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A novel and efficient operational matrix method for solving multi-term variable-order fractional differential equations

Tahereh Eftekhari[†*], **Jalil Rashidinia**[‡]

[†] *Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China*
[‡] *School of Mathematics and Computer Science, Iran University of Science and Technology (IUST), Tehran, 16846 13114, Iran*
Email(s): t.eftekhari2009@gmail.com, rashidinia@iust.ac.ir

Abstract. The main aim of this research is to present a novel and efficient method based on the Müntz-Legendre polynomials for solving differential equations involving variable-order fractional Caputo derivatives. For the first time, based on the Müntz-Legendre polynomials, a formula for the operational matrix of the variable-order fractional Caputo differential operator is derived. By using this operational matrix via the collocation method, we convert the proposed problem into a system of equations. Then, we solve the obtained system by the Newton method to provide an approximate solution for the problem. Furthermore, we obtain an error bound for the approximation. Finally, we solve four test problems to confirm the reliability and effectiveness of the proposed method.

Keywords: Multi-term variable-order fractional differential equations, variable-order fractional Caputo derivative, Müntz-Legendre polynomials, operational matrix, error bound.
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1 Introduction

Over the last decades, the importance of fractional calculus has become more evident. In fractional calculus, the order of derivatives can be arbitrary numbers. The nonlinear equations involving variable-order fractional (VOF) derivatives are very important and play significant role in various fields of heat and mass transfer, economics, elasticity, biomechanics, fluid dynamics, airfoil theory, and oscillation theory [1, 3, 12]. In [18], Samko and Ross proposed a generalization of fractional integral and derivative when the order is nonconstant and depending on the time. In this context, numerous applications have been found in signal processing, control, and physics [5, 10, 16]. Afterward, the existence and uniqueness results of VOF differential equations studied by Xu and He [20]. In [8], Gupta and Kumar suggested

*Corresponding author
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Chebyshev spectral method to solve the VOF mobile-immobile advection-dispersion equation. In [14], the authors solved the VOF differential equation model of shape memory polymers. In [9], Hosseininia et al. developed Legendre wavelets to solve nonlinear VOF advection-diffusion equation with variable coefficients.

In this study, we consider the multi-term variable-order fractional differential equations (MVOFDEs) given by

$$G\left(t, f(t), {}_0^C D_t^{\beta_1(t)} f(t), {}_0^C D_t^{\beta_2(t)} f(t), \ldots, {}_0^C D_t^{\beta_n(t)} f(t)\right) = g(t), \quad t \in [0,T], \tag{1}$$

where ${}_0^C D_t^{\beta_i(t)}$, $i = 1,2,\ldots,n$, are VOF Caputo derivatives of orders $\beta_i(t)$, respectively; G is a linear or nonlinear function; and $f(t)$ is an unknown function.

Note that if $\beta_i(t)$, $i = 1,2,\ldots,n$ are constants, then Eq. (1) has the following form:

$$G\left(t, f(t), {}_0^C D_t^{\beta_1} f(t), {}_0^C D_t^{\beta_2} f(t), \ldots, {}_0^C D_t^{\beta_n} f(t)\right) = g(t), \quad t \in [0,T].$$

Since an analytic solution for the Eq. (1) is not available or it may not be directly solvable, the demand for a numerical solution to analyze the behavior of this equation becomes more pronounced to overcome the mathematical complexity of the analytical solution. In recent years, researchers developed various methods to provide numerical solutions for the MVOFDEs. Some of them include the Legendre wavelets (LWs) [4], and the second and fourth kinds of Chebyshev polynomials, i.e., SKCPs [13] and FKCPs [15], respectively.

The aim of this paper is to find approximate solutions with high precisions for the MVOFDEs. To do this, we construct a new operational matrix of the VOF Caputo differential operator based on the Müntz-Legendre polynomials (MLPs). Our approach will increase the accuracy and computational efficiency of the method based on such polynomials.

The outline of this paper is as follows: In Section 2, the definition of the VOF Caputo derivative, an introduction of MLPs, and function approximation are provided. In Section 3, a new operational matrix of VOF derivative is produced based on the MLPs. In Section 4, by using this operational matrix, we obtain an approximate solution for Eq. (1). In Section 5, an error bound for the approximation is obtained. In Section 6, four test problems are given to demonstrate the effectiveness of the presented method. Finally, in Section 7, a conclusion is given.

2 Preliminary knowledge

In this section, we present some basic definitions and properties of VOF Caputo derivative, the MLPs, and function approximation, which are used in this study.

2.1 The VOF Caputo derivative

Definition 1 ([20]). *The VOF Caputo derivative of order $\alpha(t) > 0$ is defined as*

$${}_0^C D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(q-\alpha(t))} \int_0^t (t-\tau)^{q-1-\alpha(t)} f^{(q)}(\tau) d\tau, \quad q-1 < \alpha(t) \le q, \; q \in \mathbb{N}, \; t > 0,$$

with the following properties:

$$
\begin{aligned}
&{}_{0}^{C}D_{t}^{\alpha(t)} c = 0, \quad \textit{for constant } c,\\
&{}_{0}^{C}D_{t}^{\alpha(t)} t^{j} = \begin{cases} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha(t))} t^{j-\alpha(t)}, & q \leq j \in \mathbb{N},\\ 0, & \textit{otherwise}. \end{cases}
\end{aligned} \tag{2}
$$

2.2 The MLPs

In [7], the MLPs are defined on the interval $[0,T]$ as follows:

$$
\psi_j(t) = \sum_{k=0}^{j} \eta_{k,j} \left(\frac{t}{T}\right)^{\lambda_k}, \qquad \eta_{0,0} = 1,\ \eta_{k,j} = \frac{\prod_{i=0}^{j-1} (1+\lambda_k+\lambda_i)}{\prod_{i=0,i\neq k}^{j} (\lambda_k - \lambda_i)}, \quad j \in \mathbb{N}_0,
$$

where $\lambda_k > -1/2$, and $\lambda_k \neq \lambda_j$ for $k \neq j$. In this paper, we consider $\lambda_k = k\nu$, for a real constant ν.

