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# On $q$-fractional differential problem with parameter and $q$-derivative boundary conditions

**Fatemeh Shahabi, Abdolali Neamaty***

[†] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
*Email(s): f.shahabi01@umail.umz.ac.ir, namaty@umz.ac.ir*

**Abstract.** In this paper, we study the existence of a positive solution for $q$-fractional boundary value problem by employing the fixed-point theorem. Our analysis relies on the Banach space and the fixed point theorem. Finally, we provide an example to verify our hypothesis and showcase our results.

*Keywords*: $q$-fractional differential equation, $q$-boundary value problem $q$-BVP, positive solution, $\lambda$-parameter.
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## 1 Introduction

We investigate the nonlinear fractional differential equation

$$-D_q^{\alpha} u(t) = f(\lambda, t, u(t)), \quad 0 < t \leq 1, \quad 3 < \alpha \leq 4, \;\; \lambda \geqslant 1 \tag{1}$$

with the boundary conditions (BCs)

$$D_q^{\alpha-3} u(0) = 0, \tag{2}$$

$$D_q u(1) = D_q^2 u(1) = 0, \tag{3}$$

where $\alpha$ is a real number, $f = O(1)$ for a sufficiently large parameter $\lambda$ and $f : [1,\infty) \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous. $D_q^{\alpha}$ represents the Riemann-Liouville fractional $q$-derivative ($q$-RLFD) of order $\alpha$.

In this paper, we discuss the existence of a positive solution for a fractional differential equation with BCs that involve $q$-derivatives. Our approach to prove the main theorem is based on utilizing the fixed point theorem (FPT) presented in [7]. Specifically, we modify the problem in [7] by introducing a new BCs and assuming that the function $f$ is continuous and bounded for large real parameter $\lambda$. The new

*Corresponding author
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conditions for the problem are formulated based on the $q$-derivative and we compute the Green's function associated with this problem. Moreover, the Green's function obtained under these conditions significantly influences the proof of the existence of a solution. Consequently, both the problem formulation and the proof method are novel. By utilizing the Green's function in Theorem 1 and FPT in Theorem 2, we investigate the existence of a solution to the $q$-fractional boundary value problem ($q$-FBVP), which introduces a new asymptotic condition in the $q$-fractional differential equation.

In the field of fractional differential calculus, numerous researchers have studied such equations in various applications (see [13] for examples). In particular, extensive research has been conducted to identify the existence of solutions for fractional boundary problems (FBPs). Relevant examples can be found in [5, 10, 12, 16]. For instance, in [2], the existence of positive solutions for the nonlinear singular Riemann-Liouville fractional differential equation with Riemann-Stieltjes boundary conditions was investigated. By computing Green's function, the authors provided the solution form of the problem and applied FPT theorems to establish the existence of multiple solutions.

Both linear and nonlinear Caputo-type fractional differential equations have been studied in [3]. In this paper, the problem is reformulated as an integral equation. In other words, the aim of this article is to introduce methods that provide membership of the solution of the problem in the $L^p[0,\infty]$ for positive integers $p$. The Leray-Schauder theorem is a powerful tool for solving various mathematical problems. By applying this theorem, the author was able to analyze her fractional problem in [11].

Etemad et al. in [6] investigated the existence of solutions for the following class of $q$-integro fractional difference equations with $q$-BVPs, which include BCs containing $q$-integrals of different orders:

$$(\lambda D_q^{\alpha}+(1-\lambda)D_q^{\beta})u(h)=cf(h,u(h))+dI_q^{\delta}g(h,u(h)),\quad h\in[0,1],\quad c,d\in\mathbb{R}^+.$$

The governing equation is considered along with the following $q$-integral boundary condition:

$$u(0)=0,\quad \eta\int_0^1\frac{(1-qs)^{(\zeta_1-1)}}{\Gamma_q(\zeta_1)}u(s)d_qs+(1-\eta)\int_0^1\frac{(1-qs)^{(\zeta_2-1)}}{\Gamma_q(\zeta_2)}u(s)d_qs=0,\quad \zeta_1,\zeta_2>0,$$

where $0<q<1$, $1<\alpha,\beta<2$, $0<\delta<1$, $0<\lambda\leqslant 1$, $0\leqslant\eta\leqslant 1$, $\alpha-\beta>1$, and $D_q^{\alpha}$ represents the $q$-RLFD of order $\alpha$ and $f,g:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous. While the definition of the $p$-derivative is provided in [14], this article aims to introduce the $q$-derivative of the $q$-FBVP. The definitions, theorems and lemmas utilized in this study are presented in the next section.

