging units will not always produce an efficient unit, even sometimes, from merging two efficient units, one efficient unit is not obtained [28]. Indeed, merging units is successful when the obtained unit is efficient and the managers get more profit from this merge. Usually, in merging several units, in reality, it is not possible to merge all the input or output elements of the units together, as the obtained unit may not be efficient or may go beyond the production frontier. For example, consider 5 units with one input and one output as follows in Table 1.

The $PPS_{(VRS)}$ from the observations of Table 1, are illustrated by Figure 1, which depicts input and output values of all DMUs. DMUs A, B and E are efficient, DMUs D and C are inefficient and DMU_M is obtained from merging DMUs C and D. As shown in Figure 1, DMU_M is located outside the production frontier and it can change the production frontier so that DMU_E which is superefficient, becomes

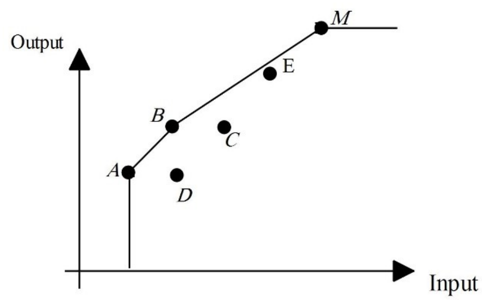

Figure 2: Production frontier and inefficiency of DMU_E after merging

inefficient. Whereas, managers do not expect that an efficient unit becomes an inefficient (see Figure 2). Additionally, Sometimes, for some outputs of DMUs a maximum value is defined and calculating more than this amount is not acceptable for managers and planners. For example, in banks usually, for paying the amount and number of the loans to customers, an upper bound is defined, therefore, in merging, the outputs amount should not be exceeded from the upper bound.
Gattoufi et al. in [14] presented the following model (2) for the horizontal merger of two units by using InvDEA.

$$\begin{array}{ll} Min & \sum_{i=1}^{m} (a_{iK} + a_{iP}) \\ S.to: & \sum_{j\in F} \lambda_j x_{ij} + \lambda_M (a_{iK} + a_{iP}) \leq \overline{\theta} (a_{iK} + a_{iP}), \quad i = 1,2,\dots,m, \\ & \sum_{j\in F} \lambda_j y_{rj} + \lambda_M (y_{rK} + y_{rP}) \geq (y_{rK} + y_{rP}), \quad r = 1,2,\dots,s, \\ & \sum_{j\in F} \lambda_j + \lambda_m = 1, \\ & \lambda_j \geq 0, \quad \forall j \in F, \quad \lambda_M \geq 0, \\ & 0 \leq a_{iK} \leq x_{iK}, \quad 0 \leq a_{iP} \leq x_{iP}, \quad i = 1,2,...,m, \end{array} \tag{2}$$

where variables a_{iK} and a_{iP} are the i-th input value of units P and K after the merger. The defined unit by merging two units P and K is called M in model (2). It is assumed that output M is equal to the sum of the two outputs of P and K.F is the set of DMUs in the post-merger evaluation process which does not include the merged DMUs and $\overline{\theta}$ is the predetermined efficiency score of the obtained new unit from the merger (it is determined by decision makers). Gattoufi et al. in [14] linearized the nonlinear model (2) by imposing $\lambda_M = 0$, which may make problem (2) infeasible. For instance, in the previous example, suppose that according to the manager's opinion $\overline{\theta} \leq 1$. If the merger of two units, C and D, is performed while assuming $\lambda_M = 0$, then model (2) becomes infeasible. Furthermore, even by assuming $\lambda_M = 1$ and $\overline{\theta} \leq 1$, model (2) remains infeasible.

As we said, in (2), DMU_M is obtained from the merger of two units, the output value of DMU_M is equal to the sum of the output of the two integrated units, as seen this may put DMU_M outside the production frontier and model (2) becomes infeasible. Amin et al. in [4] eliminated the infeasibility of (2) by changing the efficient frontier and locating DMU_M on the efficient frontier, but changing the efficient frontier may force some efficient units to leave the efficient frontier and become inefficient. On the other hand, the opinion of the inventors of DEA is that the merger should be done without changing the production possibility set [12, 28]. In this regard, by changing the feasible set of (2), they presented a model in which, by considering the prices for the inputs of the merged units, the obtained unit from

Table 2: Data of 5 single input-single output units

DMU	A	B	C	D	E
Input	1	2	3	2	4
Output	2	3	3	1	4

merging has the lowest cost [4].

$$\begin{array}{ll} Min & \sum_{i=1} c_i(a_{iK}+a_{iP}) \\ S.to: & \sum_{j\in F}\lambda_j x_{ij}+\lambda_M(x_{iK}+x_{iP})\leq(a_{iK}+a_{iP}), \quad i=1,2,...,m, \\ & \sum_{j\in F}\lambda_j y_{rj}+\lambda_M(y_{rK}+y_{rP})\geq(y_{rK}+y_{rP}), \quad r=1,2,...,s, \\ & \sum_{i=1}^m c_i(a_{iK}+a_{iP})=\frac{1}{\overline{C}}\sum_{i=1}^m c_i(a^*_{iK}+a^*_{iP}), \\ & \sum_{j\in F}\lambda_j+\lambda_M=1, \\ & \lambda_j\geq 0, \quad \forall j\in F, \quad \lambda_M\geq 0, \\ & 0\leq a_{iK}\leq x_{iK}, \quad 0\leq a_{iP}\leq x_{iP}, \quad i=1,2,...,m, \end{array} \tag{3}$$

where $\overline{C}$ is the target for cost efficiency of the merged entity and a^*_{iK}, a^*_{iP} (for each $i=1,2,\ldots,m$) are obtained from the optimal solution of the following DEA cost efficiency [12].

$$\begin{array}{ll} Min & \sum_{i=1}^m c_i(a_{iK}+a_{iP}) \\ S.to: & \sum_{j\in F}\lambda_j x_{ij}+\lambda_M\left(x_{iK}+x_{iP}\right)\leq\left(a_{iK}+a_{iP}\right), \quad i=1,2,\ldots,m, \\ & \sum_{j\in F}\lambda_j y_{rj}+\lambda_M\left(y_{rK}+y_{rP}\right)\geq\left(y_{rK}+y_{rP}\right), \quad r=1,2,\ldots,s, \\ & \sum_{j\in F}\lambda_j+\lambda_M=1, \\ & \lambda_j\geq 0, \quad \forall j\in F, \quad \lambda_M\geq 0, \\ & 0\leq a_{iK}\leq x_{iK}, \quad 0\leq a_{iP}\leq x_{iP}, \quad i=1,2,...,m. \end{array} \tag{4}$$

Problem (3) becomes infeasible for many values of $\overline{C}$, for example, based on the data in Table 2, if DMU_C and DMU_D are merged using models (3) and (4) with the input price $c_1=1$, then for any cost efficiency value of $\overline{C}<0.80$, problem (3) becomes infeasible. Consequently, the merger would be practically unsuccessful under these conditions. According to the feasible set of problem (3), the production or efficiency frontier may shift when λ_M takes a positive value. This shift in the production frontier can cause some previously efficient units to become inefficient. Moreover, in model (3), due to the existence of slack variables, the merged unit may not become a strong efficient unit, and a weak efficient unit may be obtained.

3 New non-radial merging models without changing efficient frontier

To overcome the above mentioned difficulties of merging, in this section, we present a new model in which it is able to do the horizontal merger of two or several inefficient units. In this model, the PPS and the production frontier do not change and merge planning is done based on the existing and real observations. Also the benchmark DMUs are real. Moreover, the new model is non-radial unlike the previous presented models, where in the merger process, managers can manage the efficiency value or the ratio of each of the indicators of the integrated units separately, while in the previous presented models, this ratio was the same for all indicators.