The orthogonality condition for these polynomials is as follows:

$$
\int_0^T \psi_i(t)\psi_j(t)dx = \frac{\delta_{ij}T}{2\lambda_j+1}, \qquad j \geq i,
$$

where δ_{ij} is the Kronecker delta.

2.3 The function approximation

An arbitrary and integrable function $f(t)$ in $[0,T]$ can be approximated by using the MLPs as follows:

$$
f(t) \simeq f_N(t) = \sum_{j=0}^{N} \hat{f}_j \psi_j(t) = \hat{F}^T \Psi(t), \tag{3}
$$

where

$$
\hat{f}_j = \frac{2\lambda_j+1}{T} \int_0^T f(t)\psi_j(t)\mathrm{d}t.
$$

Here, $\hat{F}$ and $\Psi(t)$ are $(N+1)\times 1$ vectors given by

$$
\hat{F} = [\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_N]^T, \tag{4}
$$

$$
\Psi(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_N(t)]^T. \tag{5}
$$

3 Operational matrix of VOF derivative

To solve the MVOFDEs numerically, we should transform the VOF differential operator, existing in the mentioned problem, into its matrix form. Therefore, we state and prove the following theorem.

Theorem 1. *Let $\Psi(t)$ be the vector of MLPs and $\alpha(t) > 0$. Then, we have*

$$ {}_0^C D_t^{\alpha(t)} \Psi(t) = \boldsymbol{D}_{\alpha(t)} \Psi(t), $$

where

$$ \boldsymbol{D}_{\alpha(t)} = L M_{\alpha(t)} L^{-1}, \tag{6} $$

is the operational matrix of VOF Caputo differential operator. Also,

$$ L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{\eta_{0,1}}{T^{\lambda_0}} & \frac{\eta_{1,1}}{T^{\lambda_1}} & 0 & \cdots & 0 \\ \frac{\eta_{0,2}}{T^{\lambda_0}} & \frac{\eta_{1,2}}{T^{\lambda_1}} & \frac{\eta_{2,2}}{T^{\lambda_2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_{0,N}}{T^{\lambda_0}} & \frac{\eta_{1,N}}{T^{\lambda_1}} & \frac{\eta_{2,N}}{T^{\lambda_2}} & \cdots & \frac{\eta_{N,N}}{T^{\lambda_j}} \end{pmatrix}, \tag{7} $$

and

$$ M_{\alpha(t)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\alpha(t))} t^{-\alpha(t)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\upsilon+1)}{\Gamma(2\upsilon+1-\alpha(t))} t^{-\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N\upsilon+1)}{\Gamma(N\upsilon+1-\alpha(t))} t^{-\alpha(t)} \end{pmatrix}. \tag{8} $$

Proof. From the definition of MLPs, we can write

$$ \psi_j(t) = \sum_{k=0}^{j} \frac{\eta_{k,j}}{T^{\lambda_k}} t^{\lambda_k} = L_j X(t), \qquad j = 0, 1, \ldots, N, \tag{9} $$

where L_j is the jth row of matrix L, which is defined in (7) and

$$ X(t) = \left(1, t^{\upsilon}, t^{2\upsilon}, \ldots, t^{N\upsilon}\right)^T. $$

From (9), it is clear that

$$ \Psi(t) = L X(t), \tag{10} $$

and therefore we can obtain

$$ X(t) = L^{-1} \Psi(t). \tag{11} $$

According to (2), (10), and (11), we get

$$ \begin{aligned} {}_0^C D_t^{\alpha(t)} \Psi(t) &= \left(L_0 {}_0^C D_t^{\alpha(t)} X(t), L_1 {}_0^C D_t^{\alpha(t)} X(t), \ldots, L_N {}_0^C D_t^{\alpha(t)} X(t)\right)^T \\ &= L {}_0^C D_t^{\alpha(t)} X(t) \\ &= L \left(0, \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\alpha(t))} t^{\upsilon-\alpha(t)}, \frac{\Gamma(2\upsilon+1)}{\Gamma(2\upsilon+1-\alpha(t))} t^{2\upsilon-\alpha(t)}, \ldots, \frac{\Gamma(N\upsilon+1)}{\Gamma(N\upsilon+1-\alpha(t))} t^{N\upsilon-\alpha(t)}\right) \\ &= L M_{\alpha(t)} X(t) = L M_{\alpha(t)} L^{-1} \Psi(t) \\ &= \mathbf{D}_{\alpha(t)} \Psi(t), \end{aligned} $$

where $M_{\alpha(t)}$ is defined in (8). Therefore, the proof is completed. □

4 The method of solution

Here, we consider the MLPs and their operational matrix to solve Eq. (1). First, by taking the operator ${}_0^C D_t^{\beta_i(t)}$ in both sides of Eq. (3), we have

$$ {}_0^C D_t^{\beta_i(t)} f(t) \simeq \hat{F}^T \mathbf{D}_{\beta_i(t)} \Psi(t), \qquad i = 1, 2, \ldots, n, \tag{12} $$

where $\hat{F}$ and $\Psi(t)$ are defined in (4) and (5), respectively. Also, $\mathbf{D}_{\beta_i(t)}$ can be obtained by using Eq. (6). Now substituting Eqs. (3) and (12) in Eq. (1) gives

$$ G\left(t, \hat{F}^T \Psi(t), \hat{F}^T \mathbf{D}_{\beta_1(t)} \Psi(t), \hat{F}^T \mathbf{D}_{\beta_2(t)} \Psi(t), \ldots, \hat{F}^T \mathbf{D}_{\beta_n(t)} \Psi(t)\right) = g(t). \tag{13} $$

By considering an appropriate N and using the points t_i, for $i = 1, 2, \ldots, N+1$, that are the roots of Legendre polynomials, we collocate the above equation and obtain a system of $N+1$ equations. We solve this system by the Newton method and determine unknown coefficients $\hat{f}_i$, $i = 0, 1, \ldots, N$, existing in Eq. (13). Substituting these coefficients in Eq. (3), an approximate solution for Eq. (1) can be obtained.