## 2 Preliminaries on fractional $q$-calculus

In this section, we present some definitions and lemmas that are used to prove our main results.

**Definition 1.** *( [12]) We say that $f(z)=O(g(z))$ as $z\to z_0$ if there exist positive constants $K$ and $\delta$ such that $|f|\leqq K|g|$ whenever $0<|z-z_0|<\delta$.*

**Definition 2.** *( [1]) If $\mu\in\mathbb{R}$ is fixed, a subset $A$ of $\mathbb{C}$ is called $\mu$-geometric if $\mu z\in A$ whenever $z\in A$.*

**Property 1.** *( [1]) If a subset $A$ of $\mathbb{C}$ is a $\mu$-geometric, then it contains all geometric sequences $\{z\mu^n\}_{n=0}^{\infty}$, $z\in A$.*

**Definition 3.** *( [1]) A function $f$ which is defined on a q-geometric set A with $0 \in A$ is called q-regular at zero if*

$$\lim_{n\to\infty} f(zq^n) = f(0), \;\; \forall\, z \in A.$$

**Definition 4.** *( [1]) A function $f$ is called q-absolutely continuous on $[0,a]$ if it is q-regular at zero, and there exists $k > 0$ such that*

$$\sum_{j=0}^{\infty} |f(tq^j) - f(tq^{j+1})| \leq k, \;\; \forall\, t \in (qa, a]. \tag{4}$$

If (4) holds then it can be extended throughout $(0,a]$. To see this, it is enough to examine the case $x \in (0,a]$. Actually, if $x \in (0,a]$, then there exists $t \in (qa,a]$ and $k \in \mathbb{N}$ such that $x = tq^k$. Then

$$\sum_{j=0}^{\infty} |f(xq^j) - f(xq^{j+1})| = \sum_{j=k}^{\infty} |f(tq^j) - f(tq^{j+1})| \leqslant \sum_{j=0}^{\infty} |f(tq^j) - f(tq^{j+1})| < \infty.$$

**Definition 5.** *( [1]) The space of all q-absolutely continuous functions defined on $[0,a]$ is $\mathscr{A}C_q[0,a]$, where $a > 0$.*

**Definition 6.** *( [7]) If f is a real continuous function on a q-geometric set A, the q-derivative of a function f is defined by*

$$\begin{aligned} (D_q f)(z) &= \frac{f(z) - f(qz)}{z - qz}, \;\; z \neq 0, \\ (D_q f)(0) &= \lim_{z\to 0} (D_q f)(z). \end{aligned}$$

Under similar conditions, higher order $q$-derivatives can be expressed as follows:

$$(D_q^0) f = f \;\; \text{and} \;\; D_q^n f = D_q(D_q^{n-1} f), \;\; n \in \mathbb{N}.$$

**Definition 7.** *( [7]) q-Integral is defined by*

$$I_q(f) = \int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \;\; 0 < |q| < 1, \;\; a \in \mathbb{R}, \tag{5}$$

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s, \;\; a \leq b \;\; a,b \in \mathbb{R}. \tag{6}$$

**Definition 8.** *( [15]) The left-sided Riemann-Liouville fractional integral (RLFI) is defined as*

$${}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \;\; t \geqslant 0,$$

*and the right-sided RLFI is written in the following form:*

$${}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau - t)^{\alpha-1} d\tau, \;\; t \leqslant b.$$

**Definition 9.** *( [7]) The function q-gamma is defined as follows:*

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}, \quad 0<q<1,$$

*that satisfies* $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

**Definition 10.** *( [7]) The q-RLFI is defined as follows:*

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{\alpha-1} f(t) d_q t, \quad \alpha \in \mathbb{R}^+, \quad x \in [0,1],$$

*and* $(I_q^0 f)(x) = f(x)$.

**Definition 11.** *( [1]) If* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$*, then the ceiling function is defined as* $\lceil x \rceil := min\{n \in \mathbb{N}_0 : x \leqslant n\}$.

