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On recovering space-dependent source term in a degenerate nonlocal parabolic equation

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Abstract. Identifying the unknown source terms in diffusion models, including nonlocal ones, is an active research area with significant applications in engineering and scientific fields such as population dynamics, biology, and physics. This study examines an inverse problem focused on recovering a space-dependent source term in a degenerate diffusion model that includes a nonlocal space term, using final-time measured data. As a first step, the inverse problem is reformulated as an optimization one by considering its solution as the minimizer of a well-defined objective function. The existence of a unique solution to the associated direct problem is discussed in a functional framework based on suitable weighted Sobolev spaces. After that, we prove the existence of a minimizer by means of standard arguments, and establish a first-order necessary optimality condition. Using this last one, we obtain some results concerning the stability and local uniqueness property. For the numerical reconstruction of the missing source term, we designed an algorithm based on the Landweber iterative method and showed its effectiveness by providing several numerical tests.

Keywords: Inverse problem, identification problem, degenerate parabolic equation, optimal control, stability.
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1 Introduction

An inverse problem is generally defined as the process of determining the cause of an observed effect using given data. Mathematically speaking, partial differential equations are used to model physical systems, for which the state inputs are generally given and well defined, such as initial/boundary data, source terms and coefficients that describe the system's physical properties. However, in many practical applications, one or more of the parameters that define the physical properties in the mathematical model are often unknown and to be determined using some additional pieces of information, so in this situation, we are led to an inverse problem.

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The present work is dedicated to study the inverse problem of recovering a space-dependent source term in a degenerate parabolic problem, which involves nonlocal term. More precisely, the degenerate parabolic problem considered is as follows:

$$\begin{cases} u_t - (a(x)u_x)_x + \int_\Omega \mathscr{K}(x,y,t)u(y,t)\mathrm{d}y = f(x), & (x,t) \in Q_T, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(0,t) = u(1,t) = 0, & t \in (0,T). \end{cases} \tag{1}$$

Here, $Q_T := \Omega \times (0,T)$ with $\Omega = (0,1)$ is the spatial domain and $T > 0$ stands for a fixed final time, u_0 is a regular function that represents the initial data, $\mathscr{K}(x,y,t) \in L^\infty(\Omega \times Q_T)$ is a given kernel. Moreover, the diffusion coefficient a is assumed to degenerate weakly, in the sense, that it vanishes at the boundary $x = 0$ and satisfies

$$\begin{cases} a \in \mathscr{C}([0,1]) \cap \mathscr{C}^1((0,1]),\ a(0) = 0, \text{ and } a(x) > 0 \text{ in } x \in (0,1], \\ \exists K \in [0,1), \text{ such that } xa'(x) \leq Ka(x), \quad \forall x \in [0,1]. \end{cases} \tag{2}$$

A typical example of the diffusion coefficient $a(\cdot)$ is $a(x) = x^\mu$ with $0 < \mu < 1$.

In this paper, we concentrate on the inverse problem of determining a spatial source term f in the problem (1) from a measurement of the data at $t = T$ given by a function z defined on Ω. To be more precise, the inverse problem under consideration is formulated as follows:

(IP) Determinate f in the problem (1), such that $u(f)(x,T) = z(x)$ for all $x \in \Omega$.

In the above formulation, $u(f)$ denotes the solution of (1) with the given term source f, and z represents experimentally obtained data, which may include noise.

Inverse problems related to initial boundary value problems governed by parabolic equations have been widely studied by various authors from many points of view, such as uniqueness, stability, numerical resolution. We refer to the monographs [5,7,10,18] for a complete overview on this topic. Amongst these kinds of problems, great interest is paid to study source identification problems, for which the theoretical analysis, as well as numerical reconstruction, are the centrepiece of investigation. In this regard, more recent attention has focused on identification source problems, for which several approaches had been introduced and developed, such as the quasi-reversibility method [13,17], Tikhonov-type's regularization method [8], spectral truncation method [9], and so on.

In the same context, an optimization-based approach has been used to handle various identification inverse problems (see for example [4, 14, 16, 19, 20]). Let us briefly describe this theory, first, one should investigate the well-posedness of the direct problem and provide some regularity results. After that, the inverse problem is relaxed, in the sense that, its solution is considered as a solution of an optimization problem. In doing so, the missing parameter is restricted to a suitable admissible set and the inverse problem is reformulated as a minimization problem of finding the minimizer of an adequate cost functional that depends on the unknown parameter to be identified.

Nonlocal models are employed for modelling complex precesses where the local approach becomes inadequate or limiting. Such kinds of models have been effectively utilised across several domains, including material sciences [2], dislocation dynamics in crystals [11], chemotaxis phenomena [3], image processing [12], and elasticity [15].

As we have pointed out, several works have been devoted for studying inverse problems related to systems of parabolic equations for both cases, with and without degeneracy. Upon a thorough examination of the literature related to this subject, it becomes evident that there is a lack of studies addressing inverse problems related to nonlocal models with degeneracy. While these models have been discussed from other perspectives in various papers (see [1]), studies focusing specifically on inverse problems in this context remain scarce.

However, there is a notable gap in studies that directly address solving inverse problems for these models, which aim to recover unknown parameters. From this perspective, the aim of this paper is to conduct a comprehensive study of the inverse problem of identifying a space-depended source term in the degenerate nonlocal model (1), where we prove the stability property on one hand, and on the other hand, we perform the numerical reconstruction of the solutions, which demonstrate that they are indeed stable.

Concerning the theoretical analysis, we use an approach based on the optimal control framework to prove the local stability as well as the uniqueness of a quasi-solution for the studied inverse problem. Our main finding can be briefly stated as follows: let f_1 and f_2 be solutions to the inverse problem **(IP)** corresponding to the observations z_1 and z_2, respectively. Then, there exists a positive constant $C > 0$ for which one has a stability estimate

$$\|f_1 - f_2\|^2_{L^2(\Omega)} \leq C\|z_1 - z_2\|^2_{L^2(\Omega)}.$$

Consequently, the uniqueness becomes a direct consequence of this stability estimate under the assumption that z_1 matches z_2. The second contribution of this work lies in the numerical reconstruction of solutions for our inverse problem **(IP)**. For this purpose, we design an efficient easy-to-implement algorithm based on the Landweber iterative method.

The outline of this paper is as follows. In Section 2, we discuss the well-posedness of the direct problem in suitable weighted Sobolev spaces. Section 3 is devoted to studying the inverse problem in the control optimal setting. First, the inverse problem is relaxed by interpreting its solution as a minimizer of an adequate cost functional, after that, we prove the existence of a minimizer and establish the first-order necessary optimality condition. In the fourth section, we present our main results concerning local stability and uniqueness. In Section 5, we reconstruct numerically the unknown source term using a Landweber-type iterative method, and also we provide two numerical tests to validate the effectiveness of the proposed algorithm. Some concluding remarks and further extensions are collected in the last section.