5 Error bound for the approximation

Here, we propose a technique for estimating the error bound of the new method.

Theorem 2. *Let*

$$ \Lambda_N = span\left\{\psi_j(t), \quad 0 \le j \le N\right\}, $$

be the finite-dimensional fractional-polynomial space. Suppose that ${}_0^C D_t^{j\nu} f(t) \in C([0,T])$*, for* $j = 0, 1, \ldots, N$. *Let* $f_N(t)$ *be the best approximation of* $f(t)$ *in* Λ_N*, then the error bound is*

$$ \|f - f_N\|_\infty \le L \frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu + 1)}, \tag{14} $$

where $L \ge \left|{}_0^C D_t^{(N+1)\nu} f(t)\right|$*, for* $t \in [0,T]$.

Proof. Since $f_N(t)$ is the best approximation of $f(t)$ in Λ_N, then we have

$$ \|f - f_N\|_\infty \le \|f - u\|_\infty, \qquad \forall u(t) \in \Lambda_N. \tag{15} $$

By considering the generalized Taylors formula $u(t) = \sum_{j=0}^{N} \frac{t^{j\nu}}{\Gamma(j\nu+1)} {}_0^C D_t^{j\nu} u(0^+)$, we obtain

$$ |f(t) - u(t)| = \left| f(t) - \sum_{j=0}^{N} \frac{t^{j\nu}}{\Gamma(j\nu+1)} {}_0^C D_t^{j\nu} u(0^+) \right| \le L \frac{t^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}. \tag{16} $$

Taking $L_\infty-$norm of both sides of inequality (16) yields

$$ \|f - u\|_\infty \le L \frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}. \tag{17} $$

Now from (15) and (17), we obtain the inequality (14). □

Theorem 3. *Let* ${}_0^C D_t^{j\nu+\beta_l} f(t) \in C([0,T])$. *If* $\left({}_0^C D_t^{\beta_l(t)} f\right)_N(t)$ *be the best approximation of* ${}_0^C D_t^{\beta_l(t)} f(t)$ *in* Λ_N, *then*

$$\left\| {}_0^C D_t^{\beta_l(t)} f - \left({}_0^C D_t^{\beta_l(t)} f\right)_N \right\|_\infty \leq L_l \frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}, \tag{18}$$

where $L_l \geq \left| {}_0^C D_t^{(N+1)\nu+\beta_l(t)} f(t) \right|$, *for* $t \in [0,T]$ *and* $l = 1,2,\ldots,n$.

Proof. Since $\left({}_0^C D_t^{\beta_l(t)} f\right)_N(t)$ is the best approximation of ${}_0^C D_t^{\beta_l(t)} f(t)$ in Λ_N, we have

$$\left\| {}_0^C D_t^{\beta_l(t)} f - \left({}_0^C D_t^{\beta_l(t)} f\right)_N \right\|_\infty \leq \left\| {}_0^C D_t^{\beta_l(t)} f - {}_0^C D_t^{\beta_l(t)} u \right\|_\infty, \qquad \forall\, u(t) \in \Lambda_N. \tag{19}$$

Considering the generalized Taylor's formula ${}_0^C D_t^{\beta_l(t)} u(t) = \sum_{j=0}^{N} \frac{t^{j\nu}}{\Gamma(j\nu+1)} \left({}_0^C D_t^{j\nu+\beta_l(t)} u\right)(0^+)$ yields

$$\begin{aligned}\left| {}_0^C D_t^{\beta_l(t)} f(t) - {}_0^C D_t^{\beta_l(t)} u(t) \right| &= \left| {}_0^C D_t^{\beta_l(t)} f(t) - \sum_{j=0}^{N} \frac{t^{j\nu}}{\Gamma(j\nu+1)} \left({}_0^C D_t^{j\nu+\beta_l(t)} u\right)(0^+) \right| \\ &\leq L_l \frac{t^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}.\end{aligned} \tag{20}$$

Taking $L_\infty-$norm of both sides of inequality (20) leads to

$$\left\| {}_0^C D_t^{\beta_l(t)} f - {}_0^C D_t^{\beta_l(t)} u \right\|_\infty \leq L_l \frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}, \tag{21}$$

Now from (19) and (21), we obtain the inequality (18). □

Theorem 4. *Let* ${}_0^C D_t^{j\nu+\beta_l(t)} f(t) \in C([0,T])$, *G be Lipschitz, and suppose that* $\left| {}_0^C D_t^{j\nu+\beta_l(t)} f(t) \right| \leq L_l$, *for* $l = 1,2,\ldots,n$. *Therefore, the error bound of the MLPs method for the Eq.* (1) *is as follows:*

$$\|E_N\|_\infty \leq (n+1)\kappa\rho \frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu+1)},$$

where $\kappa = \max_{j=0,\ldots,n} \{\kappa_j\}$ *and* $\rho = \max_{l=1,\ldots,n} \{L, L_l\}$.