3.1 New horizontal merger

For inefficient units $R=(X_R,Y_R)$ and $K=(X_K,Y_K)$ assume that DMU_M is the horizontal major merged unit of DMU_R and DMU_K where $M=(X_M=X_R+X_K, Y_M=Y_R+Y_K)$, and F is the set of DMUs after the post-merger which does not include merged DMUs. Suppose DMU_O with the input vector $x_O=(\alpha^*_{1R}+\alpha^*_{1K},\alpha^*_{2R}+\alpha^*_{2K},...,\alpha^*_{mR}+\alpha^*_{mK})$ and output vectors $y_O=(\beta^*_{1R}+\beta^*_{1K},\beta^*_{2R}+\beta^*_{2K},...,\beta^*_{sR}+\beta^*_{sK})$ is our desired unit with the predetermined efficiency $\bar{\theta}_i\leq 1$ for i-th input and $\overline{\varphi}_r\geq 1$ for r-th output of DMU_O, where for all i and r, $\alpha^*_{iR}\leq x_{iR},\alpha^*_{iK}\leq x_{iK}$, $\beta^*_{rR}\geq y_{rR}$ and $\beta^*_{rK}\geq y_{rK}$. Accordingly, DMU_G with the input *vector* $x_G=\left(\bar{\theta}_1(\alpha^*_{1R}+\alpha^*_{1K}),\bar{\theta}_2(\alpha^*_{2R}+\alpha^*_{2K}),...,\bar{\theta}_m(\alpha^*_{mR}+\alpha^*_{mK})\right)$ and the output vectors $y_G=(\overline{\varphi}_1(\beta^*_{1R}+\beta^*_{1K}),\overline{\varphi}_2(\beta^*_{2R}+\beta^*_{2K}),...,\overline{\varphi}_s(\beta^*_{sR}+\beta^*_{sK}))$ is located on the efficient frontier and is strongly efficient in $PPS_{(VRS)}$. DMU_G dominates DMU_M, since for $i=1,2,...,m$ and $r=1,2,...,s$, $\bar{\theta}_i\leq 1$ and $\overline{\varphi}_r\geq 1$, so $\alpha^*_{iG}=\bar{\theta}_i\alpha^*_{iO}\leq x_{iM}$ and $\beta^*_{rG}=\overline{\varphi}_r\beta^*_{rO}\geq y_{rM}$. Therefore, for $i=1,2,...,m$, $\frac{\alpha^*_{iG}}{x_{iM}}\leq 1$ is the relative reduction of input x_{iM}, and for $r=1,2,...,s$, $\frac{\beta^*_{rG}}{y_{rM}}\geq 1$ is the relative increase of output y_{rM}, whatever $\frac{\alpha^*_{iG}}{x_{iM}}$ is smaller than one, input x_{iM} has the more surplus amount. Moreover, whatever $\frac{\beta^*_{rG}}{y_{rM}}$ is bigger than one, it means that the output y_{rM} can be increased further.
Hence, to find the strongly efficient unit DMU_G, we minimize $\frac{\alpha_{iG}}{x_{iM}}$, for $i=1,2,...,m$, and maximize $\frac{\beta_{rG}}{y_{rM}}$, for $r=1,2,...,s$ on $PPS_{(VRS)}$ by designing the following multi-objective model, and by calculating the optimal values α^*_{iR}, α^*_{iK} for $i=1,2,...,m$ and $\beta^*_{rR},\beta^*_{rK}$ for $r=1,2,...,s$, unit G and the desired unit O are obtained:

$$\begin{array}{ll}
Min & \sum_{i=1}^{m}\bar{\theta}_i\frac{\alpha_{iR}+\alpha_{iK}}{x_{iM}}\\
Max & \sum_{r=1}^{s}\bar{\varphi}_r\frac{\beta_{rR}+\beta_{rK}}{y_{rM}}\\
S.to: & \sum_{j\in F}\lambda_j x_{ij}\leq\bar{\theta}_i(\alpha_{iR}+\alpha_{iK}),\quad i=1,2,\dots,m,\\
& \sum_{j\in F}\lambda_j y_{rj}\geq\bar{\varphi}_r(\beta_{rR}+\beta_{rK}),\quad r=1,2,\dots,s,\\
& \sum_{j\in F}\lambda_j=1,\\
& \lambda_j\geq 0, j\in F,\\
& \alpha_{iR}\leq x_{iR},\quad \alpha_{iK}\leq x_{iK},\quad i=1,2,...,m,\\
& \beta_{rR}\geq y_{rR},\quad \beta_{rK}\geq y_{rK},\quad r=1,2,...,s.
\end{array}\tag{5}$$

Theorem 1. *Suppose $y_{tL}=max\{y_{tj}|j\in F\}$ where $t\in\{1,2,...,s\}$ and $L\in F$. If $y_{tL}<y_{tR}+y_{tK}$, then problem (5) is infeasible.*

Proof. Suppose that (5) is feasible. Then, due to the fact that $\bar{\varphi}_r\geq 1$ and the constraints of (5), for $t\in\{1,2,...,s\}$, the following inequality is satisfied:

$$\sum_{j\in F}\lambda_j y_{tj}\geq\bar{\varphi}_t(\beta_{tR}+\beta_{tK})\geq y_{tR}+y_{tK}.\tag{6}$$

On the other hand, according to the assumption of the theorem, for all $j\in F$ and $t\in\{1,2,...,s\}$, $\sum_{j\in F}\lambda_j y_{tj}<\sum_{j\in F}\lambda_j y_{tL}$. Since $\sum_{j\in F}\lambda_j=1$, we have $\sum_{j\in F}\lambda_j y_{tj}<y_{tL}<y_{tR}+y_{tK}$, which this is in contradiction with (6). □

Since, according to Theorem 1, model (5) may be infeasible, we introduce the following model, with assuming $H=\{r|y_{rM}>max\{y_{rj}|j\in F\}\}$ and $\overline{H}=\{r|y_{rM}\leq max\{y_{rj}|j\in F\}\}$ which according to Theorem 1, if $H\neq\emptyset$, problem (5) is infeasible.

$$
\begin{array}{ll}
Min & \left(\sum_{i=1}^{m} \bar{\theta}_i \frac{\alpha_{iR}+\alpha_{iK}}{x_{iM}} + \sum_{r\in H} \frac{\beta_{rR}+\beta_{rK}}{y_{rM}} - \sum_{r\in \overline{H}} \bar{\varphi}_r \frac{\beta_{rR}+\beta_{rK}}{y_{rM}}\right)\\
S.to: & \sum_{j\in F} \lambda_j x_{ij} \le \bar{\theta}_i (\alpha_{iR}+\alpha_{iK}), \quad i=1,2,\ldots,m,\\
& \sum_{j\in F} \lambda_j y_{rj} \le (\beta_{rR}+\beta_{rK}), \quad r\in H,\\
& \sum_{j\in F} \lambda_j y_{rj} \ge \bar{\varphi}_r (\beta_{rR}+\beta_{rK}), \quad r\in \overline{H},\\
& \sum_{j\in F} \lambda_j = 1,\\
& \lambda_j \ge 0, \quad \forall j\in F,\\
& \alpha_{iR} \le x_{iR}, \quad \alpha_{iK} \le x_{iK}, \quad i=1,2,...,m,\\
& \beta_{rR} \le y_{rR}, \quad \beta_{rK} \le y_{rK}, \quad \forall r\in H,\\
& \beta_{rR} \ge y_{rR}, \quad \beta_{rK} \ge y_{rK}, \quad \forall r\in \overline{H},
\end{array} \tag{7}
$$

where $0 < \overline{\theta}_i \le 1 (i = 1,2,..,m)$ is a predetermined ratio (level of efficiency or the contraction factor) for i-th input of DMU_O, and $\varphi_r \geqslant 1 (r \in \overline{H})$ are the predetermined ratio (level of efficiency or expansion factor) for r-th output $(r \in \overline{H})$ of DMU_O. In other words, set H includes all indices r, in which the r-th output of unit M is greater than the maximum r-th output across all units in set F. If H is non-empty, unit M lies outside the production possibility set, and we can not find a solution for model (5) that satisfies constraints $\sum_{j\in F} \lambda_j y_{rj} \ge \bar{\varphi}_r (\beta_{rR}+\beta_{rK})$ and $\beta_{rR} \ge y_{rR}, \beta_{rK} \ge y_{rK}$ for $r = 1,2,\ldots,s$.