2 Well-posedness of the forward problem

As is well known, studying an inverse problem requires sound knowledge of its direct problem. Upon that, we should first discuss the existence of a unique solution for the forward problem (1) and the regularity of its solutions. To this end, we first need to introduce some notation and functional setting which shall be used in our paper. Let us define the weighted Sobolev spaces related to the degenerate coefficient $a(\cdot)$. Let us define

$$H^1_a(\Omega) := \{u \in L^2(\Omega) : \quad u \text{ abs. cont. in } [0,1],\ \sqrt{a}u_x \in L^2(\Omega),\ u(0) = u(1) = 0\},$$

which is a Hilbert space for the scalar product

$$(u,v)_{1,a} := \int_\Omega (uv + a u_x v_x)\,\mathrm{d}x, \quad u,v \in H_1^a(\Omega),$$

where the associated norm is given by $\|u\|_{1,a}^2 := \|u\|_{L^2(\Omega)}^2 + |u|_{1,a}^2$, here $\|\cdot\|_{L^2(\Omega)}$ denotes the usual L^2-norm of the space $L^2(\Omega)$, and $|\cdot|_{1,a}$ is a semi-norm defined on $H_a^1(\Omega)$ by $|u|_{1,a} = \|\sqrt{a}u_x\|_{L^2(\Omega)}$. Similarly, we define

$$H_a^2(\Omega) := \{u \in H_a^1(\Omega) : \quad a u_x \in L^2(\Omega)\},$$

endowed with the norm $\|u\|_{2,a}^2 := \|u\|_{1,a}^2 + \|(a u_x)_x\|_{L^2(\Omega)}^2$.

Now, we give a version of the classical integration by parts formula which is adopted to the spaces $H_a^1(\Omega)$ and $H_a^2(\Omega)$ (see [6]).

Lemma 1. *For all $(u,v) \in H_a^2(\Omega) \times H_a^1(\Omega)$, we have*

$$\int_\Omega (a u_x)_x v\,\mathrm{d}x = -\int_\Omega a u_x v_x\,\mathrm{d}x.$$

Another key ingredient in our proofs hereafter resides in the following compactness results.

Theorem 1 ([6]). *Assume that the diffusion coefficients satisfies (2), then the following embeddings are compact*

$$H_a^2(\Omega) \hookrightarrow H_a^1(\Omega) \hookrightarrow L^2(\Omega).$$

The notion of a solution for (1) is understood in a weak sense.

Definition 1. *Assume that u_0 and $f \in L^2(\Omega)$. By a weak solution of (1), we mean a function*

$$u \in L^2(0,T;H_a^1(\Omega)), \quad u_t \in L^2(0,T;L^2(\Omega)),$$

satisfying

$$\iint_{Q_T} u_t \phi\, dxdt + \iint_{Q_T} a u_x \phi_x\, dxdt + \iint_{Q_T} \left(\int_\Omega \mathscr{K}(x,\xi,t) u(\xi,t)\, d\xi \right) \phi\, dxdt = \iint_{Q_T} f\phi\, dxdt, \tag{3}$$

for all $\phi \in L^2(0,T;H_a^1(\Omega))$ and $u(0) = u_0$ is taken in the distributions sense.

The existence of a unique weak solution to the problem (1) is guaranteed by the following theorem [1].

Theorem 2. *Assume that $u_0, f \in L^2(\Omega)$. Then problem (1) admits a unique weak solution such that*

$$u \in C([0,T],L^2(\Omega) \cap L^2(0,T;H_a^1(\Omega)), \quad u_t \in L^2(0,T;L^2(\Omega)).$$

Moreover, if $u_0 \in H_a^1(\Omega)$, then we have

$$\sup_{0\le t\le T} \|u(t)\|_{1,a}^2 + \int_0^T |u(t)|_{1,a}^2 dt + \int_0^T \|(a u_x(t))_x\|^2 dt + \int_0^T \|u_t(t)\|_{L^2(\Omega)}^2 dt \le C, \tag{4}$$

where $C > 0$ is a positive constant that depends on T, u_0, f and $\mathscr{K}$.

3 Optimal control approach

The inverse problem to be studied in this paper consists in determining the term f in the degenerate parabolic equation (1), so that the solution $u(f)$ matches or at least fairly approaches to an observed final measurement z, that is, $u(f)(x,T) = z(x)$ for all $x \in \Omega$.

Let us consider an admissible set $\mathbb{U}_{ad}$ to be specified later, and introduce the operator $\mathscr{T} : \mathbb{U}_{ad} \to L^2(\Omega)$ given by

$$\mathscr{T}(f)(x) = u(f)(x,T), \quad \text{for all } x \in \Omega.$$

In this situation, the inverse problem **(IP)** can be reformulated in the form of operator equation, that is, to find $f \in \mathbb{U}_{ad}$ such that

$$\mathscr{T}(f) = z. \tag{5}$$

Incorrect or inaccurate measurements can misalign the equality in the equation (5), which leads, in a general manner, to an ill-posed problem due to the lack of stability. Therefore, to bypass this drawback, one may relax the notion of the solution by seeking a quasi-solution or best-approximate solution that satisfies a regularized formulation of the original problem. In our situation, one way to achieve that is to consider the inverse problem using the optimal control settings.

Considering the uncertainty in the data measurement, consequently, the equality in the above operator equation is far from being achieved exactly. To overcome this drawback, the inverse problem needs to be relaxed by seeking quasi-solution in some sense. To this end, we treat the inverse problem under consideration by interpreting its solution as the solution of an adequate optimal control problem, which which reads as follows: Find f^* the solution of the following optimization problem

$$\mathbb{J}(f^*) = \min_{f \in \mathbb{U}_{ad}} \mathbb{J}(f), \text{ subject to } u \text{ a solution of (1)}. \tag{6}$$

where

$$\mathbb{J}(f) := \frac{1}{2}\|u(f)(\cdot,T) - z\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|f\|^2_{L^2(\Omega)}$$

is a cost functional defined on the admissible set

$$\mathbb{U}_{ad} := \{f \in L^2(\Omega), \quad \|f\|_{L^2(\Omega)} \leq C_0\},$$

$u(f)$ stands for the solution of the problem (1) corresponding to the source term f, z is the given observation which as we mentioned previously it describes the final measurement of the desired data, and λ is a positive regularisation parameter.

Next, in the following theorem we show that the optimization problem has a solution.