Proof. Since G is Lipschitz with the constant μ, we can write

$$\begin{aligned}|E_N| &= \left| G\left(t, f_N(t), \left({}_0^C D_t^{\beta_1(t)} f\right)_N(t), \ldots, \left({}_0^C D_t^{\beta_n(t)} f\right)_N(t)\right) - g(t) \right| \\ &= \left| G\left(t, f_N(t), \left({}_0^C D_t^{\beta_1(t)} f\right)_N(t), \ldots, \left({}_0^C D_t^{\beta_n(t)} f\right)_N(t)\right) - G\left(t, f(t), {}_0^C D_t^{\beta_1(t)} f(t), \ldots, {}_0^C D_t^{\beta_n(t)} f(t)\right) \right| \\ &\leq \kappa_0 |f(t) - f_N(t)| + \kappa_1 \left| {}_0^C D_t^{\beta_1(t)} f(t) - \left({}_0^C D_t^{\beta_1(t)} f\right)_N(t) \right| + \ldots + \kappa_n \left| {}_0^C D_t^{\beta_n(t)} f(t) - \left({}_0^C D_t^{\beta_n(t)} f\right)_N(t) \right| \\ &\leq \kappa \|f - f_N\|_\infty + \kappa \left\| {}_0^C D_t^{\beta_1(t)} f - \left({}_0^C D_t^{\beta_n(t)} f\right)_N \right\|_\infty + \ldots + \kappa \left\| {}_0^C D_t^{\beta_n(t)} f - \left({}_0^C D_t^{\beta_n(t)} f\right)_N \right\|_\infty.\end{aligned}$$

By using Theorems 2 and 3, we obtain

$$\|E_N\|_\infty \le \kappa(L+L_1+\ldots+L_n)\frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}.$$

Let

$$\rho=\max_{l=1,\ldots,n}\{L,L_l\}.$$

Therefore, we get

$$\|E_N\|_\infty \le (n+1)\kappa\rho\frac{T^{(N+1)\nu}}{\Gamma((N+1)\nu+1)},$$

and the proof is completed. □

6 Illustrative examples

In this section, four test problems are presented and examined on an Intel(R) Core(TM) i5-2450M CPU @ 2.50GHz Processor with 4 GB of RAM using Maple 2018 on Windows 7 (64 bit) operating system. In this section, $\hat{n}$ denotes the number of bases used to solve these problems. In order to show the accuracy and computational efficiency of our method, we use

$$|f(t)-f_N(t)|,\quad t\in[0,T],\ N\in\mathbb{N},$$

and

$$\max_{i=0,1,\ldots,N}\{|f(t_i)-f_N(t_i)|\},$$

which are the absolute errors in the solutions and the maximum absolute errors, respectively, where points $t_i,\ i=0,1,\ldots,N$ are roots of the Legendre polynomial in the interval $[0,T]$.

Example 1. Consider the following MVOFDE [6, 13, 15]:

$$\begin{cases} a\,{}_0^CD_t^{\alpha(t)}f(t)+b(t)\,{}_0^CD_t^{\beta_1(t)}f(t)+c(t)\,{}_0^CD_t^{\beta_2(t)}f(t)+d(t)\,{}_0^CD_t^{\beta_3(t)}f(t))+e(t)f(t)=g(t),\\ f(0)=2,\ f'(0)=0,\end{cases}\tag{22}$$

where

$$a=1,\ b(t)=t^{\frac{1}{2}},\ c(t)=t^{\frac{1}{3}},\ d(t)=t^{\frac{1}{4}},\ e(t)=t^{\frac{1}{5}},$$

and

$$g(t)=-a\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}-b(t)\frac{t^{2-\beta_1(t)}}{\Gamma(3-\beta_1(t))}-c(t)\frac{t^{2-\beta_2(t)}}{\Gamma(3-\beta_2(t))}-d(t)\frac{t^{2-\beta_3(t)}}{\Gamma(3-\beta_3(t))}+e(t)\left(2-\frac{t^2}{2}\right),$$

with the exact solution $f(t)=2-\frac{t^2}{2}$.

We solve this example in different cases for the VOF parameters defined as:

Table 1: Maximum absolute errors with $N=1$, $\nu=2$, and different values of T, for Cases 1 and 2 of Example 1.

T	Case 1			Case 2		
	MLPs	JPs [6]	SKCPs [13]	MLPs	JPs [6]	SKCPs [13]
	$N=1$	$n=3$	$n=5$	$N=1$	$n=5$	$n=5$
2	0	0	1.3323e−15	3e−24	4.8849e−14	2.2959e−13
4	5e−24	0	3.1974e−14	4e−24	1.9007e−13	7.3275e−15

Table 2: CPU time (s) for Cases 1 and 2 of Example 1 with $N=1$, $\nu=2$, and different values of T.

T	Case 1	Case 2
2	0.343	0.250
4	0.344	0.218

Case 1: $\alpha(t)=2t$, $\beta_1(t)=\frac{t}{3}$, $\beta_2(t)=\frac{t}{4}$, $\beta_3(t)=\frac{t}{5}$.

Case 2: $\alpha(t)=2$, $\beta_1(t)=1.234$, $\beta_2=1$, $\beta_3(t)=0.333$.

By considering the Case 2, the Eq. (22) becomes a multi-term constant-order fractional differential equation.

The maximum absolute errors and CPU times, respectively, are reported in Tables 1 and 2, for Cases 1 and 2 with $N=1$, $\nu=2$, and different values of T. From Table 1, we see that by selecting $\hat{n}=N+1=2$ numbers of MLPs, we obtain more accurate results than the methods of [6, 13], used $\hat{n}=n+1=6$ numbers of Jacobi polynomials (JPs) and $\hat{n}=n+1=4,6$ numbers of the second kind of Chebyshev polynomials (SKCPs), respectively, to solve this problem. These tables and Figure 1 illustrate the efficiency and accuracy of our method.