The lower and upper bounds of $0 < \overline{\theta}_i \le 1 (i = 1,2,..,m)$ and $\overline{\varphi}_r \ge 1 (r \in \overline{H})$ are obtained from the following model:

$$
\begin{array}{ll}
Min & (\sum_{i=1}^{m} \theta_i + \sum_{r\in H} \varphi_r - \sum_{r\in \overline{H}} \varphi_r)\\
S.to: & \sum_{j\in F} \lambda_j x_{ij} \le \theta_i (x_{iR}+x_{iK}), \quad i=1,2,\ldots,m,\\
& \sum_{j\in F} \lambda_j y_{rj} \le \varphi_r (y_{rR}+y_{rK}), \quad \forall r\in H,\\
& \sum_{j\in F} \lambda_j y_{rj} \ge \varphi_r (y_{rR}+y_{rK}), \quad \forall r\in \overline{H},\\
& \sum_{j\in F} \lambda_j = 1,\\
& \lambda_j \ge 0, \quad \forall j\in F,\\
& 0 < \theta_i \le 1, \quad i=1,2,...,m,\\
& 0 < \varphi_r \le 1, \quad \forall r\in H,\\
& \varphi_r \ge 1, \quad \forall r\in \overline{H}.
\end{array} \tag{8}
$$

Theorem 2. *Problem (8) is feasible.*

Proof. We claim that the solution $\theta_i = 1$ $(i = 1,2,...,m)$, $\varphi_r = 1 (r \in H \cup \overline{H})$, and for all $j \in F, \lambda_j \ge 0$, such that $\sum_{j\in F} \lambda_j = 1$, is a feasible solution of (8). To this end, since DMU_R and DMU_K are members of $PPS_{(VRS)}$, for all i, we have $\sum_{j\in F} \lambda_j x_{ij} + \lambda_R x_{iR} + \lambda_K x_{iK} \le x_{iR}, \sum_{j\in F} \lambda_j x_{ij} + \lambda_R x_{iR} + \lambda_K x_{iK} \le x_{iK}$, and $\sum_{j\in F} \lambda_j + \lambda_R + \lambda_K = 1$. Since, for all $j \in F$, $\lambda_j \ge 0$ and $\sum_{j\in F} \lambda_j = 1$, $\lambda_R = 0, \lambda_K = 0$, and

$$
\sum_{j\in F} \lambda_j x_{ij} \le x_{iR} + x_{iK}, \quad i=1,2,...,m, \quad \sum_{j\in F} \lambda_j = 1, \quad \lambda_j \ge 0, \quad j \in F. \tag{9}
$$

Also, since for each $r \in H$, $y_{rM} > max\{y_{rj} | j \in F\}$, $\sum_{j\in F} \lambda_j y_{rM} > \sum_{j\in F} \lambda_j y_{rj}$, and $\sum_{j\in F} \lambda_j = 1$, we have

$$
\sum_{j\in F} \lambda_j y_{rj} < y_{rM} = y_{rR} + y_{rK}, \quad \forall r \in H. \tag{10}
$$

Moreover, according to $y_{rM} \le max\{y_{rj} | j \in F\} = y_{rT} (T \in F)$ for all $r \in \overline{H}$, based on definition of

$PPS_{(VRS)}$, we have

$$\begin{aligned} y_{rR}+y_{rK}=y_{rM}\leq y_{rT}\leq \sum_{j\in F}\lambda_j y_{rj},\\ S.to:\sum_{j\in F}\lambda_j=1,\quad \lambda_j\geq 0,\quad j\in F. \end{aligned} \tag{11}$$

From relations (9), (10) and (11), one can conclude that for all i, r and $j\in F$, $\theta_i=1,\varphi_r=1$ and $\lambda_j\geq 0$ such that $\sum_{j\in F}\lambda_j=1$, are satisfied in set of constraints in (8). □

Theorem 3. *Suppose $(\lambda_j^*,\theta_i^*,\varphi_r^*)$ is the optimal solution of (8). If $\theta_i^*=\varphi_r^*=1$ for $i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$, then DMU_M is located on the efficient frontier and it is strongly efficient.*

Proof. Let $(\lambda_j^*,\theta_i^*,\varphi_r^*)$ for $j\in F, i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$, be the optimal solution of (8). Due to the properties of optimal solution in linear programming, this solution makes the set of first, second, and third inequality constraints of (8) active. This claim can be proved by contradiction as follows. Suppose in the set of the first constraints of (8), the constraint related to t-th input is strict inequality, i.e., $\sum_{j\in F}\lambda_j^* x_{tj}<\theta_t^*(x_{tR}+x_{tK})$. Therefore, there is $\gamma_t>0$ such that $\sum_{j\in F}\lambda_j^* x_{tj}+\gamma_t(x_{tR}+x_{tK})=\theta_t^*(x_{tR}+x_{tK})$, as a result $\sum_{j\in F}\lambda_j^* x_{tj}=(x_{tR}+x_{tK})(\theta_t^*-\gamma_t)$. Therefore, $(\lambda_j^*,\theta_i^*,(\theta_t^*-\gamma_t),\varphi_r^*)$ for $j\in F, i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$ is a feasible solution of (8), and the objective function value of (8) for this feasible solution is equal to $\sum_{i=1,i\neq t}^m\theta_i^*+(\theta_t^*-\gamma_t)+\sum_{r\in H}\varphi_r^*-\sum_{r\in\overline{H}}\varphi_r^*<\sum_{i=1,i\neq t}^m\theta_i^*+\theta_t^*+\sum_{r\in H}\varphi_r^*-\sum_{r\in\overline{H}}\varphi_r^*$, which is in contradiction by the optimality of $(\lambda_j^*,\theta_i^*,\varphi_r^*) for j\in F,\ i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$. So, $\sum_{j\in F}\lambda_j^* x_{ij}=\theta_i^*(x_{iR}+x_{iK})$ and $\sum_{j\in F}\lambda_j^* y_{rj}=\varphi_r^*(y_{rR}+y_{rK})$ for $i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$. In other words, the virtual DMU with inputs $\theta_i^*(x_{iR}+x_{iK})$ for $i=1,2,\ldots,m$ and outputs $\varphi_r^*(y_{rR}+y_{rK})$ for $r\in(H\cup\overline{H})$ is located on the efficient frontier and hence it is strongly efficient. Now, consider $\theta_i^*=\varphi_r^*=1$ for $i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$. Hence DMU_M that is located on the efficient frontier is strongly efficient. In other words, from the major merger of two units R and K a strongly efficient unit is obtained. □

Now, suppose θ_i^* for $i=1,2,\ldots,m$ and φ_r^* for $r\in\overline{H}$ are the optimal solution of (8). Problem (7) is always feasible, if $\overline{\theta}_i$ and $\overline{\varphi}_r$ such that $\theta_i^*\leq\overline{\theta}_i\leq 1(i=1,2,..,m)$ and $\varphi_r^*\geq\overline{\varphi}_r\geq 1(r\in\overline{H})$ are determined in (7). In this setting, decision makers can select predetermined efficiency levels within the specified ranges. Moreover, if the upper bound value of 1 is chosen for all $\overline{\theta}_i$ and $\overline{\varphi}_r$, the resulting new unit will be strongly efficient.