Theorem 3. *Under the assumptions of Theorem 2, the optimization problem (6) has at least a solution $f^* \in \mathbb{U}_{ad}$, namely,*

$$\mathbb{J}(f^*) = \min_{f \in \mathbb{U}_{ad}} \mathbb{J}(f).$$

Proof. According to Theorem 2, problem (1) has a unique weak solution, therefore, we claim that $\mathbb{J}(f) \geq 0$ for all $f \in \mathbb{U}_{ad}$, so its infinitum exists, let us denote it by d. Now, consider a minimizing sequence $(f_n)_{n \in \mathbb{N}^*}$, which satisfies

$$d < \mathbb{J}(f_n) \leq d + \frac{1}{n}, \quad \forall n \geq 1. \tag{7}$$

From the above claim, clearly $\mathbb{J}(f_n)$ is uniformly bounded, and in view of the explicit formula of $\mathbb{J}$ we deduce that $\|f_n\| \leq C$ for all $n \in \mathbb{N}$, so there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$, denoted $(f_n)_{n\in\mathbb{N}}$ for seeking of clarity, such that it converges weakly to some limit $f^* \in L^2(\Omega)$, namely,

$$f_n \rightharpoonup f^*. \tag{8}$$

On the other hand, since $\mathbb{U}_{ad}$ is a closed convex subset of $L^2(\Omega)$, hence it is weakly closed, and consequently $f^* \in \mathbb{U}_{ad}$. Let $u(f_n)$ be the solution of (1) with f_n replaced by f. According to the estimate (4), we claim that $(u(f_n))_{n\mathbb{N}}$ and $(u_t(f_n))_{n\mathbb{N}}$ are uniformly bounded in $L^2(0,T;H_a^2(\Omega))$ and $H^1(0,T,L^2(\Omega))$. Thus, there exists a sub-sequences of $(u(f_n))_{n\mathbb{N}}$ and $(u_t(f_n))_{n\mathbb{N}}$ such that

$$\begin{aligned} &u(f_n) \rightharpoonup u^* \text{ weakly in } L^2(0,T;H_a^2(\Omega)),\\ &u_t(f_n) \rightharpoonup u_t^* \text{ weakly in } H^1(0,T;L^2(\Omega)), \end{aligned} \tag{9}$$

Using Lions-Aubin theorem [21, Theorem 3.1.1.] with the following settings

$$p_0 = p_1 = 2, \quad X_0 = H_a^2(\Omega), \quad X = H_a^1(\Omega), \quad X_1 = L^2(\Omega),$$

it follows immediately that the following embedding

$$H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H_a^2(\Omega)) \hookrightarrow L^2(0,T;H_a^1(\Omega)),$$

is compact. Moreover, the space $H^1(0,T;L^2(\Omega)$ is compactly embedded in $C(0;T;L^2(\Omega)$. Consequently, in view of the these compactness results, the weak convergence in (9) becomes

$$\begin{aligned} &u(f_n) \to u^* \text{ strongly in } L^2(0,T;H_a^1(\Omega)),\\ &u_t(f_n) \to u_t^* \text{ strongly in } C(0,T;L^2(\Omega)). \end{aligned} \tag{10}$$

Next, we shall prove that $u(f^*) = u^*$. Notice that $u(f_n)$ satisfies

$$\begin{aligned} \iint_{Q_T} u(f_n)_t \phi \,dxdt + \iint_{Q_T} a u(f_n)_x \phi_x \,dxdt \\ + \iint_{Q_T} \left(\int_\Omega \mathscr{K}(x,\xi,t) u(f_n)(\xi,t) d\xi \right) \phi \,dxdt = \iint_{Q_T} f_n \phi \,dxdt, \end{aligned} \tag{11}$$

for all $\phi \in L^2(0,T;H_a^1(\Omega))$. Owing to the convergence results of $(u(f_n))_{n\in\mathbb{N}}$ and $(u(f_n)_t)_{n\in\mathbb{N}}$, it is immediate to prove

$$\begin{aligned} &\lim_{n\to\infty} \iint_{Q_T} u_t(f_n)(x,t)\phi(x,t)\,dxdt = \iint_{Q_T} u_t^*(x,t)\phi(x,t)\,dxdt,\\ &\lim_{n\to\infty} \iint_{Q_T} a u(f_n)_x(x,t)\phi_x(x,t)\,dxdt = \iint_{Q_T} a u_x^*(x,t)\phi_x(x,t)\,dxdt. \end{aligned} \tag{12}$$

Furthermore, it is easy to check that

$$\lim_{n\to\infty} \iint_{Q_T} \left(\int_\Omega \mathscr{K}(x,\xi,t) u(f_n)(\xi,t) d\xi \right) \phi \,dxdt = \iint_{Q_T} \left(\int_\Omega \mathscr{K}(x,\xi,t) u^* \xi,t) d\xi \right) \phi \,dxdt. \tag{13}$$

Indeed, one has

$$\begin{aligned} I &:= \left| \iint_{Q_T} \left(\int_\Omega \mathscr{K}(x,\xi,t) u(f_n)(\xi,t) d\xi \right) \phi dxdt - \iint_{Q_T} \left(\int_\Omega K(x,\xi,t) u^*(\xi,t) d\xi \right) \phi dxdt \right| \\ &= \left| \iint_{\Omega_T} \left(\int_\Omega \mathscr{K}(x,\xi,t)(u(f_n)(\xi,t) - u^*(\xi,t)) d\xi \right) \phi dxdt \right| \\ &\leq \|\mathscr{K}\|_{L^\infty(\Omega\times Q_T)} \|u(f_n) - u^*\|_{L^2(Q_T)} \|\phi\|_{L^2(Q_T)}, \end{aligned}$$

since $u(f_n) \to u^*$ strongly in $L^2(0,T;L^2(\Omega))$, $I \to 0$ as $n \to \infty$, which matches (13). Using (11),(12) and (13), we find upon passing to limit in (11) when $n \to 0$ that

$$\iint_{Q_T} u_t^* \phi dxdt + \iint_{Q_T} a u_x^* \phi_x dxdt + \iint_{Q_T} \left(\int_\Omega \mathscr{K}(x,\xi,t) u^*(\xi,t) d\xi \right) \phi dxdt = \iint_{Q_T} f^* \phi dxdt.$$

Moreover, by employing standard arguments one can easily show that $u^*(\cdot,0) = u_0$ in the distributions sense, therefore, u^* is a weak solution for (1) with f^* instead of f, from which it follows that $u^* = u(f^*)$ regarding the uniqueness property of the solution.

Now, we pass to show that f^* is a solution to the optimization problem (6). Obviously, we have

$$\lim_{n\to\infty} \|u(f_n)(T) - z\|^2_{L^2(\Omega)} = \|u(f^*)(T) - z\|^2_{L^2(\Omega)}, \tag{14}$$

as we know that $u(f_n)(T) \to u(f^*)(T)$ strongly in $L^2(\Omega)$.