Example 2. Consider the following MVOFDE [6, 13, 15]:

$$\begin{cases} a\,{}_0^C D_t^{\alpha(t)} f(t) + b\,{}_0^C D_t^{\beta(t)} f(t) + c f(t) = g(t), \\ f(0)=f_0,\ f'(0)=f_1. \end{cases}$$

We solve this example in different cases for the VOF parameters defined as:

Case 1 [15]:

$a=1, b=2, c=4,$

$f_0=f_1=0,$

$\alpha(t)=2t,\ \beta(t)=\frac{1+t}{2},$

$g(t)=\frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}+\frac{4t^{2-\beta(t)}}{\Gamma(3-\beta(t))}+4t^2,$

the exact solution: $f(t)=t^2$.

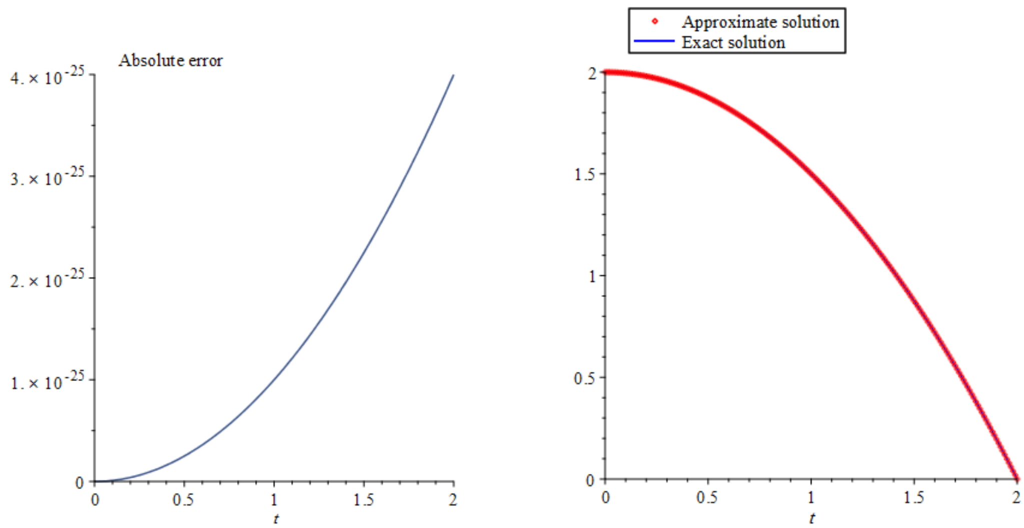


Figure 1: Plots of the absolute error (left), the exact and approximate solutions (right), with $N = 1$, $\nu = 2$, and $T = 2$, for Case 1 of Example 1.

Case 2 [4, 15]:

$a = 1$, $b = -10$, $c = 1$,

$f_0 = 5$, $f_1 = 10$,

$\alpha(t) = \frac{t+2e^t}{7}$, $\beta(t) = 1$,

$g(t) = 10\left(\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \frac{t^{1-\beta(t)}}{\Gamma(2-\beta(t))}\right) + 5t^2 - 90t - 95$,

the exact solution: $f(t) = 5(1+t)^2$.

Case 3 [2, 17]:

$a = 1$, $b = 3$, $c = -1$,

$f_0 = 1$,

$\alpha(t) = \frac{1+\cos^2(x)}{4}$, $\beta(t) = 1$,

$g(t) = e^t\left(3 - \frac{\Gamma(1-\alpha(t),t)}{\Gamma(1-\alpha(t))}\right)$,

the exact solution: $f(t) = e^t$.

Case 4 (The damped mechanical oscillator) [15, 19]:

$a = 1$, $b = 2$, $c = 4$,

$f_0 = f_1 = 0$,

Table 3: Absolute errors with $N = 2$, $\nu = 1$, and $T = 1$, for Case 2 of Example 2.

t	Present method	LPs [17]	FKCPs method [15]	LWs method [4]	FDS [4]
	$N = 2$	$N = 2$	$n = 2$	$k = 2, M = 4$	$N = 20$
0.2	3.518e − 21	2.66454e − 15	1.818101e − 12	8.091305e − 12	4.737e − 2
0.4	3.594e − 21	3.55271e − 15	1.817213e − 12	2.024535e − 9	7.718e − 2
0.6	3.670e − 21	1.77636e − 15	1.820765e − 12	9.564669e − 10	7.891e − 2
0.8	3.750e − 21	3.55271e − 15	1.818989e − 12	1.696030e − 10	4.821e − 2
1.0	3.830e − 21	3.55271e − 15	1.818989e − 12	1.734222e − 10	9.251e − 3

Table 4: Maximum absolute errors with different values of N, ν, and T, for Cases 1-5 of Example 2.

T	Case 1			Case 2		Case 3		Case 4	Case 5
	$N = 4, \nu = \frac{1}{2}$	$N = 2, \nu = 1$	$N = 1, \nu = 2$	$N = 4, \nu = \frac{1}{2}$	$N = 2, \nu = 1$	$N = 15, \nu = 1$	$N = 20, \nu = 1$	$N = 1, \nu = 2$	$N = 1, \nu = 2$
1	1.475300e − 23	3.900e − 25	1e − 25	3.744000e − 20	3.80e − 21	1.515086e − 16	1.086580e − 19	6e − 25	8.11e − 24
2	7.575871e − 23	1.048e − 23	9e − 25	3.370904e − 20	1.04e − 21	7.349818e − 12	1.307470e − 19	0	4.00e − 24

$\alpha(t) = 2$, $\beta(t) = 1$,

$g(t) = 2 + 4t + 4t^2$,

the exact solution: $f(t) = t^2$.