**Definition 12.** *( [7]) The q-RLFD of a function* $f$ *of order* $\alpha$ *is defined by*

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \ \forall \alpha > 0.$$

**Remark 1.** *( [8]) The following formulas that will be used later:*

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \tag{7}$$

$$_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \tag{8}$$

$$({}_xD_q \int_0^x f(x,t) d_q t)(x) = \int_0^x {}_xD_q f(x,t) d_q t + f(qx,x). \tag{9}$$

**Note 1.** *( [7]) For* $q \in (0,1)$*, the notation* $[a]_q$ *is defined as*

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

**Definition 13.** *( [7]) For* $\alpha \in \mathbb{R}$*, we define*

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

**Definition 14.** *( [4]) If* $X$ *and* $Y$ *are Banach spaces and* $A \in \mathscr{B}(X,Y)$*, then* $A$ *is completely continuous if for any sequence* $\{x_n\}$ *in* $X$ *such that* $x_n \to x$ *weakly, it follows that* $||Ax_n - Ax|| \to 0$.

**Definition 15.** *( [1]) Let* $\eta$ *be a real number and let* $p$ *be a positive number.* $\mathscr{L}^p_{q,\eta}[0,a]$ *is the space of all functions f defined on* $(0,a]$ *satisfying*

$$||f||_{p,\eta} := \sup_{x\in(0,a]} (\int_0^x t^\eta |f(t)|^p d_q t)^{\frac{1}{p}} < \infty.$$

For simplicity, we shall use the symbols $L^p_q[0,a]$ , $\mathscr{L}^p_q[0,a]$ and $||f||_p$ to denote $L^p_{q,0}[0,a]$ , $\mathscr{L}^p_{q,0}[0,a]$ and $||f||_{p,0}$, respectively.

**Definition 16.** *( [1])* $\mathscr{A}C_q^{(n)}[0,a]$*,* $n \in \mathbb{N}$*, denotes the space of all functions f on* $[0,a]$ *such that* $f, D_q f, \ldots, D_q^{n-1} f$ *are q-regular at zero and* $D_q^{n-1} f \in \mathscr{A}C_q[0,a]$.

**Lemma 1.** *( [7]) Let* $\alpha, \beta \geq 0$ *and* $f$ *be a function defined on [0,1]. Then* $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$ *and* $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Therefore, we will conclude the following lemma.

**Lemma 2.** *( [7]) Let* $\alpha > 0$ *and* $p$ *be a positive integer. Then, the following equality holds:*

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0). \tag{10}$$

## 3 Main results

In this section, we study the existence of positive solution for the $q$-FBVP (1) - (3).

**Theorem 1.** *If $y \in C[0,1]$ and $3 < \alpha \leq 4$, then the problem $-D_q^{\alpha}u(t) = y(t)$ with boundary conditions $D_q^{\alpha-3}u(0) = 0$ and $D_q u(1) = D_q^2 u(1) = 0$ has a solution $u(t) = \int_0^1 G(t,qs)y(s)d_qs$, where the Green's function $G$ is*

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-3} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}\right)t^{\alpha-1} - (t-s)^{\alpha-1} \\ +\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-s)^{\alpha-3} + \frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}\right)t^{\alpha-2}, \quad 0 \leq s \leq t \leq 1 \\ \\ \left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-3} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}\right)t^{\alpha-1} \\ +\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-s)^{\alpha-3} + \frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}\right)t^{\alpha-2}, \quad 0 \leq t \leq s \leq 1 \end{cases} \tag{11}$$

*and $[\alpha-1]_q = \alpha_1(q), [\alpha-2]_q = \alpha_2(q)$, $[\alpha-3]_q = \alpha_3(q)$.*

*Proof.* By using Lemmas 1 and 2, the following relationship can be obtained:

$$-I_q^{\alpha}y(t) = u(t) - \sum_{j=1}^{4} D_q^{\alpha-j}u(0)\frac{t^{\alpha-j}}{\Gamma_q(\alpha-j+1)},$$

and we have

$$u(t) = \frac{D_q^{\alpha-1}u(0)}{\Gamma_q(\alpha)}t^{\alpha-1} + \frac{D_q^{\alpha-2}u(0)}{\Gamma_q(\alpha-1)}t^{\alpha-2} + \frac{D_q^{\alpha-3}u(0)}{\Gamma_q(\alpha-2)}t^{\alpha-3} + \frac{D_q^{\alpha-4}u(0)}{\Gamma_q(\alpha-3)}t^{\alpha-4} - \int_0^t \frac{(t-qs)^{\alpha-1}}{\Gamma_q(\alpha)}y(s)d_qs.$$