Now, by means of (14), the weak convergence of f^* in $L^2(\Omega)$ and the weak semi-continuity in the L^2-norm, we can obtain

$$\begin{aligned} \liminf_{n\to\infty} \mathbb{J}(f_n) &= \lim_{n\to\infty} \left(\frac{1}{2} \|u(f_n)(T) - z\|^2_{L^2(\Omega)} + \lambda \|f_n\|^2_{L^2(\Omega)} \right) \\ &\geq \frac{1}{2} \|u(f^*)(T) - z\|^2_{L^2(\Omega)} + \alpha \|f^*\|^2_{L^2(\Omega)} \\ &= \mathbb{J}(f^*). \end{aligned}$$

From this last inequality together with (7), we obtain

$$\inf_{f\in\mathbb{U}_{ad}} \mathbb{J}(f) \leq \mathbb{J}(f^*) \leq \liminf_{n\to\infty} \mathbb{J}(f_n) = \lim_{n\to\infty} \mathbb{J}(f_n) = \inf_{f\in\mathbb{U}_{ad}} \mathbb{J}(f),$$

thus, $f^* \in \mathbb{U}_{ad}$ is a solution for the optimization problem (6). This completes the proof. □

Next, we focus on establishing a first-order necessary optimality condition, which has an indispensable role in proving our main results later.

Theorem 4. *Let f^* be a solution to problem (6) corresponding to u^* solution of (1). Then, f^* satisfies the following variational inequality for all $h \in \mathbb{U}_{ad}$*

$$\int_\Omega (u^*(x,T) - z(x)) \theta(x,T) dx + \lambda \int_\Omega f^*(h - f^*) dx \geq 0, \tag{15}$$

where θ is a solution to the following problem

$$\begin{cases} \theta_t - (a\theta_x)_x + \int_\Omega \mathscr{K}(x,\xi,t)\theta(\xi,t) d\xi = h - f^*, & x \in \Omega, \quad t \in (0,T), \\ \theta(x,0) = 0, & x \in \Omega, \\ \theta(0,t) = \theta(1,t) = 0, & t \in (0,T). \end{cases} \tag{16}$$

Proof. Let f^* be a solution to the problem (6) subject to u^* weak solution of (1) with f^* instead of f. Let $\varepsilon \in [0,1], h \in \mathbb{U}_{ad}$, and denote $f_\varepsilon := f^* + \varepsilon(h - f^*)$. Under this notation, we have

$$\mathbb{J}_\varepsilon := \mathbb{J}(f_\varepsilon) = \frac{1}{2}\|u_\varepsilon(T) - z\|^2 + \frac{\lambda}{2}\|f_\varepsilon\|^2.$$

Take the Fréchet derivative of $\mathbb{J}_\varepsilon$, we get

$$\frac{\mathrm{d}\mathbb{J}_\varepsilon}{\mathrm{d}\varepsilon} = \int_\Omega \big(u_\varepsilon(x,T) - z(x)\big)\frac{\partial u_\varepsilon}{\partial \varepsilon}dx + \lambda \int_\Omega f_\varepsilon(h - f^*)dx.$$

Since f^* is an optimal solution, it satisfies

$$\frac{\mathrm{d}\mathbb{J}_\varepsilon}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{J}(f^* + \varepsilon(h - f^*))\bigg|_{\varepsilon=0} \geq 0, \quad \forall h \in \mathbb{U}_{ad}. \tag{17}$$

Let us set $\theta = \dfrac{\partial u_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0}$, so by a straightforward calculation we can verify that θ solves (16), moreover, keeping in mined $f_\varepsilon = f^*$ and $u_\varepsilon = u^*$ for $\varepsilon = 0$, (17) yields

$$\int_\Omega (u^*(x,T) - z(x))\theta(x,T)dx + \lambda \int_\Omega f^*(h - f^*)dx \geq 0,$$

which is exactly the desired result. □

4 Stability results

In the context of inverse problems, *stability* generally refers to the sensitivity of the solution with respect to variations in the input data. More formally, stability can be defined in terms of whether small perturbations in the given data, $z(x)$, result in correspondingly small changes in the recovered source term, $f(x)$. Establishing the stability property for an inverse problem typically involves proving an estimate, in suitable norms, of the form

$$\|f - \tilde{f}\|_X \leq C\|z - \tilde{z}\|_Y,$$

where f and $\tilde{f}$ are two possible solutions corresponding to the measurements z and $\tilde{z}$, respectively. This inequality quantifies how an error in the data, measured in the norm $\|z - \tilde{z}\|_Y$, propagates to an error in the recovered source term, measured in the norm $\|f - \tilde{f}\|_X$, with the constant C encapsulating the stability characteristics of the problem.

After proving the existence of a solution to the regularized optimization problem which may be considered as a quasi-solution to the inverse problem (1), we now turn to analyse the stability property of this solution.

Theorem 5. *Let f and $\tilde{f}$ be solutions to the problem (6) corresponding to the measurements z and $\tilde{z}$ respectively. Then, the following stability estimate holds*

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \frac{1}{2\lambda}\|z - \tilde{z}\|^2_{L^2(\Omega)}. \tag{18}$$

Proof. Let f and $\tilde{f}$ be two solutions for (6) subject to u and $\tilde{u}$, respectively. Let us rewrite the variational inequality (15) by choosing f as f^* and $\tilde{f}$ as h, so we shall have

$$\int_\Omega (u(x,T)-z(x))\theta(x,T)dx+\lambda\int_\Omega f(\tilde{f}-f)dx \geq 0. \tag{19}$$

Similarly, taking f^* as $\tilde{f}$ and h as f respectively in (15) gives

$$\int_\Omega (\tilde{u}(x,T)-\tilde{z}(x))\tilde{\theta}(x,T)dx+\alpha\int_\Omega \tilde{f}(f-\tilde{f})dx \geq 0. \tag{20}$$

By combining (19) and (20), we obtain

$$\lambda\|f-\tilde{f}\|^2_{L^2(\Omega)} \leq \int_\Omega (u(x,T)-z(x))\theta(x,T)dx+\int_\Omega (\tilde{u}(x,T)-\tilde{z}(x))\tilde{\theta}(x,T)dx. \tag{21}$$

Now, let us set $U=u-\tilde{u}$ and $\Theta=\theta+\tilde{\theta}$, so U and Θ satisfy the following boundary values problems, respectively

$$\begin{cases} U_t-(aU_x)_x+\int_\Omega \mathscr{K}(x,\xi,t)U(\xi,t)dt=f-\tilde{f}, & x\in\Omega, \quad t\in(0,T),\\ U(x,0)=0, & x\in\Omega,\\ U(0,t)=U(1,t)=0, & t\in(0,T), \end{cases} \tag{22}$$

and

$$\begin{cases} \Theta_t-(a\Theta_x)_x+\int_\Omega \mathscr{K}(x,\xi,t)\Theta(\xi,t)d\xi=0, & x\in\Omega, \quad t\in(0,T),\\ \Theta(x,0)=0, & x\in\Omega,\\ \Theta(0,t)=\Theta(1,t)=0, & t\in(0,T). \end{cases} \tag{23}$$