Theorem 4. *Problem (7) is feasible.*

Proof. Suppose $(\lambda_j^*,\theta_i^*,\varphi_r^*)$ for $j\in F, i=1,2,\ldots,m$ and $r\in(H\cup\overline{H})$ are the optimal solution of (8). Since $\theta_i^*\leq\overline{\theta}_i\leq 1(i=1,2,..,m)$, $\varphi_r^*\leq 1(r\in H)$ and $\varphi_r^*\geq\overline{\varphi}_r\geq 1(r\in\overline{H})$, we have

$$\begin{aligned} &\textstyle\sum_{j\in F}\lambda_j^* x_{ij}\leq\theta_i^*(x_{iR}+x_{iK})\leq\overline{\theta}_i(x_{iR}+x_{iK}),\quad i=1,2,\ldots,m,\\ &\textstyle\sum_{j\in F}\lambda_j^* y_{rj}<\varphi_r^*(y_{rR}+y_{rK})\leq(y_{rR}+y_{rK}),\quad \forall r\in H,\\ &\textstyle\sum_{j\in F}\lambda_j^* y_{rj}\geq\varphi_r^*(y_{rR}+y_{rK})\geq\overline{\varphi}_r(y_{rR}+y_{rK}),\quad \forall r\in\overline{H},\\ &\textstyle\sum_{j\in F}\lambda_j^*=1,\quad \lambda_j^*\geq 0,\quad j\in F. \end{aligned} \tag{12}$$

(12) shows that for all $i=1,2,\ldots,m, r=1,2,\ldots,s$, $\left(\alpha_{iR}=x_{iR},\alpha_{iK}=x_{iK},\beta_{rR}=y_{rR},\beta_{rK}=y_{rK},\lambda_j^*\right)$ is a feasible solution of (7). □

Corollary 1. *In model (8), the first, second, and third sets of constraints are active in the optimal solution, therefore we have*

$$\begin{array}{ll} \sum_{j\in F}\lambda_j^* x_{ij} = \bar{\theta}_i\,(\alpha_{iR}^*+\alpha_{iK}^*), & i=1,2,...,m,\\ \sum_{j\in F}\lambda_j^* y_{rj} = (\beta_{rR}^*+\beta_{rK}^*), & \forall r\in H,\\ \sum_{j\in F}\lambda_j^* y_{rj} = \overline{\varphi}_r\,(\beta_{rR}^*+\beta_{rK}^*), & \forall r\in \overline{H}. \end{array} \tag{13}$$

(13) shows that DMU_G with inputs $\overline{\theta}_i\,(\alpha_{iR}^*+\alpha_{iK}^*)\,(i=1,2,\ldots,m)$ and outputs $(\beta_{rR}^*+\beta_{rK}^*)\,(r\in H)$, $\overline{\varphi_r}\,(\beta_{rR}^*+\beta_{rK}^*)\,(r\in \overline{H})$ is strongly efficient.

Theorem 5. *Suppose $n(H)$ and $n(\overline{H})$ are the members number of sets H and $\overline{H}$, respectively. Also $(\lambda^*,\alpha_O^*,\beta_O^*)$ is the optimal solution of (7). Then ϑ, which is defined as follows, is the efficiency score of DMU_O:*

$$\vartheta = \frac{n(\overline{H})\left(\sum_{i=1}^m \frac{\sum_{j\in F}\lambda_j^* x_{ij}}{\alpha_{iO}^*} + \sum_{r\in H}\frac{\sum_{j\in F}\lambda_j^* y_{rj}}{\beta_{rO}^*}\right)}{(m+n(H))\sum_{r\in\overline{H}}\frac{\sum_{j\in F}\lambda_j^* y_{rj}}{\beta_{rO}^*}}. \tag{14}$$

Proof. According to the Corollary 1 we have

$$\frac{\sum_{j\in F}\lambda_j^* x_{ij}}{\alpha_{iO}^*} = \bar{\theta}_i\ (i=1,2,...,m),\quad \frac{\sum_{j\in F}\lambda_j^* y_{rj}}{\beta_{rO}^*} = 1\ (r\in H),\quad \frac{\sum_{j\in F}\lambda_j^* y_{rj}}{\beta_{rO}^*} = \bar{\varphi}_r\ (r\in\overline{H}). \tag{15}$$

Since, $0<\overline{\theta}_i\le 1(i=1,2,..,m)$ and $\overline{\varphi}_r\ge 1(r\in\overline{H})$, we have

$$0<\sum_{i=1}^m\frac{\sum_{j\in F}\lambda_j^* x_{ij}}{\alpha_{iO}^*}+\sum_{r\in H}\frac{\sum_{j\in F}\lambda_j^* y_{rj}}{\beta_{rO}^*} = \sum_{i=1}^m\bar{\theta}_i+\sum_{r\in H}1\le(m+n(H)),\ \sum_{r\in\overline{H}}\frac{\sum_{j\in F}\lambda_j^* y_{rj}}{\beta_{rO}^*} = \sum_{r\in\overline{H}}\bar{\varphi}_r\ge n(\overline{H}). \tag{16}$$

Based on (16), it is clear that for ϑ in (14) we have $0<\vartheta\le 1$. Now, by the definition of ϑ, if $\alpha_{tO}^*,(t\in\{1,2,...,m\})$ and $\beta_{qO}^*,(q\in H)$ are increased, the value of ϑ is decreased. Also, by decreasing $\beta_{qO}^*,(q\in H)$ the value of ϑ, would be increased. Therefore, ϑ is the efficiency score of DMU_O, and regarding to (15), one can define ϑ as $\vartheta = \frac{\frac{1}{m+n(H)}(\sum_{i=1}^m\bar{\theta}_i+n(H))}{\frac{1}{n(\overline{H})}\sum_{r\in\overline{H}}\overline{\varphi_r}}$. □

Corollary 2. *Suppose in (7) $\vartheta=1$, then DMU_O is strongly efficient.*

Managers always want the units under their management have MPSS, indeed, in both types of the PPS, (i.e. $PPS_{(CRS)}$ and $PPS_{(VRS)}$), they want that their units be efficient [12]. In the proposed model for merging, this fact can also be considered for this purpose, as shown in the following theorem.

Theorem 6. *Suppose that in model (7), $\overline{\theta}_i=\overline{\varphi}_r=1$ for all i and $r\in\overline{H}$, then DMU_O is an MPSS unit.*

Proof. To determine that DMU_O is an MPSS unit, we apply model (1) for DMU_O and it is as follows:

$$\begin{array}{ll} Max & \frac{\psi}{\xi} \\ S.to: & \sum_{j\in F\cup O}\lambda_j X_j - \xi X_O + S^- = 0, \\ & \sum_{j\in F\cup O}\lambda_j Y_j - \psi Y_O - S^+ = 0, \\ & \sum_{j\in F\cup O}\lambda_j = 1, \quad {}^\circ\lambda_j \geq 0, \quad j\in F\cup O, \\ & \psi \geq 1, \quad 0 < \xi \leq 1, \\ & \lambda_j \geq 0, \quad S^-, S^+ \geq 0, \\ & X_j = (x_{ij}, y_{rj}), \quad i = 1,2,\ldots,m, \quad r\in H, \quad j\in F\cup O, \\ & Y_j = (y_{rj}), \quad r\in \overline{H}, \\ & X_O = (\alpha^*_{iO}, \beta^*_{rO}), \quad i = 1,2,\ldots,m, \quad r\in H, \\ & Y_O = (\beta^*_{rO}), \quad r\in \overline{H}. \end{array} \tag{17}$$

Since $\overline{\theta}_i = \overline{\varphi}_r = 1$ for all i and $r \in \overline{H}$, therefore, based on Corollary 1, we have

$$\sum_{j\in F}\lambda^*_j x_{ij} = (\alpha^*_{iR} + \alpha^*_{iK}), \quad i = 1,2,\ldots,m, \quad \sum_{j\in F}\lambda^*_j y_{rj} = (\beta^*_{rR} + \beta^*_{rK}), \quad r \in (H\cup\overline{H}). \tag{18}$$