According to the initial condition $D_q^{\alpha-3}u(0) = 0$, since $3 < \alpha \leq 4$, we have $D_q^{\alpha-4}u(0) = 0$. Then

$$u(t) = A_1 t^{\alpha-1} + A_2 t^{\alpha-2} - \int_0^t \frac{(t-qs)^{\alpha-1}}{\Gamma_q(\alpha)}y(s)d_qs, \tag{12}$$

where for $i = 1,2$, we have $A_i = \frac{D_q^{\alpha-i}u(0)}{\Gamma_q(\alpha-i)}$. Now by first $q$-derivative of $u$ and by relation (8), we get

$$D_q u(t) = [\alpha-1]_q A_1 t^{\alpha-2} + [\alpha-2]_q A_2 t^{\alpha-3} - \int_0^t D_q \frac{(t-qs)^{\alpha-1}}{\Gamma_q(\alpha)}y(s)d_qs.$$

According to (9), we have

$$D_q u(t) = [\alpha-1]_q A_1 t^{\alpha-2} + [\alpha-2]_q A_2 t^{\alpha-3} - \int_0^t \frac{[\alpha-1]_q(t-qs)^{\alpha-2}}{\Gamma_q(\alpha)}y(s)d_qs. \tag{13}$$

For the second condition of BCs, we get the second $q$-derivative of $u$, so

$$D_q^2 u(t) = [\alpha-1]_q[\alpha-2]_q A_1 t^{\alpha-3} + [\alpha-2]_q[\alpha-3]_q A_2 t^{\alpha-4} - \int_0^t \frac{[\alpha-1]_q[\alpha-2]_q (t-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s. \tag{14}$$

We put the value of $t=1$ in (13) and (14). Therefore from (3), we have

$$\begin{aligned}&[\alpha-1]_q A_1 + [\alpha-2]_q A_2 - \textstyle\int_0^1 \frac{[\alpha-1]_q(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s \\ &\quad = [\alpha-1]_q[\alpha-2]_q A_1 + [\alpha-2]_q[\alpha-3]_q A_2 - \textstyle\int_0^1 \frac{[\alpha-1]_q[\alpha-2]_q(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s = 0.\end{aligned}$$

Let $C_1 = [\alpha-1]_q = \alpha_1(q)$, $C_2 = [\alpha-2]_q = \alpha_2(q)$, $C_3 = \alpha_1(q)\alpha_2(q)$ and $C_4 = \alpha_2(q)\alpha_3(q)$, then we have a system of equations of the form:

$$\begin{cases} C_1 A_1 + C_2 A_2 - \displaystyle\int_0^1 \frac{[\alpha-1]_q(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s = 0, \\ C_3 A_1 + C_4 A_2 - \displaystyle\int_0^1 \frac{[\alpha-1]_q[\alpha-2]_q(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s = 0. \end{cases}$$

For this system, multiplying the first term by $-\frac{C_4}{C_2}$ and then adding the two equations, the following equation is obtained:

$$\left(-\frac{C_4 C_1}{C_2} + C_3\right) A_1 + \frac{C_4}{C_2} \int_0^1 \frac{[\alpha-1]_q(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s - \int_0^1 \frac{[\alpha-1]_q[\alpha-2]_q(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s = 0.$$

Then, we find the values of $A_1$ and $A_2$ as follows:

$$\begin{aligned} A_1 &= \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)} \int_0^1 \frac{(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)} \int_0^1 \frac{(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s, \\ A_2 &= \frac{\alpha_1(q)}{-\alpha_2(q)-\alpha_3(q)} \int_0^1 \frac{(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s + \frac{\alpha_3(q)\alpha_1(q)}{\alpha_2(q)(\alpha_2(q)-\alpha_3(q))} \int_0^1 \frac{(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s \\ &\quad + \frac{\alpha_1(q)}{\alpha_2(q)} \int_0^1 \frac{(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s. \end{aligned}$$