Obviously, problem (23) admits a zero-solution $\Theta=0$ as a unique solution, thus $\theta=-\tilde{\theta}$ for almost every where in Ω. On the other hand, by noticing that $\tilde{\theta}$ is the unique weak solution of the problem

$$\begin{cases} \tilde{\theta}_t-(a\tilde{\theta}_x)_x+\int_\Omega \mathscr{K}(x,\xi,t)\tilde{\theta}(\xi,t)d\xi=f-\tilde{f}, & x\in\Omega, \quad t\in(0,T),\\ \tilde{\theta}(x,0)=0, & x\in\Omega,\\ \tilde{\theta}(0,t)=\tilde{\theta}(1,t)=0, & t\in(0,T), \end{cases}$$

we deduce, according to uniqueness of solution, that $U=-\theta$. As matter of fact, inequality (20) becomes after performing some manipulations on the term

$$\lambda\|f-\tilde{f}\|^2_{L^2(\Omega)} \leq \int_\Omega U(x,T)\theta(x,T)dx+\int_\Omega (z(x)-\tilde{z}(x))\theta(x,T)dx,$$

from which it follows that

$$\lambda\|f-\tilde{f}\|^2_{L^2(\Omega)} \leq -\int_\Omega |\theta(\cdot,T)|^2dx+\int_\Omega (z-\tilde{z})\theta(\cdot,T)dx. \tag{24}$$

Using of Cauchy-Schwartz and Young inequalities yields

$$\lambda\|f-\tilde{f}\|^2 \leq -\|\theta(T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|z-\tilde{z}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\theta(T)\|^2_{L^2(\Omega)}.$$

hence, we arrive at

$$\|f-\tilde{f}\|^2_{L^2(\Omega)} \le \frac{1}{2\lambda}\|z-\tilde{z}\|^2_{L^2(\Omega)}.$$

This completes the proof of the theorem. □

Corollary 1. *Let f_1 and f_2 be solutions to the problem (6) corresponding to the observations z_1 and z_2, respectively. Assume that $z_1 = z_2$, then $f_1 = f_2$.*

5 Numerical identification

After the theoretical analysis of the inverse problem, in this section, we deal with a numerical reconstruction of the unknown source term f. To this end, we design a Landeweber-type iterative algorithm. It is worth noting that the Landweber method is chosen due to its simplicity and efficiency in handling ill-posed inverse problems, particularly when direct regularization techniques such as Tikhonov regularization are computationally expensive or difficult to implement

5.1 Landweber iteration method

For computational reasons, we focus on the numerical reconstruction of the source term f in the following problem, which is more general than (1):

$$\begin{cases} \eta_t - (a\eta_x)_x + \int_\Omega \mathscr{K}(x,\xi,t)\eta(\xi,t)\mathrm{d}\xi = f(x) + F(x,t), & x\in\Omega, \quad t\in(0,T), \\ \eta(x,0) = u_0(x), & x\in\Omega, \\ \eta(0,t) = \eta(1,t) = 0, & t\in(0,T), \\ \eta(x,T) = z_\eta(x), & x\in\Omega. \end{cases} \tag{25}$$

Based on the superposition principle, we write $\eta = u + \tilde{\eta}$, where u and $\tilde{\eta}$ represent solutions to the following problems.

$$\begin{cases} u_t - (au_x)_x + \int_\Omega \mathscr{K}(x,\xi,t)u(\xi,t)\mathrm{d}\xi = f(x), & x\in\Omega, \quad t\in(0,T), \\ u(x,0) = 0, & x\in\Omega, \\ u(0,t) = u(1,t) = 0, & t\in(0,T), \\ u(x,T) = z(x) = z_\eta(x,T) - \tilde{\eta}(x,T), & x\in\Omega. \end{cases} \tag{26}$$

and

$$\begin{cases} \tilde{\eta}_t - (a\tilde{\eta}_x)_x + \int_\Omega \mathscr{K}(x,\xi,t)\tilde{\eta}(\xi,t)\mathrm{d}\xi = F(x,t), & x\in\Omega, \quad t\in(0,T), \\ \tilde{\eta}(x,0) = u_0(x), & x\in\Omega, \\ \tilde{\eta}(0,t) = \tilde{\eta}(1,t) = 0, & t\in(0,T). \end{cases} \tag{27}$$

Consequently, the inverse problem of identifying f in (25) from the final measured data $\eta(\cdot,T) = z_\eta$ requires to handle two problems. The first one is the inverse problem of identifying f in (26) from the observation $u(\cdot,T) = z$, and the other is a direct problem (27) which consists of finding the data $\tilde{\eta}$. The main reason for such decomposition lies in the need to make the operator equation (5) linear, so that, the Landweber method is employed without any complications.

Assume that f solves the operator equation (5). It is a well-known fact that f is a solution of the normal equation $\mathscr{T}^*\mathscr{T}(f) = \mathscr{T}^*(z)$, where $\mathscr{T}^*$ denotes the adjoint of $\mathscr{T}$. This last equation can be transformed into a fixed point equation, that is,

$$f = f - \beta\mathscr{T}^*(\mathscr{T}(f) - z),$$

where $0 < \beta < \frac{1}{\|\mathscr{T}\|^2}$ is an adjust parameter. On the basis of this last one, we construct the Landweber iteration as follows:

$$\begin{aligned} f_{m+1} &= f_m - \beta\mathscr{T}^*(\mathscr{T}(f_m) - z) \\ &= f_m - \beta\mathscr{T}^*[u_m(\cdot,T) - z], \end{aligned} \tag{28}$$

where u_m stands for the solution of (26) with f_m instead of f. It is worth noticing that the fixed-point iteration (28) is convergent for any initial choice $f_0 \in D(\mathscr{T})$ if $0 < \beta < 1/\|\mathscr{T}\|^2$ (see [5]).

In order to apply the iterative Landweber method, it is necessary to specify the form of the adjoint operator $\mathscr{T}^*$. As a matter of fact, $\mathscr{T}^*$ is characterized as follows.

Proposition 1. *Let $w \in L^2(\Omega)$. Assume that $\vartheta \in L^2(0,T;H_a^1(\Omega))$ solves the following problem*

$$\begin{cases} \vartheta_t - (a\vartheta_x)_x + \displaystyle\int_\Omega \mathscr{K}(\xi,x,t)\vartheta(\xi,t)d\xi = w, & x \in \Omega, \quad t \in (0,T), \\ \vartheta(x,0) = 0, & x \in \Omega, \\ \vartheta(0,t) = \vartheta(1,t) = 0, & t \in (0,T). \end{cases} \tag{29}$$

then, we have $\mathscr{T}^*(w) = \vartheta(\cdot,T)$.