Case 5 (Bagley-Torvik equation) [11, 15]:

$a = b = c = 1$,

$f_0 = f_1 = 0$,

$\alpha(t) = 2$, $\beta(t) = \frac{3}{2}$,

$g(t) = t^2 + 2 + 4\sqrt{\frac{t}{\pi}}$,

the exact solution: $f(t) = t^2$.

The absolute errors are reported in Table 3 for Case 2 with $N = 2$, $\nu = 1$, and $T = 1$. From this table, we see that by selecting $\hat{n} = N + 1 = 3$ numbers of MLPs, we obtain more accurate results than the methods of [4, 15, 17], used $\hat{n} = 2^k M = 20$ numbers of Legendre wavelets (LWs), $\hat{n} = N = 20$ in finite difference scheme (FDS), $\hat{n} = n + 1 = 3$ numbers of the fourth kind of Chebyshev polynomials (FKCPs), and $\hat{n} = N + 1 = 3$ numbers of Lagrange polynomials (LPs), respectively, to solve this problem. The maximum absolute errors and CPU times, respectively, are reported in Tables 4 and 5, for Cases 1-5 with different values of N, ν, and T. These tables and Figures 2-4 illustrate the efficiency and accuracy of our method.

Example 3. Consider the following MVOFDE:

$$\begin{cases} {}_0^C D_t^{\alpha(t)} f(t) + {}_0^C D_t^{\beta_1(t)} f(t) {}_0^C D_t^{\beta_2(t)} f(t) + (f(t))^2 = g(t), \\ f(0) = f'(0) = f''(0) = 0, \end{cases}$$

Table 5: CPU time (s) for Cases 1-5 of Example 2 with different values of N, ν, and T.

T	Case 1			Case 2		Case 3		Case 4	Case 5
	$N=4,\ \nu=\frac{1}{2}$	$N=2,\ \nu=1$	$N=1,\ \nu=2$	$N=4,\ \nu=\frac{1}{2}$	$N=2,\ \nu=1$	$N=15,\ \nu=1$	$N=20,\ \nu=1$	$N=1,\ \nu=2$	$N=1,\ \nu=2$
1	0.234	0.312	0.234	0.312	0.327	0.890	1.810	0.156	0.171
2	0.343	0.296	0.250	0.296	0.328	0.873	1.653	0.218	0.187

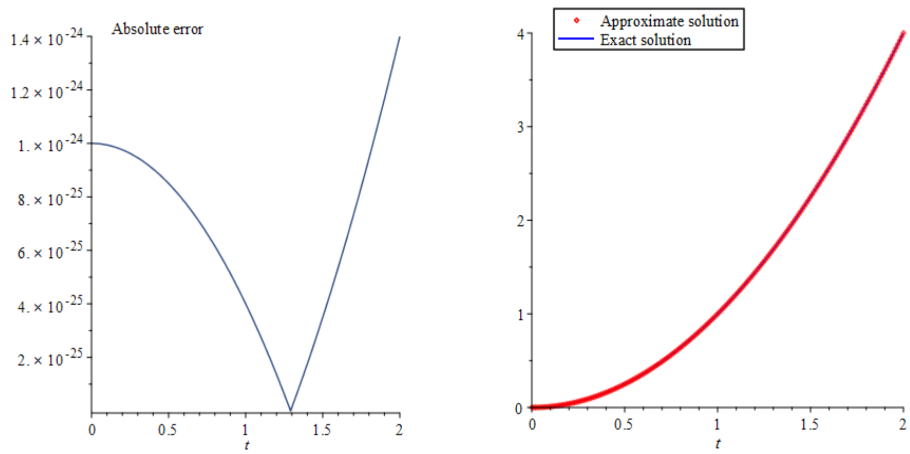


Figure 2: Plots of the absolute error (left), the exact and approximate solutions (right), with $N=1$, $\nu=2$, and $T=2$, for Case 1 of Example 2.

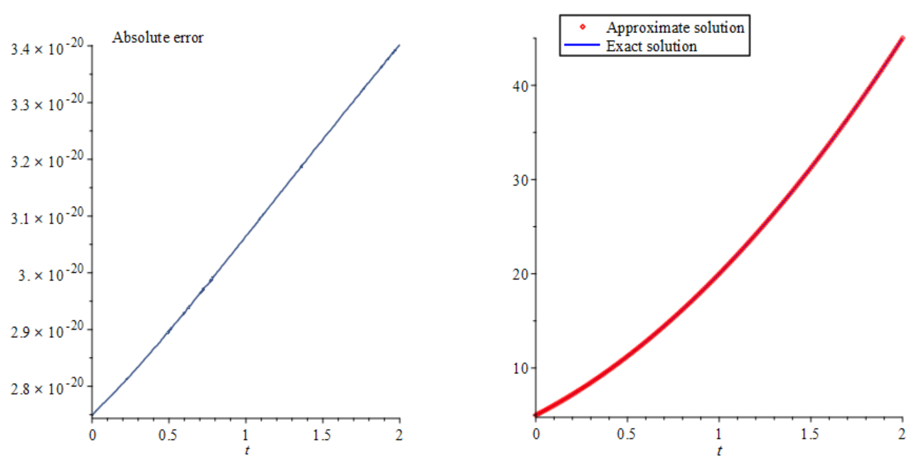
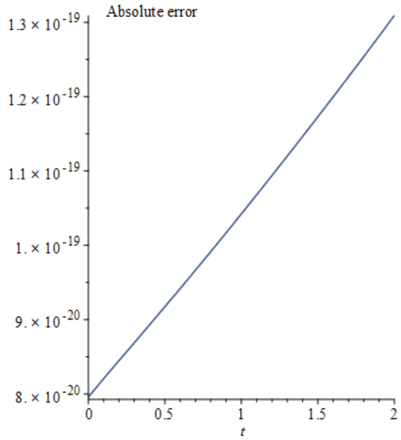