Now by substituting $A_1$ and $A_2$ in the relation (12), we get

$$\begin{aligned} u(x) &= \left( \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)} \int_0^1 \frac{(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)} \int_0^1 \frac{(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s \right) t^{\alpha_1} \\ &\quad + \left( \frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)} \int_0^1 \frac{(1-qs)^{\alpha-3}}{\Gamma_q(\alpha)} y(s) d_q s + \frac{\alpha_3(q)\alpha_1(q)}{\alpha_2(q)(\alpha_2(q)-\alpha_3(q))} \int_0^1 \frac{(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s \right) t^{\alpha-2} \\ &\quad + \frac{\alpha_1(q)}{\alpha_2(q)} \int_0^1 \frac{(1-qs)^{\alpha-2}}{\Gamma_q(\alpha)} y(s) d_q s t^{\alpha-2} - \int_0^t \frac{(t-qs)^{\alpha-1}}{\Gamma_q(\alpha)} y(s) d_q s. \end{aligned}$$

By replacing integral $\int_0^1$ with $\int_0^t + \int_t^1$, we have

$$\begin{aligned}
u(x) = \frac{1}{\Gamma_q(\alpha)} \Bigg[ \int_0^t \Bigg( & \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}t^{\alpha-1} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}t^{\alpha-1} \\
& + \frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}t^{\alpha-2} + \frac{\alpha_3(q)\alpha_1(q)}{\alpha_2(q)(\alpha_2(q)-\alpha_3(q))}(1-qs)^{\alpha-2}t^{\alpha-2} \\
& + \frac{\alpha_1(q)}{\alpha_2(q)}(1-qs)^{\alpha-2}t^{\alpha-2} - (t-qs)^{\alpha-1} \Bigg) y(s) d_q s \\
& + \int_t^1 \Bigg( \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}t^{\alpha-1} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}t^{\alpha-1} \\
& + \frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}t^{\alpha-2} + \frac{\alpha_3(q)\alpha_1(q)}{\alpha_2(q)(\alpha_2(q)-\alpha_3(q))}(1-qs)^{\alpha-2}t^{\alpha-2} \\
& + \frac{\alpha_1(q)}{\alpha_2(q)}(1-qs)^{\alpha-2}t^{\alpha-2} \Bigg) y(s) d_q s \Bigg] = \int_0^1 G(t,qs)y(s)d_q s,
\end{aligned}$$

then the Green's function is obtained as

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-3}t^{\alpha-1} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1} \\ +\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-s)^{\alpha-3}t^{\alpha-2} + \frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-2}, & 0 \le s \le t \le 1 \\ \\ \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-3}t^{\alpha-1} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-1} \\ +\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-s)^{\alpha-3}t^{\alpha-2} + \frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases}$$

□

**Theorem 2.** *( [9]) Let $\mathscr{B}$ be a Banach space and $C \subset \mathscr{B}$ be a cone on $\mathscr{B}$. Also, let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $\mathscr{B}$ with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. If $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ is a completely continuous operator so that*

$$\|Ty\| \le \|y\|, \ \forall \ y \in C \cap \partial\Omega_1$$
$$\|Ty\| \ge \|y\|, \ \forall \ y \in C \cap \partial\Omega_2,$$

*then $T$ has at least one fixed point in $C \cap \overline{(\Omega_2} \setminus \Omega_1)$.*

**Lemma 3.** *The function $G$ defined in Theorem 1 satisfies the following conditions;*

$$G(t,qs) \ge 0, \quad G(t,qs) \le G(1,qs),$$

$$G(t,qs) \ge t^{\alpha-1}G(1,qs), \ \ 0 \le t,s \le 1.$$

*Proof.* From (11), we can define two functions $g_1$ and $g_2$ as follows:

$$\begin{aligned}
g_1(t,s) = & \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-3}t^{\alpha-1} - \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1} \\
& + \frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-s)^{\alpha-3}t^{\alpha-2} + \frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-2}, \qquad 0 \le s \le t \le 1
\end{aligned}$$

$$g_2(t,s)=\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-3}t^{\alpha-1}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-1} +\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-s)^{\alpha-3}t^{\alpha-2}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-s)^{\alpha-2}t^{\alpha-2}. \qquad 0\le t\le s\le 1$$