Proof. Let v be a solution for the following problem

$$\begin{cases} -v_t - (av_x)_x + \displaystyle\int_\Omega \mathscr{K}(\xi,x,t)v(\xi,t)d\xi = w, & x \in \Omega, \quad t \in (0,T), \\ v(x,T) = 0, & x \in \Omega, \\ v(0,t) = v(1,t) = 0, & t \in (0,T). \end{cases}$$

Multiply the first equation of (26) by v, perform an integration by parts and make use the initial condition $u(\cdot,0) = 0$ and final condition $v(\cdot,T) = 0$, we obtain

$$\begin{aligned} \iint_{Q_T} w(x)u(x,t)dxdt &= \iint_{Q_T} \left(u_t v + au_x v_x + v\int_\Omega \mathscr{K}(x,\xi,t)u(\xi,t)d\xi \right) dxdt \\ &= \iint_{Q_T} \left(-uv_t + av_x u_x + u\int_\Omega \mathscr{K}(\xi,x,t)v(\xi,t)d\xi \right) dxdt, \end{aligned}$$

from which it follows by claiming that u solves (26) in the sense of (17) that

$$\int_\Omega w(x)\int_0^T u(x,t)dtdx = \int_\Omega f(x)\int_0^T v(x,t)dtdx. \tag{30}$$

Let us make the variable transformation $\tau = T - t$ and set $\vartheta(\cdot,\tau) = v(\cdot, T-t)$. Thus, it is readily to check that ϑ solves the following problem

$$\begin{cases} \vartheta_\tau - (a\vartheta_x)_x + \int_\Omega \mathscr{K}(\xi,x,\tau)\vartheta(\xi,\tau)d\xi = w, & x \in \Omega, \quad \tau \in (0,T), \\ \vartheta(x,0) = 0, & x \in \Omega, \\ \vartheta(0,\tau) = \vartheta(1,\tau) = 0, & \tau \in (0,T). \end{cases} \tag{31}$$

Meanwhile, the right-hand side of (30) is transformed into

$$\begin{aligned} \int_\Omega f(x)\int_0^T v(x,t)dtdx &= \int_\Omega f(x)\int_0^T v(x,T-\tau)d\tau dx \\ &= \int_\Omega f(x)\int_0^T \vartheta(x,\tau)d\tau dx, \end{aligned}$$

from which it follows that

$$\int_\Omega w(x)\int_0^T u(x,t)dtdx = \int_\Omega f(x)\int_0^T \vartheta(x,t)dtdx.$$

Noticing that the above identity holds for any $T > 0$, we deduce

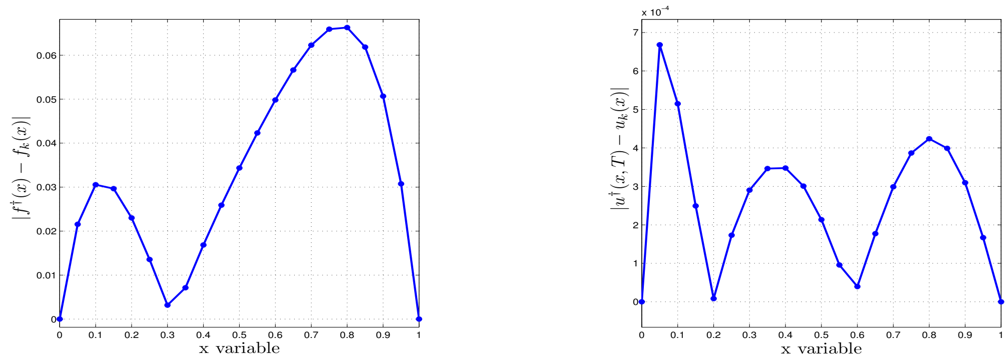
$$\int_\Omega w(x)u(x,T)dtdx = \int_\Omega f(x)\xi(x,T)dtdx. \tag{32}$$

Now, the above integral identity can be rewritten as $(\mathscr{T}(f),w)_{L^2(\Omega)} = (f,\vartheta(\cdot,T))_{L^2(\Omega)}$, then we immediately obtain $\mathscr{T}(w) = \vartheta(\cdot,T)$. $\square$

To summarize, we outline below the main steps of the proposed algorithm used to numerically recover the unknown source term f in problem (26).

Algorithm 1 iterative Landweber method

Inputs : $\beta > 0$ adjacent parameter, $\varepsilon > 0$ tolerance.
Outputs $(u^\dagger, f^\dagger)$ solution of the inverse problem.
Step 1: Set $k = 0$ and choose initial guesses f_0.
Step 2: Obtain the solution u_0 by solving problem (26), where $f = f_0$.
Step 3: Obtain the solution ϑ_0 by solving problem (29) where $w = u_0(\cdot,T) - z$.
Step 4: For $k \geq 1$, set $f_{k+1} = f_k - \beta\vartheta_k(\cdot,T)$.
Step 5: Obtain the solution u_{k+1} by solving problem (26), where $f = f_{k+1}$.
if $\|u_{k+1}(T) - z\| < \varepsilon$ **then**
  take $u^\dagger = u_{k+1}$ and $f^\dagger = f_{k+1}$.
else
  Update $k = k+1$ and go to **Step 4**.
end if



(a) $|f_k(x) - f^\dagger(x)|$. **(b)** $|u_k(x,T) - u^\dagger(x,T)|$.

Figure 1: Numerical results obtained by $k = 1000$ and $\beta = 2$ for Example 1.

5.2 Numerical results and discussions

In this subsection, we perform some numerical computations to reconstruct the space-dependent source component $f(x)$ based on the iterative Landweber method described previously. The numerical realization of Algorithm 1 is mainly based on two steps: (i) solve the problem (26), and (ii) solve the adjoint problem (29). In doing so, we utilize the finite difference method to discretize (26) and (29).

In our computations, the discretization step for time and space variables in the finite difference schema are given by $\Delta t = T/n$ and $\Delta x = 1/m$, respectively. The grid points in the space domain Ω are denoted by $x_i = i\Delta x, i = 0, \cdots, m$ and the grid points in the time interval $[0,T]$ are $t_j = j\Delta t, j = 0, \cdots, m$. The value of u at a given grid point (x_i, t_j) is denoted $u^{i,j} = u(x_i, t_j)$.

In our numerical tests, the error $E_m(k)$ is considered as a function of the number of iteration k, and it is defined by

$$E_m(k) := \|f - f_k\|_{L^2(\Omega)} \approx \left(\frac{1}{m} \sum_{i=0}^{m} (f^i - f_k^i)^2 \right)^{\frac{1}{2}},$$

where $f^i = f(x_i)$ is the exact value of the source term f at $x_i \in \Omega$ and $f_k^i = f_k(x_i)$ is its approximation at the k-th iteration.