Comparing the first and second sets of constraints in model (17) with (18), give that $X_O = (\alpha^*_{iR} + \alpha^*_{iK}, \beta^*_{rR} + \beta^*_{rK})(i = 1,2,\ldots,m, r\in H)$, $Y_O = (\beta^*_{rO})(r\in\overline{H})$ and $(\lambda_j, \xi, \psi, S^-, S^+) = (\lambda^*_j, 1, 1, 0, 0)$ with $\lambda^*_O = 0$ and $\sum_{j\in F\cup O}\lambda^*_j = 1$ is a feasible solution of (17). Now suppose that this solution is not optimal for (17). Then, by contradiction, let the feasible solution $(\lambda^{**}_j, \xi^{**}, \psi^{**}, S^{-**}, S^{+**})$ that $\xi^{**} < 1, \psi^{**} = 1, S^{-**} = S^{+**} = 0$ and $\lambda^{**}_j \geq 0, \sum_{j\in F\cup O}\lambda^{**}_j = 1$, is the optimal solution of (17). Therefore, $\sum_{j\in F\cup O}\lambda^{**}_j X_j - X_O < 0$ Now, since $\sum_{j\in F\cup O}\lambda^*_j X_j = X_O$, $\sum_{j\in F\cup O}\lambda^{**}_j X_j < \sum_{j\in F\cup O}\lambda^*_j X_j$ and consequently $\sum_{j\in F\cup O}\lambda^{**}_j < \sum_{j\in F\cup O}\lambda^*_j = 1$ which this is in contradiction with the feasibility of $\lambda^{**}_j (j\in F\cup O)$. Therefore $\xi^{**} = 1$. In the same way, it is proved that $\psi^{**} = 1$ and $S^{-**} = S^{+**} = 0$. Thus, the optimal value of (17) is one and based on Definition 5, DMU_O is an MPSS unit. □

Theorem 7. *In (7), if DMU_O is strongly efficient, then the obtained profit from DMU_O is more than the obtained profit from each unit R and K.*

Proof. Since unit $K \in PPS_{(VRS)}$ is inefficient, and DMU_O is strongly efficient, according to Corollary 1, we have

$$(\alpha^*_{iR} + \alpha^*_{iK}) = \sum_{j\in F}\lambda^*_j x_{ij} \leq x_{iK},\ i = 1,2,\ldots,m, \quad (\beta^*_{rR} + \beta^*_{rK}) = \sum_{j\in F}\lambda^*_j y_{rj} \geq y_{rK},\ r\in(H\cup\overline{H}) \tag{19}$$

Since unit $K \in PPS_{(VRS)}$ is inefficient, in (19), at least one of the constraints holds strictly. Now, suppose the prices of i-th input and r-th output of units are c_i and p_r. Therefore from (19) we have $\sum_{i=1}^m c_i(\alpha^*_{iR} + \alpha^*_{iK}) < \sum_{i=1}^m c_i x_{iK} and \sum_{r=1}^s p_r(\beta^*_{rR} + \beta^*_{rK}) > \sum_{r=1}^s p_r y_{rK}$. Thus, we can conclude that

$$\sum_{r=1}^s p_r(\beta^*_{rR} + \beta^*_{rK}) - \sum_{i=1}^m c_i(\alpha^*_{iR} + \alpha^*_{iK}) > \sum_{r=1}^s p_r y_{rK} - \sum_{i=1}^m c_i x_{iK}. \tag{20}$$

(20) shows that the obtained profit from unit O is more than the obtained profit from unit K. In the same way, it is proved that the obtained profit from unit O is more than the obtained profit from unit R. □

Now, we can list the obtained results from the above-discussed topics about the proposed merger model, as follows:

1. In the optimal state of model (7), the surplus amount is equal to zero, so there is no congestion or surplus amount in the obtained unit inputs from the merger.

2. The virtual DMU_G with inputs $\bar{\theta}_i\,(\alpha^*_{iR}+\alpha^*_{iK})$ (for each i) and outputs $(\beta^*_{rR}+\beta^*_{rK})\,(r\in H)$, $\bar{\varphi}_r\,(\beta^*_{rR}+\beta^*_{rK})$ $(r\in\overline{H})$ is a strongly efficient unit. Therefore, if $\overline{\theta}_i=\overline{\varphi}_r=1$, then for all and $r\in(H\cup\overline{H})$, DMU_O with inputs $(\alpha^*_{iR}+\alpha^*_{iK})$ and outputs $(\beta^*_{rR}+\beta^*_{rK})$ is a strongly efficient unit and it is an MPSS unit as well.

3. In model (7), for merging units, managers have this opportunity to have a role in setting the value of each of the inputs and outputs which are members of $\overline{H}$. This could be done by choosing predetermined ratios.

4. From an economic point of view, the obtained DMU_O in comparison to other aggregations of units K and R, has the most profit.

5. Model (7) is input- output-oriented, while, the previously presented models are input-oriented or output-oriented not both.

We can extend the merging model (7) to merge more than two units, for instance, for three units P, R and K, it is enough to solve the following model:

$$\begin{array}{ll}
Min & \left(\sum_{i=1}^{m}\overline{\theta}_i\frac{\alpha_{iP}+\alpha_{iR}+\alpha_{iK}}{x_{iM}}+\sum_{r\in H}\frac{\beta_{rP}+\beta_{rR}+\beta_{rK}}{y_{rM}}-\sum_{r\in\overline{H}}\overline{\varphi}_r\frac{\beta_{rP}+\beta_{rR}+\beta_{rK}}{y_{rM}}\right)\\
S.to: & \sum_{j\in F}\lambda_j x_{ij}\le\overline{\theta}_i\,(\alpha_{iP}+\alpha_{iR}+\alpha_{iK}),\quad i=1,2,\ldots,m,\\
& \sum_{j\in F}\lambda_j y_{rj}\le(\beta_{rP}+\beta_{rR}+\beta_{rK}),\quad \forall r\in H,\\
& \sum_{j\in F}\lambda_j y_{rj}\ge\overline{\varphi}_r\,(\beta_{rP}+\beta_{rR}+\beta_{rK}),\quad \forall r\in\overline{H},\\
& \sum_{j\in F}\lambda_j=1,\\
& \lambda_j\ge 0,\quad \forall j\in F,\\
& \alpha_{iP}\le x_{iP},\quad \alpha_{iR}\le x_{iR},\quad \alpha_{iK}\le x_{iK},\quad i=1,2,...,m,\\
& \beta_{rP}\le y_{rP},\quad \beta_{rR}\le y_{rR},\quad \beta_{rK}\le y_{rK},\quad \forall r\in H,\\
& \beta_{rP}\ge y_{rP},\quad \beta_{rR}\ge y_{rR},\quad \beta_{rK}\ge y_{rK},\quad \forall r\in\overline{H}.
\end{array}\tag{21}$$

3.2 New acquisition merger model

An acquisition merger is referred to as a business transaction in which one unit buys or gains all or part of other units [11]. Suppose unit P for increasing its power and efficiency score wants to gain all or part of units R and K. Therefore for determining the necessary amount inputs and outputs values of units R and K, we convert model (21) to the following model:

$$\begin{array}{ll}
Min & \left(\sum_{i=1}^{m}\overline{\theta}_i\frac{\alpha_{iR}+\alpha_{iK}}{x_{iM}}+\sum_{r\in H}\frac{\beta_{rR}+\beta_{rK}}{y_{rM}}-\sum_{r\in\overline{H}}\overline{\varphi_r}\frac{\beta_{rR}+\beta_{rK}}{y_{rM}}\right)\\
S.to: & \sum_{j\in F}\lambda_j x_{ij}\le\overline{\theta}_i\,(x_{iP}+\alpha_{iR}+\alpha_{iK}),\quad i=1,2,...,m,\\
& \sum_{j\in F}\lambda_j y_{rj}\le(y_{rP}+\beta_{rR}+\beta_{rK}),\quad \forall r\in H,\\
& \sum_{j\in F}\lambda_j y_{rj}\ge\overline{\varphi_r}\,(y_{rP}+\beta_{rR}+\beta_{rK}),\quad \forall r\in\overline{H},\\
& \sum_{j\in F}\lambda_j=1,\\
& \lambda_j\ge 0,\quad \forall j\in F,\\
& \alpha_{iR}\le x_{iR},\quad \alpha_{iK}\le x_{iK},\quad i=1,2,...,m,\\
& \beta_{rR}\le y_{rR},\quad \beta_{rK}\le y_{rK},\quad \forall r\in H,\\
& \beta_{rR}\ge y_{rR},\quad \beta_{rK}\ge y_{rK},\quad \forall r\in\overline{H}.
\end{array}\tag{22}$$

