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Comparative study of numerical methods for singularly perturbed boundary turning point problems with mixed boundary conditions

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Abstract. A comparative study on numerical methods for Singularly Perturbed Boundary Turning Point Problems (SPBTPPs) featuring discontinuous source terms are examined. The study involves developing and analyzing two specific numerical techniques: the finite difference method and a hybrid difference method incorporating a Shishkin-type mesh. This approach demonstrates notable capabilities, exhibiting almost first-order and second-order convergence for the finite difference and hybrid difference methods, respectively. Numerical results are given to support the theoretical findings.

Keywords: Boundary turning point problem, interior and boundary layer, finite and hybrid difference methods, piecewise uniform mesh.
AMS Subject Classification 2010: 65L11, 65L12, 65L20,65L70

1 Introduction

Singularly Perturbed Boundary Turning Point Problems (SPBTPPs) find applications in various scientific and engineering domains, where the dynamics of a system exhibit abrupt changes near specific points. These scenarios often involve a small parameter multiplying higher-order derivatives, leading to challenges in analyzing and solving differential equations. For instance, in fluid dynamics, singularly perturbed boundary turning point problems may arise when studying the behaviour of a fluid flow near

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critical points, such as stagnation points or boundary layer transitions. In electronics, these problems can be encountered when analyzing circuits with rapidly changing characteristics near certain points. Additionally, applications in chemical engineering, biomechanics, and ecological modelling may involve singularly perturbed systems with turning points. Successfully addressing these problems is crucial for gaining insights into the intricate dynamics of real-world phenomena and for designing efficient and accurate models in diverse fields of study such as [4, 12, 13]. The development of analytical and numerical techniques tailored to these specific problems contributes to advancements in understanding and predicting complex behaviors in various scientific disciplines are discussed in [18–20]. Also the recent literature has developed several computational methods for singularly perturbed differential equations [5, 22–25].

Let us take the following class of singularly perturbed differential equations with boundary-turning point on the domain $G = (0,1)$:

$$\begin{cases} P_\varepsilon y_\varepsilon(t) = \varepsilon y''_\varepsilon(t) + a_\varepsilon(t) y'_\varepsilon(t) - b(t) y_\varepsilon(t) = f(t), \;\; t \in G_- \cup G_+, \\ P_0 y_\varepsilon(0) = b_1 y_\varepsilon(0) - \varepsilon b_2 y'_\varepsilon(0) = A, \\ R_1 y_\varepsilon(1) = b_3 y_\varepsilon(1) + \varepsilon b_4 y'_\varepsilon(1) = B, \\ a_\varepsilon(t) \geq 0, \; b(t) \geq \delta > 0, \; |[f](d)| \leq C, \\ b_3, b_1 > 0, \; b_3 + \varepsilon b_4 \geq 1 \text{ and } b_1 - \varepsilon b_2 \geq 1. \end{cases} \tag{1}$$

In this context, $a_\varepsilon(t)$ and $b(t)$ are defined as functions that exhibit sufficient smoothness over the closed domain $\bar{G}$. Furthermore, $f(t)$ is characterized by adequate smoothness across the domains $G_- \cup G_+$, except a jump discontinuity at the point d. This discontinuity is quantified by $[f](d) = f(d+) - f(d-)$, where A and B represent known constants.

Additionally, we establish certain assumptions regarding the convection coefficient $a_\varepsilon(t)$ as follows:

$$\begin{cases} a_\varepsilon(0) = 0, \\ a_\varepsilon(t) \geq \beta_\varepsilon(t) := \theta(1 - e^{-\frac{r}{\varepsilon}x}), \;\; r \geq 2\theta > 0 \\ \int\limits_{t=0}^{x} |a'_\varepsilon(t)| dt \leq C, \\ d_\varepsilon(t) := a_0(t) - a_\varepsilon(t) \;\text{ satisfies } |d_\varepsilon(t)| \leq |d_\varepsilon(0)| e^{-\frac{\theta}{2\varepsilon}x}, \end{cases} \tag{2}$$

where $a_0(t) := \lim\limits_{\varepsilon \to 0} a_\varepsilon(t)$, $a_0(0) := \lim\limits_{t \to 0} a_0(t)$ and $a_0 \in C^2(\bar{G})$. Since the convection coefficient $a_\varepsilon(t)$ vanishes at $x = 0$, this problem has a boundary turning point at $t = 0$. Additionally, there is a jump discontinuity in the differential equation's source term at a location inside the domain. Therefore, the solution of (1) exhibits two layers namely boundary and interior layer in the neighborhood of the turning point $t = 0$ and the point of discontinuity $t = d$, respectively.