Figure 3: Plots of the absolute error (left), the exact and approximate solutions (right), with $N=4$, $\nu=\frac{1}{2}$, and $T=2$, for Case 2 of Example 2.

where

$$g(t) = t^6 + 6\frac{t^{3-\alpha(t)}}{\Gamma(4-\alpha(t))} + 36\frac{t^{6-\beta_1(t)-\beta_2(t)}}{\Gamma(4-\beta_1(t))\Gamma(4-\beta_2(t))},$$

with the exact solution $f(t) = t^3$.



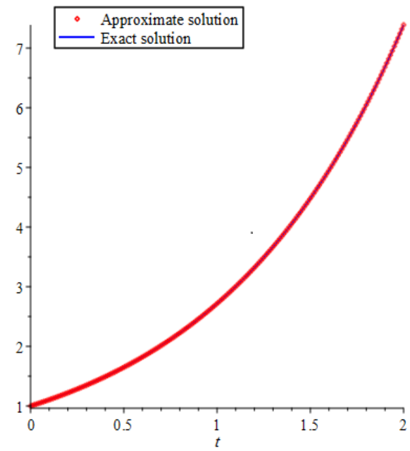


Figure 4: Plots of the absolute error (left), the exact and approximate solutions (right), with $N = 20$, $\nu = 1$, and $T = 2$, for Case 3 of Example 2.

Table 6: Maximum absolute errors with $N = T = 3$ and $\nu = 1$, for Cases 1 and 2 of Example 3.

MLPs	
Case 1	Case 2
7.37583e − 22	2.7e − 23

Table 7: Maximum absolute errors with $N = 3$ and $\nu = T = 1$, for Case 3 of Example 3.

MLPs	SKCPs [13]			
$N = 3$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
2.7710e − 22	1.2628e − 15	1.5910e − 14	4.7362e − 13	1.2801e − 11

We solve this example in different cases for the VOF parameters defined as:

Case 1: $\alpha(t) = t^2$, $\beta_1(t) = \sin(t)$, $\beta_2(t) = 0.9$.

Case 2: $\alpha(t) = t^3 + t$, $\beta_1(t) = e^t$, $\beta_2(t) = \frac{t}{4}$.

Case 3 [13]: $\alpha(t) = 2.5$, $\beta_1(t) = 1.5$, $\beta_2(t) = 0.9$.

The maximum absolute errors are reported in Tables 6 and 7 for Cases 1-3 with $N = 3$, $\nu = 1$, and different values of T. From Table 7, we see that by selecting $\hat{n} = N + 1 = 4$ numbers of MLPs, we obtain more accurate results than the method of [13], used $\hat{n} = n + 1 = 4, 5, 6, 7$ numbers of SKCPs to solve this problem. Also, the CPU times are reported in Table 8 for Cases 1-3 with $N = T = 3$ and $\nu = 1$. These tables and Figure 5 illustrate the efficiency and accuracy of our method.

Table 8: CPU time (s) for Cases 1-3 of Example 3 with $N = T = 3$ and $\nu = 1$.

Case 1	Case 2	Case 3
0.156	0.249	0.219

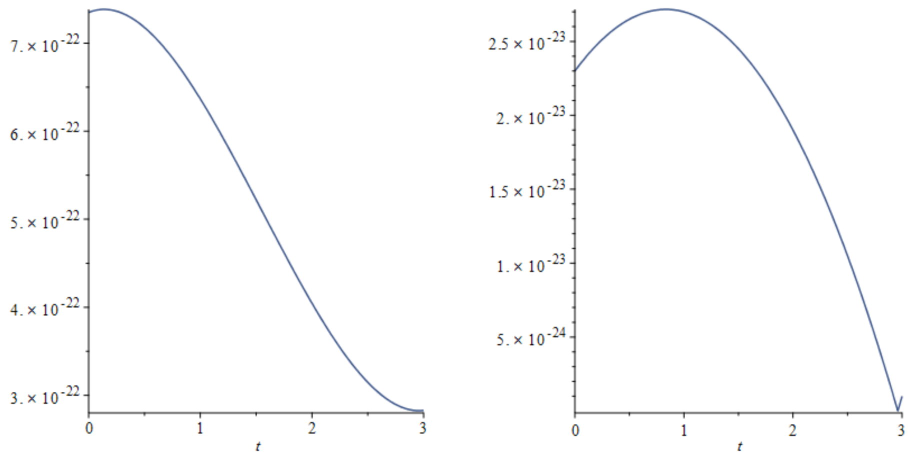


Figure 5: Plots of the absolute errors with $N = T = 3$ and $\nu = 1$, for Case 1 (left), Case 2 (right) of Example 3.

Example 4. Consider the following MVOFDE:

$$\begin{cases} a(t)\,{}_0^CD_t^{\alpha(t)}f(t) + b(t)\,{}_0^CD_t^{\beta(t)}f(t) + c(t)f(t) = g(t), \\ f(0) = 0,\ f'(0) = 1, \end{cases}$$

where $a(t) = t^2$, $b(t) = \frac{t}{2}$, $c(t) = 3t$, and

$$g(t) = \sum_{i=0}^{\infty}(-1)^i\left(a(t)\frac{t^{2i+1-\alpha(t)}}{\Gamma(2i+2-\alpha(t))} + b(t)\frac{t^{2i+1-\beta(t)}}{\Gamma(2i+2-\beta(t))} + c(t)\frac{t^{2i+1}}{\Gamma(2i+2)}\right),$$

with the exact solution $f(t) = \sin(t) = \sum_{i=0}^{\infty}(-1)^i\frac{t^{2i+1}}{\Gamma(2i+2)}$.