Table 3: Data and efficiency scores of 5 banks (data are based on billion Rials)

DMU	Bank 1	Bank 2	Bank 3	Bank 4	Bank 5
Input 1(operating costs)	120	80	90	85	70
Input 2(employee salaries)	50	40	45	42	38
Output (amount of loans granted)	500	250	200	230	240
Efficiency score	1	0.96	0.84	0.90	1

Table 4: Comparison of the merger of Banks 3 and 4 using the presented model (21) and models [4, 14]

Model	DMU	New bank	New bank profit
Models [4, 14]	Input 1(operating costs)	116	265
	Input 2(employee salaries)	49	
	Output (amount of loans granted)	430	
Model (21)	Input 1(operating costs)	120	280
	Input 2(employee salaries)	50	
	Output (amount of loans granted)	450	

All the presented theorems for the horizontal merger in the previous sub-section also hold for the acquisition merger model (22).

3.3 Comparison with the merger models [4, 14]

In this section, by presenting a simple example of five banks with two inputs and one output, we demonstrate the application of the proposed models and compare their performance with the merger models introduced in [4, 14]. According to Table 3, consider five banks with two inputs-operating costs and employee salaries-and one output, the amount of loans granted. As shown in Table 3, Banks 1 and 5 are efficient, while the others are inefficient. To create a new efficient bank with efficiency score 1, we merge the inefficient Banks 3 and 4 using the proposed model (21) and models [4, 14]. The results of these mergers are presented in Table 4.

As presented in Table 4, the horizontal merger of Banks 3 and 4 using the proposed model (21) (in this merger H is empty) produces an efficient new bank with higher profitability compared to the new bank generated based on the models proposed by [4, 14]. The new bank obtained through model (21) is an MPSS bank. According to model (8), the upper and lower bounds for the predetermined efficiencies $\overline{\theta}_1$, $\overline{\theta}_2$, and $\overline{\varphi}_1$ are as $\overline{\theta}_1 \in [0.685, 1], \overline{\theta}_2 \in [0.574, 1]$ and $\overline{\varphi}_1 \in [1, 1.046]$. Assuming $\overline{\theta}_1 = 0.7$, $\overline{\theta}_2 = 0.8$, and $\overline{\varphi}_1 = 1$, we obtain a new unit from the horizontal merger of Banks 3 and 4 with the efficiency score 0.75 by using model (21) (see Table 5).

By selecting different values for the predetermined efficiencies within the specified range, new banks with desired efficiency levels using the proposed model (21) are obtained. In other words, managers can control all inputs and outputs during the merger process, whereas such capability is not provided to managers in models [4, 14].

Now, suppose that Bank 4 acquires a part of Bank 3 to form an efficient new bank. For this purpose,

Table 5: Horizontal merger of Banks 3 and 4 using the presented model (21) ($\overline{\theta}_1 = 0.7$, $\overline{\theta}_2 = 0.8$, and $\overline{\varphi}_1 = 1$)

Model	DMU	New bank	Efficiency score
	Input 1(operating costs)	171.42	
Model (21)	Input 2(employee salaries)	62.5	0.75
	Output (amount of loans granted)	450	

Table 6: Inputs and output of Bank 3 after acquisition by Bank 4

Model	DMU	Bank3	New bank	New bank profit
	Input 1(operating costs)	35	120	
Model (21)	Input 2(employee salaries)	8	50	280
	Output (amount of loans granted)	220	450	

using model (22), the input and output amounts from Bank 3 that are acquired by Bank 4 are obtained and shown in Table 6.

In general, the presented models offer several important advantages over previous models. First, they are linear and non-radial, allowing managers to incorporate their judgments in determining the value of each input and output of the merged units when forming a new unit. Second, the models make it possible to generate a strongly efficient unit and achieve MPSS. Third, in the proposed models, the efficient frontier remains unchanged, in other words, the efficiency scores of units that were efficient before the merger do not change. Finally, the presented models are both input-oriented and output-oriented, meaning they simultaneously determine the optimal inputs and optimal outputs for the merged units.

4 Application of the new merger models to the Iranian banks

In this section, we apply the presented models in the previous section for merging banks from the set of Iranian banks in 2022. Sepah Bank is the first bank that was established in 1925 with Iranian capitals, for the financial affairs of the military personnel [25]. Currently, the number of bank branches in Iran has increased a lot, at least twice the world standard. According to statistics, in many countries, there are between 10 and 15 branches for every 100,000 adults to provide optimal banking services, while in Iran, there are about 31 bank branches for every 100,000 adults [30]. This statistic also shows that the number of branches in Iran is much higher than level of high-income countries. But this multiplicity of bank branches in Iran is considered a threat to the country's economy, and the critics of this issue believe that funds allocated to the establishment of bank branches should be spent directly on increasing production and productivity of the country. Also, they believe that reducing the number of bank branches is one of the requirements for establishing the modern dynamic banking industry in Iran, so that the country's economy achieves the necessary growth and development. In this regard, special attention should be paid to the merger of banks. Also, the merger causes banks to force, in the competitive environment, to reduce costs and they present their services with better quality and cheaper to the customers. Our case study in this paper is 23 Iranian banks, in which their year 2022 data are collected from Iran banking institute [17]. Like [4, 14], the two considered inputs for the analysis in this study are

Table 7: Data and efficiency scores of the Iranian banks in 2022 (data are based on billion Rials)

DMU number	Bank name	Inputs (X1)	Inputs (X2)	Outputs (Y1)	Outputs (Y2)	Efficiency score
1	Ayandeh	503716	29810	252061	5786	0.63646
2	Gardeshgari	129029	9383	121179	22655	0.97131
3	EghtesadNovin	191292	19195	217119	18533	0.84769
4	Mehre	8490	18990	19470	26692	1.00000
5	IranVenezuela	204	232	2348	4	1.00000
6	Karafarin	51590.342	11156	71051	4867	0.84776
7	Keshavarzi	214919.112	84136	276994	22292	0.77705
8	Khavarmianeh	37902	4094	59610	2965	1.00000
9	Maskan	146502	67604	204944	9153	0.80040
10	Melal	74666	8393	60296	13240	0.60887
11	Mellat	446217	245631	798033	54946	1.00000
12	Parsian	226920	25020	234405	22964	0.76149
13	Pasargad	311805.555	31270	411117	87156	1.00000
14	Postbank	27110	12099	41263	10073	0.88234
15	Refah	219180.216	80643	281755	19979	0.78797
16	Saderat	420384.805	133358	464283	93103	1.00000
17	Saman	102146.712	20302	121039	23278	0.81377
18	Sarmaye	76995	5454	12412	867	0.17029
19	Shahr	171444	24655	156707	61414	1.00000
20	Sina	42448	15928	60414	5518	0.82659
21	Tejarat	352874.423	147788	522491	57491	0.89848
22	Tosee-Saderat	53303.789	7812	48384	2918	0.56562
23	Tosee-Taavon	31209	15365	39500	7952	0.71291

interest expenses (X1) and non-interest expenses (X2). Interest expenses include expenses for deposits and other borrowed funds, expenses of commissions and doubtful receivables, expenses of investment and transactions in foreign currency and income tax cost, while non-interest expenses include the costs of personnel salaries, administrative and general expenses, depreciation expenses, and other expenses. The two considered outputs for the analysis are interest income (Y1) and non-interest income (Y2). Interest income includes income from loans and deposits, income from investments and foreign exchange transactions. The non-interest income includes service charges on loans and transactions, commissions income, and other operating income. The list of the collected inputs and outputs of 23 Iranian banks are shown in Table 7. We calculate the technical efficiency scores of banks by using the BCC model under $PPS_{(VRS)}$ and the DEA-solver software are used to solve the model. The results are shown in Table 7.