For the numerical tests with noisy data, we simulate the noisy input by adding a random perturbation, namely,

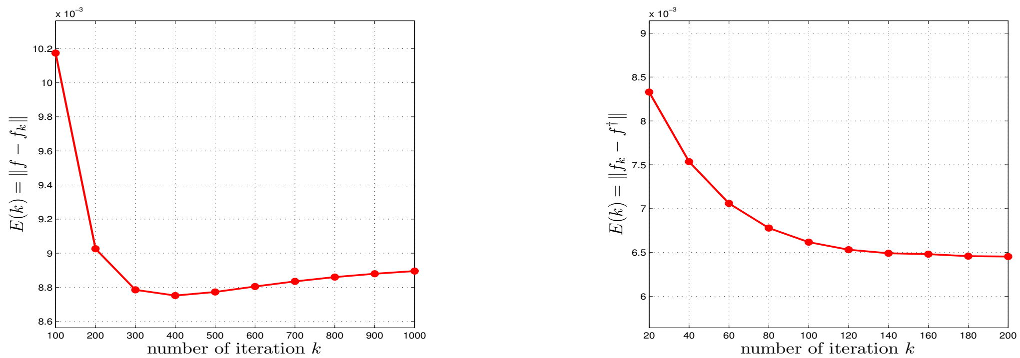
$$z^\delta = z + \delta z \cdot rand(-1,1), \tag{33}$$

where $\delta > 0$ stands for the noise level and $rand(-1,1)$ denotes a random perturbation uniformly distributed within $[-1,1]$.

Next, we present several numerical examples to validate the effectiveness of the designed algorithm.

Example 1. In this first example, let us fix $T = 1$ and consider problem (25) with the following configuration inputs

$$\begin{cases} F(x,t) = \pi^2 x^{\frac{1}{2}} \sin(\pi x) - \dfrac{\pi}{2\sqrt{x}} \cos(\pi x), \quad K(x,y,t) = \sin(\pi(x-y))e^{-t}, \quad a(x) = x^{\frac{1}{2}}, \\ u(x,0) = \sin(\pi x), \quad z(x) = \sin(\pi x)e^1. \end{cases} \tag{34}$$



(a) $\|f_k - f^{\dagger}\|_{L^2(\Omega)}$ in Example 1. **(b)** $\|f_k - f^{\dagger}\|_{L^2(\Omega)}$ in Example 2.

Figure 2: Reconstruction performance in term of iterations number k.

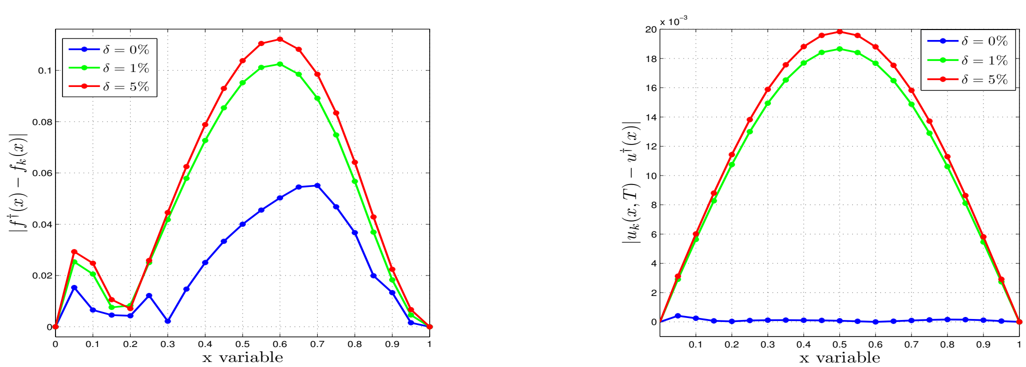
The exact solution for the inverse problem (25)-(34) is given explicitly

$$u^{\dagger}(x,t) = e^t \sin(\pi x), \quad f^{\dagger}(x) = -\frac{1}{2}\cos(\pi x).$$

For the inversion process, we use moderate discretization parameters, setting $\Delta t = 10^{-3}$ and $\Delta x = 5 \cdot 10^{-2}$ with initial guess $f_0(x) = x - \frac{1}{2}$. Figure 1 shows, on the left in (a), the comparison between the exact solution $f^{\dagger}$ (source term) and the reconstructed solution f_k in Example 1 after $k = 1000$ iterations.

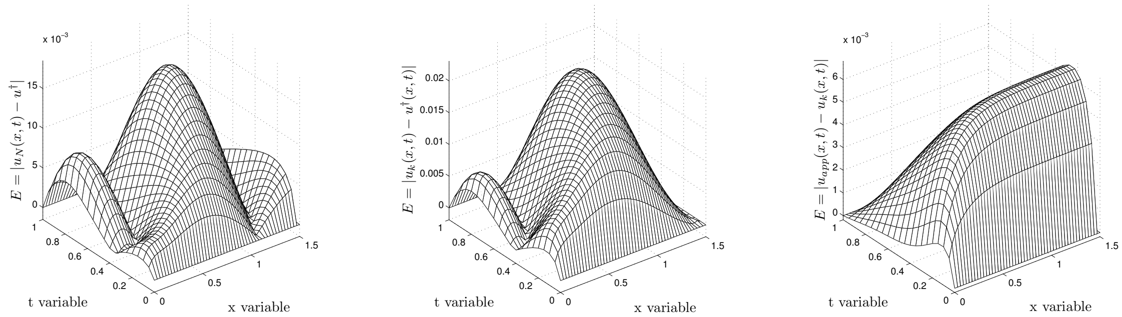
Figure 2-(a) illustrates the behavior of the error $E_m(k) = \|f^{\dagger} - f_k\|_{L^2(\Omega)}$ as a function of the iteration number k. It is obvious that $E_m(k)$ decreases as k increases, until $k = 400$. From this point on, discretization error commuted during the resolution of (26) and (29) dominates.

We perform the numerical implementations using the input noise data z^{δ} given by (33). The results are shown in Figure 3. Those results are obtained by $k = 400$ iterations. It can be seen that the reconstructed solutions are satisfactory for the exact observation and $u_k(\cdot, T)$ matches well the observation z, but they are deteriorated for large noise level.