Here, we consider the first 9 terms of the Taylor series for $\sin(t)$ and solve this example in different cases for the VOF parameters defined as:

Case 1: $\alpha(t) = \sin(2t)$, $\beta(t) = \cos(2t)$.

Case 2: $\alpha(t) = \tan(2t)$, $\beta(t) = \exp\left(-\frac{t}{5}\right)$.

Case 3: $\alpha(t) = \frac{1}{2}$, $\beta(t) = \frac{1}{3}$.

The absolute errors are reported in Table 9, for Cases 1-3 with $N = 20$, $T = \frac{\pi}{2}$, and different values of ν. Also, the maximum absolute errors and CPU times, respectively, are reported in Tables 10 and 11, for Cases 1-3 with $N = 20$, and different values of ν and T.

These tables and Figure 6 illustrate the efficiency and accuracy of our method.

Table 9: Absolute errors with $T = \frac{\pi}{2}$, $N = 20$, and different values of ν, for Cases 1-3 of Example 4.

t	Case 1		Case 2		Case 3	
	$\nu = \frac{1}{2}$	$\nu = 1$	$\nu = \frac{1}{2}$	$\nu = 1$	$\nu = \frac{1}{2}$	$\nu = 1$
0.2	2.695408e − 12	3.205700e − 21	1.483390e − 15	8.3e − 24	3.748483e − 10	1e − 25
0.4	1.476285e − 12	2.140800e − 21	4.971374e − 16	4.2e − 24	3.611199e − 10	7e − 25
0.6	1.123431e − 12	1.791300e − 21	2.848580e − 16	3.1e − 24	3.627596e − 10	9e − 25
0.8	9.034017e − 13	1.558100e − 21	7.691559e − 17	2.3e − 24	3.684002e − 10	5e − 25
1.0	8.442655e − 13	1.556400e − 21	8.802630e − 17	2.1e − 24	3.753657e − 10	5e − 25

Table 10: Maximum absolute errors with $N = 20$, and different values of ν and T, for Cases 1-3 of Example 4.

T	Case 1		Case 2		Case 3	
	$\nu = \frac{1}{2}$	$\nu = 1$	$\nu = \frac{1}{2}$	$\nu = 1$	$\nu = \frac{1}{2}$	$\nu = 1$
$\frac{\pi}{2}$	9.676548e − 12	9.481294e − 21	8.647143e − 14	2.1325e − 23	7.300364e − 10	1.100e − 24
π	9.333023e − 11	7.662000e − 22	1.559643e − 12	2.7200e − 23	7.300364e − 10	7.662e − 22

Table 11: CPU time (s) for Cases 1-3 of Example 4 with $N = 20$, and different values of ν and T.

T	Case 1		Case 2		Case 3	
	$\nu = \frac{1}{2}$	$\nu = 1$	$\nu = \frac{1}{2}$	$\nu = 1$	$\nu = \frac{1}{2}$	$\nu = 1$
$\frac{\pi}{2}$	3.291	2.511	3.260	2.543	0.635	0.639
π	3.182	2.512	3.323	2.480	0.718	0.608

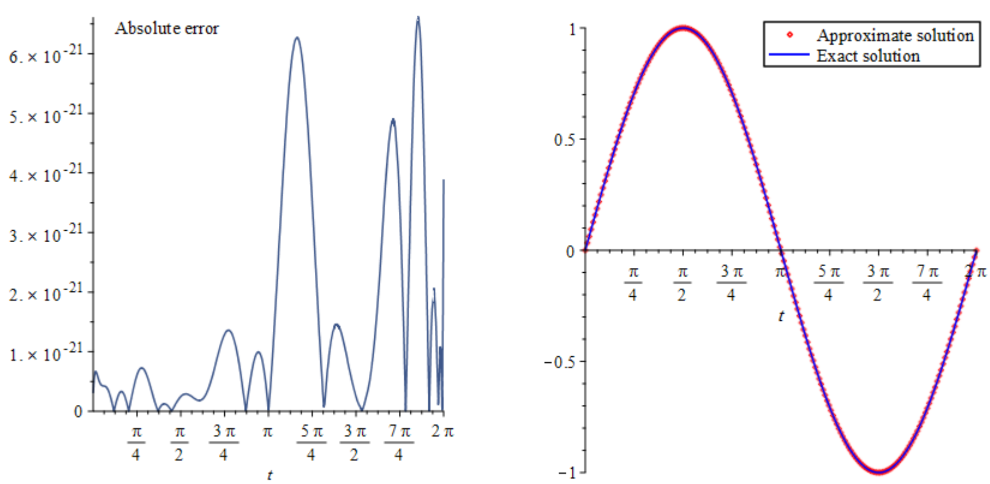


Figure 6: Plots of the absolute error (left), the exact and approximate solutions (right), with $N = 20$, $\nu = 1$, and $T = 2\pi$, for Case 2 of Example 4.

7 Conclusion

In this paper, a new and efficient operational matrix method proposed to obtain approximate solutions for the MVOFDEs. In the presented method, by using the obtained operational matrix via the collocation method, the equation is reduced to a system of equations and then solved by the Newton method. We provided an error bound for the approximation. Furthermore, the method is evaluated by solving four test problems and showed that by applying lower numbers of MLPs we get more accurate results than the other methods used for solving these problems. It can be concluded that the proposed method can be considered as an efficient numerical tool to solve the MVOFDs.

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