According to the definition of $g_1$, we have

$$\begin{aligned}
g_1(t,qs)&=\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)t^{\alpha-1}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)t^{\alpha-2}-(t-qs)^{\alpha-1}\\
&=\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}-(1-q\frac{s}{t})^{\alpha-1}\right)t^{\alpha-1}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)t^{\alpha-2}\\
&\geqslant\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}-(1-qs)^{\alpha-1}\right)t^{\alpha-1}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)t^{\alpha-2}.
\end{aligned}$$

We also have

$$(1-s)^{\alpha-1}<(1-s)^{\alpha-2}<(1-s)^{\alpha-3}, \tag{15}$$

therefore $g_1(t,qs)\geqslant 0$. Let $t=0$, then $g_2(0,qs)=0$. Similar to the previous relationships, this shows that $g_2(t,qs)\geqslant 0$. For $s\in[0,1]$ we show that ${}_tDg_1(t,qs)\geqslant 0$ (${}_tD=\frac{d}{dt}$ is the classical derivative with respect to the variable $t$). We have

$$\begin{aligned}
&{}_tDg_1(t,qs)\\
&=\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-1)t^{\alpha-2}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-2)t^{\alpha-3}-(\alpha-1)(t-qs)^{\alpha-2}\\
&=\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-1)t^{\alpha-2}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-2)t^{\alpha-3}-(\alpha-1)t^{\alpha-2}(1-\frac{qs}{t})^{\alpha-2}\\
&\geqslant\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-1)t^{\alpha-2}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-2)t^{\alpha-3}-(\alpha-1)t^{\alpha-2}(1-qs)^{\alpha-2}\\
&=\left(\frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}-\frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}-(1-qs)^{\alpha-2}\right)(\alpha-1)t^{\alpha-2}\\
&\quad+\left(\frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}+\frac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-2)t^{\alpha-3}.
\end{aligned}$$

According to (15), $g_1(t,qs)$ is an increasing function with respect to $t$. Also, we have

$$\begin{aligned}
{}_tDg_2(t,qs) &= \left(\tfrac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3} - \tfrac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-1)t^{\alpha-2} \\
&\quad + \left(\tfrac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3} + \tfrac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2}\right)(\alpha-2)t^{\alpha-3} \\
&\geqslant \left(\tfrac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)} - \tfrac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}\right)(1-qs)^{\alpha-2}(\alpha-1)t^{\alpha-2} \\
&\quad + \left(\tfrac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)} + \tfrac{\alpha_1(q)}{\alpha_2(q)-\alpha_3(q)}\right)(1-qs)^{\alpha-2}(\alpha-2)t^{\alpha-3}.
\end{aligned}$$

According to (15), we can get ${}_tDg_2(t,qs) \geqslant 0$. Then $g_2(t,qs)$ is increasing with respect to $t$, therefore $G(t,qs)$ is an increasing function with respect to $t$ for fixed $s \in [0,1]$. Suppose now that $qs \leq t$. Then

$$\begin{aligned}
\frac{G(t,qs)}{G(1,qs)} &= \frac{z_1t^{\alpha-1} - z_2t^{\alpha-1} + z_3t^{\alpha-2} + z_4t^{\alpha-2} + z_5t^{\alpha-2} - (t-qs)^{\alpha-1}}{z-(1-qs)^{\alpha-1}} \\
&\geqslant \frac{z_1t^{\alpha-1} - z_2t^{\alpha-1} + z_3t^{\alpha-1} + z_4t^{\alpha-1} + z_5t^{\alpha-1} - t^{\alpha-1}(1-q\frac{s}{t})^{\alpha-1}}{z-(1-qs)^{\alpha-1}} \\
&= t^{\alpha-1}\frac{z_1 - z_2 + z_3 + z_4 + z_5 - (1-\frac{q}{t}s)^{\alpha-1}}{z-(1-qs)^{\alpha-1}} \geqslant t^{\alpha-1}\tfrac{z-(1-qs)^{\alpha-1}}{z-(1-qs)^{\alpha-1}} = t^{\alpha-1},
\end{aligned}$$