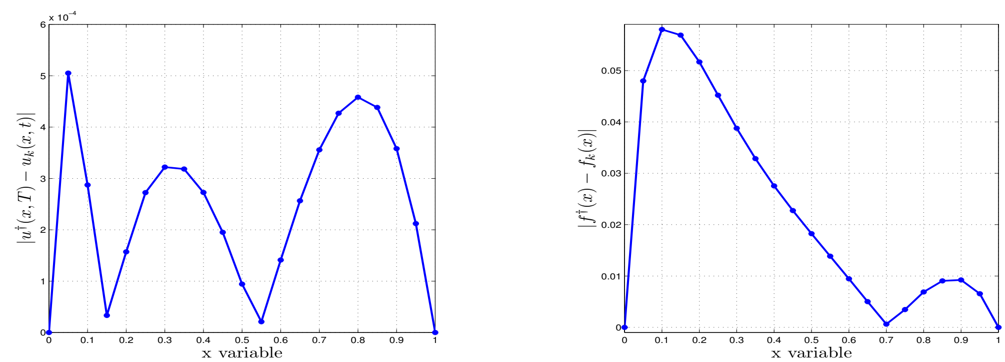
(a) $|f_k(x) - f^{\dagger}(x)|$. **(b)** $|u_k(x,T) - z(x)|$.

Figure 3: Numerical results obtained by $k = 400$ and $\beta = 2$ for different noise levels



(a) $E = |f_k(x) - f^\dagger(x)|$. **(b)** $E = |u_k(x,T) - z(x)|$. **(c)** $E = |u_k(x,T) - z(x)|$.

Figure 4: Comparison between $u^\dagger, u_{app}$ and u_k.



(a) $|f_k(x) - f^\dagger(x)|$. **(b)** $|u_k(x,T) - z(x)|$.

Figure 5: Numerical results obtained by $k = 200$ and $\beta = 1$ for Example 2.

Example 2. In this example, we take $T = 3/2$ and consider the problem (25) with the following configuration inputs

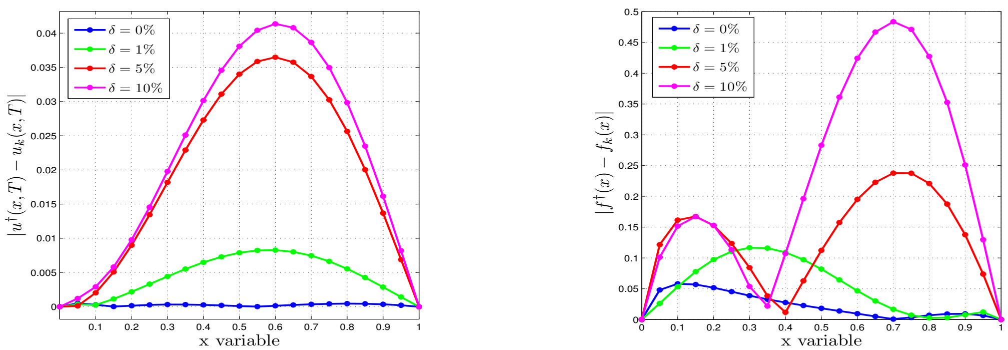
$$\begin{cases} F(x,t) = \pi^2 x^{\frac{1}{2}} \sin(\pi x) - \dfrac{\pi}{2\sqrt{x}} \cos(\pi x), \quad K(x,y,t) = \sin(\pi(x-y))e^{-t}, \quad a(x) = x^{\frac{1}{2}}, \\ u(x,0) = \sin(\pi x), \quad z(x) = \sin(\pi x)e^1. \end{cases} \tag{35}$$

The exact solution of (25)-(35) is given by the expression

$$u^\dagger(x,t) = -20x^2(x-1)\sin(\pi t), \quad f^\dagger(x) = \frac{1}{6}(5x^2 - 2).$$

The numerical reconstruction of $u(x,T)$ and $f(x)$ for Example 2 are shown in Figure 5. For inversion process, we take $\Delta t = 10^{-3}$ and $\Delta x = 5 \cdot 10^{-2}$ with initial guess $f_0(x) = \frac{1}{6}(5x+2)$. We observe that the numerical results obtained are almost coincide with the exact data.

Figure 4 illustrates a comparison between u_k reconstructed solution with $k = 200$ iterations, u_{app} approximate solution of the direct problem and $u^\dagger$ exact data. Figure 2-(b) exhibits the distribution



(a) $|f_k(x) - f^\dagger(x)|$. **(b)** $|u_k(x,T) - z(x)|$.

Figure 6: Numerical results obtained by $k = 100$ and $\beta = 1$ for different noise levels.

of error $E_m(k) = \|f^\dagger - f_k\|_{L^2(\Omega)}$ as a function of the iteration number k. It is noticeable that $E_m(k)$ is decreasing as k increasing until k reaches 200, after that, the discretization error dominated (see Table 1). In Figure 6, we present the numerical errors $E_m(k)$ of the Example (2) for different noise levels. Clearly the reconstructions are satisfactory for the zero-noise data (exact input data), and the error $E_k(m)$ increases when the noise level in the observation data z_δ increases.

Table 1: Comparison between $u^\dagger, u_{app}$ and u_k.

$\|u_k - u^\dagger\|_\infty$	$\|u_k - u_{app}\|_\infty$	$\|u_{app} - u^\dagger\|_\infty$
$2.122 \cdot 10^{-2}$	$6.845 \cdot 10^{-3}$	$1.701 \cdot 10^{-2}$

Remark 1.

The initial guess f_0 is chosen during the inversion process as a first-degree polynomial, such that its value at the end points satisfies the compatible conditions

$$\begin{cases} f(0) = -(a(u_0)_x)_x|_{x=0} - \displaystyle\int_\Omega K(0,y,0)u_0(y)dy + F(0,0), \\ f(1) = -(a(u_0)_x)_x|_{x=1} - \displaystyle\int_\Omega K(1,y,0)u_0(y)dy + F(1,0). \end{cases}$$

6 Conclusion

In this paper, we investigated an inverse problem for a degenerate diffusion equation with a space-dependant non-local term from both aspects theoretically and numerically. More precisely, we have tackled the identification issue of space-dependant source term from a final measured data. The underlying inverse problem is solved in the control optimal framework, by which, the inverse problem is rewritten as an optimization problem. Then unknown source term is characterized as the solution of an optimization problem minimizing a suitable cost functional. We establish the existence of an optimal solution and derive a necessary optimality condition via the adjoint problem. On the base of this last one, the stability, as well as, the conditional uniqueness of a solution to the inverse problem are deduced. From numerical analysis angle, we had successfully designed a Landweber-type iterative method for the numerical reconstruction of solutions to our inverse problem.

It turned out that the Landweber method needs to be iterated continuously to obtain accurate results, as the error decreases with an increasing number of iterations. However, numerical tests showed that the proposed algorithm is relatively stable in the presence of data noise, confirming the validity of this method for solving such problems.

It is promising to extend our approach to inverse problems for problems involving general memory terms in the form $\int_{\Omega} K(x,y,t)\Psi(u,u_x,u_{xx})dy$. On the other hand, this kind of problem awaits further investigation on the numerical aspect in the future.

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