According to the obtained efficiency scores, among 23 banks, 7 banks are efficient and are located on the efficiency frontier. Among the non-state banks, Banks 18, 10, and 1 have the lowest efficiency scores, and Banks 22 and 23 have the lowest efficiency scores among the state banks. In three non-state Banks 1, 10, and 18, and the state Bank 22, the difference between the sum of incomes and the sum of expenses is a negative value and hence every four banks are losing.

According to the previous discussions, now, we are going to merge the three non-state Banks 1, 10, and 18 together, and three inefficient non-state Banks 1, 15, and 21 together. Also, Bank 23, for increasing its efficiency score, is placed in the situation of the buyer or applicant of all or part of Bank 22 and is merged with it. We do these merger using model (21) and (22).

4.1 Horizontal merger of inefficient non-state banks (H is empty)

Consider the virtual Bank 24 which its inputs and outputs are obtained from the integration of the inputs and outputs of three Banks 1, 10, and 18 in Table 7. Inputs and outputs of Bank 24 are $X1 = 655377$, $X2 = 43657$, $Y1 = 324769$ and $Y2 = 324769$. By adding Bank 24 to the PPS of Iranian banks, its efficiency score is 0.563, which shows Bank 24 is an inefficient bank. Since $Max\{y_{1j}|j \in F\} = 798033$ and $Max\{y_{2j}|j \in F\} = 93103$, hence set H is empty and $\overline{H} = \{1,2\}$. Now, by using model (21) and assuming different levels of efficiency for each input and output, we obtain the new bank from merging three Banks 1, 10, and 18. At first, we calculate the upper and lower bounds of the predetermined ratios of $\overline{\theta}_i(i = 1,2,\ldots,m)$ and $\overline{\varphi}_r(r \in \overline{H})$ by using model (8), which is obtained as $\overline{\theta}_1 \in [0.4758,1], \overline{\theta}_2 \in [0.7163,1], \overline{\varphi}_1 \in [1,1.2659]$ and $\overline{\varphi}_2 \in [1,4.3812]$.

Table 8 gives the minimum amount of inputs and the maximum amount of outputs from each Bank 1, 10, and 18 which must be kept the predetermined ratio or level of efficiency for each input and output. α^*_{1j} and α^*_{2j} are the optimal amount of inputs 1 and 2, of the merged $DMU_j(j = 1,10,18)$, respectively. Also, β^*_{1j} and β^*_{2j} are the optimal amount of outputs 1 and 2 of the merged $DMU_j(j = 1,10,18)$, respectively. In the first row of Table 8, it is assumed that the efficiency of the new bank is equal to one, therefore the obtained new bank is strongly efficient and is an MPSS unit. In this case, the 100% reduction in both inputs and the 100% increase in both outputs of the merged units have been done, and the biggest profit is earned by the new bank. The profit of the new bank is more than the profit of each of the three merged banks. In this case, the inputs and outputs of the new bank are $(311810,31260)$ and $(411120,87160)$, respectively, which compared to Bank 24 its inputs are less and its outputs are more.

In the second row, 100% decrease in interest expense and 100% increase in non-interest income in the inputs and outputs of the merged unit are not considered. For this reason, $\overline{\theta}_1 = 1, \overline{\theta}_2 = 0.8, \overline{\varphi}_1 = 1.5$ and $\overline{\varphi}_2 = 1$ is considered. In the next rows, the optimal input and output values for all three merged units are determined by choosing different predetermined ratios.

4.2 Horizontal merger of non-state banks (H is not empty)

In merging, the input and output of virtual Bank 25, which is obtained by aggregating three Banks 1, 15 and 21, are equal to (1075771, 258241) and (1056307, 83256), respectively. Since $Max\{y_{1j}|j \in F\} = 798033$ and $Max\{y_{2j}|j \in F\} = 93103$, $H = 1$ and $\overline{H} = \{2\}$. If we add Bank 25 to the list of Iranian banks, given that the first output value of Bank 25 is greater than the maximum value of the first outputs of 23 banks, therefore, Bank 25 will be efficient [12]. This will cause firstly, the efficiency frontier to change and at least one of the banks (such as Bank 11) become inefficient, secondly, in practice, it may be impossible to achieve this output, i.e., interest income 1056307. Now, in order that, the efficient frontier does not change, we merge three banks by using model (21) as horizontal. The obtained results from this merger are shown in Table 9. In the merger of three banks, the upper and lower bounds of the predetermined ratios of $\overline{\theta}_i(i = 1,2)$ and $\overline{\varphi}_2(2 \in \overline{H})$ which are obtained from (8), are as $\overline{\theta}_1 \in$

Table 8: Optimal input and output of Banks 1, 10, and 18 after horizontal merger

Inputs and outputs predetermined ratios	Minimum amount of inputs	Maximum amount of outputs	New bank profit	New bank efficient score
$\overline{\theta}_1=1, \overline{\theta}_2=1$ $\overline{\varphi}_1=1, \overline{\varphi}_2=1$	$\alpha^*_{11}=160140, \alpha^*_{110}=74670$ $\alpha^*_{118}=77000, \alpha^*_{21}=17420$ $\alpha^*_{210}=8390, \alpha^*_{218}=5450$	$\beta^*_{11}=252060, \beta^*_{110}=60300$ $\beta^*_{118}=98760, \beta^*_{21}=5790$ $\beta^*_{210}=13240, \beta^*_{218}=68130$	155210	1
$\overline{\theta}_1=1, \overline{\theta}_2=0.8,$ $\overline{\varphi}_1=1.5, \overline{\varphi}_2=1,$	$\alpha^*_{11}=52310, \alpha^*_{110}=74670$ $\alpha^*_{118}=77000, \alpha^*_{21}=29810,$ $\alpha^*_{210}=8390, \alpha^*_{218}=5450,$	$\beta^*_{11}=252060, \beta^*_{110}=60300$ $\beta^*_{118}=12410, \beta^*_{21}=5790$ $\beta^*_{210}=13240, \beta^*_{218}=27940,$	124110	0.720
$\overline{\theta}_1=1, \overline{\theta}_2=1,$ $\overline{\varphi}_1=1.3, \overline{\varphi}_2=1.2,$	$\alpha^*_{11}=163990, \alpha^*_{110}=74670,$ $\alpha^*_{118}=77000, \alpha^*_{21}=23560,$ $\alpha^*_{210}=8390, \alpha^*_{218}=5450,$	$\beta^*_{11}=252060, \beta^*_{110}=60300$ $\beta^*_{118}=12410, \beta^*_{21}=5790,$ $\beta^*_{210}=13240, \beta^*_{218}=52840,$	43580	0.800
$\overline{\theta}_1=0.7, \overline{\theta}_2=1,$ $\overline{\varphi}_1=1, \overline{\varphi}_2=1.25,$	$\alpha^*_{11}=293780, \alpha^*_{110}=74670,$ $\alpha^*_{118}=77000, \alpha^*_{21}=17420,$ $\alpha^*_{210}=8390, \alpha^*_{218}=5450,$	$\beta^*_{11}=252060, \beta^*_{110}=60300,$ $\beta^*_{118}=98760, \beta^*_{21}=5790,$ $\beta^*_{210}=13240, \beta^*_{218}=50700,$	4140	0.75