where

$$\begin{aligned}
z_1 &= \frac{\alpha_2(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-3}, z_2 = \frac{\alpha_3(q)}{\alpha_2(q)-\alpha_3(q)}(1-qs)^{\alpha-2} \\
z_3 &= \frac{\alpha_1(q)}{\alpha_3(q)-\alpha_2(q)}(1-qs)^{\alpha-3}, z_4 = \frac{\alpha_3(q)\alpha_1(q)}{\alpha_2(q)(\alpha_2(q)-\alpha_3(q))}(1-qs)^{\alpha-2} \\
z_5 &= \frac{\alpha_1(q)}{\alpha_2(q)}(1-qs)^{\alpha-2}, z = z_1 - z_2 + z_3 + z_4 + z_5.
\end{aligned}$$

Also, for $t \leq qs$ we have

$$\begin{aligned}
\frac{G(t,qs)}{G(1,qs)} &= \frac{z_1t^{\alpha-1} - z_2t^{\alpha-1} + z_3t^{\alpha-2} + z_4t^{\alpha-2} + z_5t^{\alpha-2}}{z} \\
&\geqslant \frac{z_1t^{\alpha-1} - z_2t^{\alpha-1} + z_3t^{\alpha-1} + z_4t^{\alpha-1} + z_5t^{\alpha-1}}{z} \\
&= t^{\alpha-1}.
\end{aligned}$$

□

In view of Lemma 3, we can conclude the following lemma.

**Lemma 4.** *If* $0 < \tau < 1$, *then* $\min_{t\in[\tau,1]} G(t,qs) \geq \tau^{\alpha-1}G(1,qs)$, $\forall s \in [0,1]$.

We know $\mathscr{B} = C[0,1]$ is a Banach space by norm $||u|| = \sup_{t\in[0,1]} |u(t)|$. Suppose $\tau = q^n$ with $n \in \mathbb{N}$, we define $C \subset \mathscr{B}$ as follows;

$$C = \{u \in \mathscr{B} : u(t) \geq 0, \min_{t\in[\tau,1]} u(t) \geq \tau^{\alpha-1}||u||\}$$

and the operator $T : C \to \mathscr{B}$ is defined by

$$Tu(t) = \int_0^1 G(t,qs)f(\lambda,s,u(s))d_qs.$$

**Remark 2.** *Since G and f are positive and continuous functions, we conclude that operator $T : C \to \mathscr{B}$ is completely continuous. Furthermore for $u \in C$, $(Tu)(x) \geq 0$ on [0,1] and we have*

$$\begin{aligned}\min_{t\in[\tau,1]}(Tu)(t) &= \min_{t\in[\tau,1]} \textstyle\int_0^1 G(t,qs)f(\lambda,s,u(s))d_qs \\ &\geq \tau^{\alpha-1} \textstyle\int_0^1 G(1,qs)f(\lambda,s,u(s))d_qs \\ &= \tau^{\alpha-1}||Tu||,\end{aligned}$$

*where* $||Tu|| = \max_{t\in[0,1]} \int_0^1 G(t,qs)f(\lambda,s,u(s))d_qs.$

We will use the following two symbols:

$$M = \int_0^1 G(1,qs)d_qs, \quad N = \max_{t\in[0,1]} \int_\tau^1 G(t,qs)d_qs.$$

Now, we show the main result of the existence of a solution of the problem (1) - (3). Let $0 < r_1 < r_2$, we define the following two sets:

$$\Omega_1 = \{u \in C : ||u|| < r_1\}, \quad \Omega_2 = \{u \in C : ||u|| < r_2\}.$$

**Theorem 3.** *Assume $\tau = q^n$ with $n \in \mathbb{N}$ and $f(\lambda,s,u(s))$ is a continuous positive function. If there are two constants $r_2 > r_1 > 0$ so that*

$$\begin{aligned}M \max_{s\in[0,1],u\in C\cap\partial\Omega_1} f(\lambda,s,u(s)) \leq r_1, \\ N \min_{s\in[\tau,1],u\in C\cap\partial\Omega_2} f(\lambda,s,u(s)) \geq r_2,\end{aligned}$$

*then problem (1) - (3) has a solution u such that $u > 0$ for $t \in (0,1]$.*

*Proof.* Since the operator $T : C \to C$ is completely continuous, we only prove that equation $Tu = u$ has a solution $u$ that satisfies $u(t) \geq 0$ for $t \in (0,1]$. For $u \in C \cap \partial\Omega_2$, we have $\|u\| = r_2$. Considering Lemma 4, we have

$$\begin{aligned}||Tu|| &= \max_{0\leq t\leq 1} \int_0^1 G(t,qs)f(\lambda,s,u(s))d_qs \geq \max_{0\leq t\leq 1} \int_\tau^1 G(t,qs)d_qs \min_{s\in[\tau,1],u\in C\cap\partial\Omega_2} f(\lambda,s,u(s)) \\ &= N \min_{s\in[\tau,1],u\in C\cap\partial\Omega_2} f(\lambda,s,u(s)) \geq r_2,\end{aligned}$$

therefore $\|Tu\| \geq r_2 = \|u\|$. For $u \in C \cap \partial\Omega_1$, we have $\|u\| = r_1$ on [0,1]. By using Lemma 3 it follows that

$$\begin{aligned}||Tu|| &= \max_{0\leq t\leq 1} \int_0^1 G(t,qs)f(\lambda,s,u(s))d_qs \leqslant \max_{0\leq t\leq 1} \int_0^1 G(1,qs)f(\lambda,s,u(s))d_qs \\ &= M \max_{s\in[0,1],u\in C\cap\partial\Omega_1} f(\lambda,s,u(s)) \leqslant r_1,\end{aligned}$$

then we get $\|Tu\| \leqslant r_1 = \|u\|$. Theorem 2 states that the operator $T$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$. Therefore $u(t) > 0$ for $t \in (0,1]$ and $u$ is a solution of problem (1) - (3), so the proof is complete. □

**Example 1.** Let $\alpha = 3.5$ and $q = \tau = 0.1$. Define function $f(\lambda,t,u) = \frac{3u}{\lambda} + u^2$, where $u$ is a continuous function on $[0,1]$, therefore $f = O(1)$, for a positive real large parameter $\lambda$. We have

$$M = \int_0^1 G(1,qs)d_q s \approx \frac{1}{1.17} \times 0.07 = 0.05 \geqslant 0.$$

Moreover, there is the following approximation

$$\int_0^{0.1} G(1,qs)d_q s \approx 0.008,$$

and $\int_{0.1}^1 G(1,qs)d_q s = \int_0^1 G(1,qs)d_q s - \int_0^{0.1} G(1,qs)d_q s$. By $\tau = 0.1$, we have $\int_{0.1}^1 G(1,qs)d_q s \approx 0.04$, and

$$N = \max_{0\leq t\leq 1} \int_\tau^1 G(t,qs)d_q s \geqslant \int_{0.1}^1 G(1,qs)d_q s \approx 0.04 \geqslant 0.$$

Now, define $r_1 = 16$ and $r_2 = 28000$. Then, note that

$$M \max_{s\in[0,1],u\in C\cap\partial\Omega_1} f(\lambda,s,u(s)) \leqslant 0.05 \times 16(3+16) = 15.2 \leqslant r_1,$$
$$N \min_{s\in[\tau,1],u\in C\cap\partial\Omega_2} f(\lambda,s,u(s)) \geqslant 0.04 \times (0.03)^2 \times 28000 \times 28000 = 28224 \geqslant r_2.$$

Therefore, by using Theorem 3, the following problem

$$(D_{0.1}^{3.5}u)(x) = \frac{3u}{\lambda}(x) + u^2(x), \qquad 0 < x \leqslant 1,$$

with the following conditions

$$D_{0.1}u(1) = D_{0.1}^2 u(1) = 0, \qquad D_{0.1}^{0.5}u(0) = 0$$

has a positive solution in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

## 4 Conclusion

By considering a $q$-fractional boundary value problem, the fixed point theorem and Green's function of problem (1) - (3), it was proved that the problem has a positive solution. For this problem, we assume that the given function is bounded with respect to parameter $\lambda$. We used $q$-derivative calculus to get the Green's function of this problem. In the method of proving the results, an operator is introduced. We show that this operator has a fixed point and thus this fixed point becomes a solution to the problem.

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