Table 9: Optimal input and output of Banks 1, 15 and 21 after horizontal merger

Inputs and outputs predetermined ratios	Minimum amount of inputs	Maximum amount of outputs	New bank profit	New bank efficient score
$\overline{\theta}_1=1, \overline{\theta}_2=1$ $\overline{\varphi}_2=1$	$\alpha^*_{11}=0, \alpha^*_{115}=0,$ $\alpha^*_{121}=290540, \alpha^*_{21}=0,$ $\alpha^*_{215}=0, \alpha^*_{221}=30270$	$\beta^*_{11}=0, \beta^*_{115}=0,$ $\beta^*_{121}=372570, \beta^*_{21}=5790,$ $\beta^*_{215}=19980, \beta^*_{221}=57490$	135020	1
$\overline{\theta}_1=0.85, \overline{\theta}_2=0.90$ $\overline{\varphi}_2=1.01$	$\alpha^*_{11}=0, \alpha^*_{115}=0,$ $\alpha^*_{121}=347150, \alpha^*_{21}=0,$ $\alpha^*_{215}=0, \alpha^*_{221}=33870$	$\beta^*_{11}=0, \beta^*_{115}=0,$ $\beta^*_{121}=380800, \beta^*_{21}=5790,$ $\beta^*_{215}=19980, \beta^*_{221}=57490$	83040	0.907
$\overline{\theta}_1=0.80, \overline{\theta}_2=0.60$ $\overline{\varphi}_2=1.03$	$\alpha^*_{11}=0, \alpha^*_{115}=27320,$ $\alpha^*_{121}=352870, \alpha^*_{21}=0,$ $\alpha^*_{215}=0, \alpha^*_{221}=51520$	$\beta^*_{11}=0, \beta^*_{115}=0,$ $\beta^*_{121}=397260, \beta^*_{21}=5790,$ $\beta^*_{215}=19980, \beta^*_{221}=57490$	48810	0.776

$[0.2898, 1], \overline{\theta}_2 \in [0.1211, 1]$ and $\overline{\varphi}_2 \in [1, 1.0468]$.

In the first row of Table 9, the obtained bank from the merger of three banks, is the strongly efficient and MPSS unit. This new bank with inputs (290540, 30270) and outputs (372570, 83260) has the most profit which is more than any of Banks 1, 15 and 21. This new bank, while its first input is less than the first input of the inefficient Bank 25, is strongly efficient. In the next rows, for different values of $\overline{\theta}_i (i=1,2)$ and $\overline{\varphi}_2 (2 \in \overline{H})$, other new banks are obtained. In general, the virtual DMUs with inputs and outputs $(\overline{\theta}_1(\alpha^*_{11}+\alpha^*_{115}+\alpha^*_{121}), \overline{\theta}_2(\alpha^*_{21}+\alpha^*_{215}+\alpha^*_{221}))$ and $((\beta^*_{11}+\beta^*_{115}+\beta^*_{121}), \overline{\theta}_2(\beta^*_{21}+\beta^*_{215}+\beta^*_{221}))$ are located on the efficient frontier.

Table 10: Optimal input and output of Banks 22 after acquisition merger

Inputs and outputs predetermined ratios	Minimum amount of inputs	Maximum amount of outputs	New bank profit	New bank efficient score
$\overline{\theta}_1=1,\overline{\theta}_2$=1 $\overline{\varphi}_1=1,\overline{\varphi}_2=1$	$\alpha^*_{122}=53304,\alpha^*_{222}=6703$	$\beta^*_{122}=78132,\beta^*_{222}=33895$	52898	1
$\overline{\theta}_1=1,\overline{\theta}_2$=0.98 $\overline{\varphi}_1=1.3,\overline{\varphi}_2=1.2$	$\alpha^*_{122}=53304,\alpha^*_{222}=7153$	$\beta^*_{122}=50986,\beta^*_{222}=26920$	18327	0.792
$\overline{\theta}_1=1,\overline{\theta}_2$=1 $\overline{\varphi}_1=1.5,\overline{\varphi}_2=1$	$\alpha^*_{122}=53304,\alpha^*_{222}=7812$	$\beta^*_{122}=48384,\beta^*_{222}=9601$	-2253	0.80

4.3 Acquisition merger of Bank 22 by Bank 23 (H is empty)

In the sequence, suppose Bank 23 is the buyer or applicant of all or part of Bank 22. If Bank 23 gains 100 percent of Bank 22, i.e., we aggregate the inputs and outputs of the two banks together, Bank 26 is obtained with inputs (84513, 23177) and outputs (87884, 10870), which by adding it to the PPS of Iranian banks, its efficiency score becomes 0.648 and is inefficient. Now using model (22), we merge two Banks 22 and 23, in which 100 percent of the inputs and outputs of Bank 23 are used. The results are shown in Table 10. In the merger these two banks, set H is empty and $\overline{H}=\{1,2\}$, also, the upper and lower bounds of the predetermined ratios of $\overline{\theta}_i(i=1,2)$ and $\overline{\varphi}_r(r\in\overline{H})$ which are obtained by using model (8), are as $\overline{\theta}_1\in[1,1],\overline{\theta}_2\in[0.9521,1],\overline{\varphi}_1\in[1,1.3385]$ and $\overline{\varphi}_2\in[1,3.8497]$.

In the first row of Table 10, the obtained bank from the merger of two Banks 22 and 23 is strongly efficient and MPSS unit. This new bank with inputs (84513, 22068) and outputs (117632, 41847) has the most profit which is more than each one of Banks 22 and 23. This new bank compared to Bank 26 has less input and more output. In the next rows of Table 10 with different values for $\overline{\theta}_i(i=1,2)$ and $\overline{\varphi}_r(r\in\overline{H})$, the new banks are obtained by merging Banks 22 and 23. Note that in the last row of the Table 10 with values $\overline{\theta}_1=1,\overline{\theta}_2=1,\overline{\varphi}_1=1.5$ and $\overline{\varphi}_2=1$, profit of the new bank is negative, therefore, the merger is not economical and the value of $\overline{\varphi}_1$ should be reduced until the share of interest income of Bank 22 in the merger is increased and the profit of the new bank becomes positive. Indeed, managers with choosing different choices of the predetermined ratio value, can manage each of the indicators alone and predict the profit amount of the obtained unit from the merger, this is one of the important advantages of our presented method.

Note: In this example, since the data scales were identical and exhibited no differences, we used the original data directly in merging units. However, if the data scales of the units differ substantially, we apply normalization methods to transform the data to values between zero and one, and in the presented models, we use the normalized data.

5 Conclusion

One of the proposals that managers do to rescue inefficient units in the production possibility set is the major merger of two or more inefficient units together and convert them into one new unit. However, in this merger, the efficiency score of the new unit is not always higher than the efficiency score of the merged inefficient units. Moreover, sometimes a major merger changes the efficient frontier of the production possibility set and is far from the expected one in the reality. We believe that the efficiency frontier should not be changed in the merger process as it may cause that the units which are efficient before the merger become inefficient units now. Also, logically, the expected results from the merger should not be unreachable. In this paper, we present new models for merging units as horizontal and acquisition in which in addition to unchanging the efficiency frontier, managers have a role in determining the value of each of the indicators of the merged units to gain the suitable profits. The presented models have some important advantages compared to the previous models, including: the presented models are linear and non-radial (indeed, managers can apply their opinions in determining each of the inputs and outputs of the merged units to create a new unit), it is possible to produce a strongly efficient unit and MPSS using the presented models, in the presented methods the efficient frontier does not change (in other words, the efficiency score of the units that were efficient before merging does not change), the presented models are input-oriented and output-oriented (i.e., at the same time that the optimal inputs of the merged units are obtained, the optimal outputs are found as well). Moreover, we believe that these presented models can be used to merge units that have a network structure as well. Also, the presented models can be used to merge units with negative, interval, and fuzzy data, these points can be considered for the next studies.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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