Singularly perturbed differential equations with turning points describe many mathematical models [7, 26] . The authors in [27] have discussed all possible cases modeling turning point behavior. O' Riordan and Quinn [16] have developed parameter uniform numerical method for SPBTPPs. Motivated by the above works, Janani Jayalakshmi and Tamilselvan [8, 10, 11] have developed higher order numerical method for SPBTPPs.

Numerical methods for solving singularly perturbed parabolic partial differential equations with boundary turning point problems was discussed in [1, 9]. Pratima Rai and Kapil K. sharma in [21] have proposed an efficient numerical scheme for a class of singularly perturbed parabolic problems with boundary turning point and retarded arguments on a Shishkin mesh. A linear singularly perturbed turning point problem with interior layer have been studied by O' Riordan and Quinn [17]. Our aim is to compare the results of two finite difference methods used to solve the problem (1). Theorem 1 ensures the existence of a solution for equation (1). Section 2 addresses the uniqueness, stability, and Shishkin decomposition for evaluating the continuous solution and its derivatives. Section 3 focuses on the numerical study, including mesh discretization and error analysis. In Section 4, the theoretical results are experimentally validated, with a comparative discussion of both methods.

The constant C employed in this work is both generic and positive. The discrete norm, defined as $\|u\|_{\bar{\Omega}} = \max_{t\in\bar{\Omega}} |u(t)|$, is employed in the convergence of numerical solutions to analytical solutions.

Theorem 1. *A solution exists for the stated problem (1) in $y_\varepsilon \in C^1(G)\cap C^2(G_-\cup G_+)$.*

Proof. The demonstration is accomplished through construction. Consider specific solutions, denoted as y_1 and y_2, for the given differential equations:

$$\begin{aligned} \varepsilon y_1''(t)+a_\varepsilon(t)y_1'(t)-b(t)y_1(t) &= f(t), \quad t\in G_- \text{ and} \\ \varepsilon y_2''(t)+a_\varepsilon(t)y_2'(t)-b(t)y_2(t) &= f(t), \quad t\in G_+. \end{aligned}$$

Consider the function

$$y(t) = \begin{cases} y_1(t)+(P_0 y_\varepsilon(0)-P_0 y_1(0))\phi_1(t)+l\phi_2(t), & t\in G_-, \\ y_2(t)+m\phi_1(t)+(R_1 y_\varepsilon(1)-R_1 y_2(1))\phi_2(t), & t\in G_+, \end{cases}$$

where $\phi_1(t)$, $\phi_2(t)$ respectively satisfies the following equations

$$\begin{cases} \varepsilon\phi_1''(t)+a_\varepsilon(t)\phi_1'(t)-b(t)\phi_1(t)=0, & t\in G, \\ P_0\phi_1(0)=1, \quad R_1\phi_1(1)=0, \end{cases}$$

$$\begin{cases} \varepsilon\phi_2''(t)+a_\varepsilon(t)\phi_2'(t)-b(t)\phi_2(t)=0, & t\in G, \\ P_0\phi_2(0)=0, \quad R_1\phi_2(1)=1, \end{cases}$$

where l and m are constants, ensuring that $y(t)$ belongs to the space $C^1(G)$. Any function adhering to this structure fulfils the conditions:

$$\begin{cases} \varepsilon y_\varepsilon''(t)+a_\varepsilon(t)y_\varepsilon'(t)-b(t)y_\varepsilon(t)=f(t), & t\in G_-\cup G_+, \\ P_0 y(0)=P_0 y_\varepsilon(0) \text{ and} \\ R_1 y(1)=R_1 y_\varepsilon(1). \end{cases}$$

Note that for all $t\in G$, $0<\phi_i(t)<1,\ i=1,2$. Thus $\phi_1(t),\ \phi_2(t)$ cannot have an internal maximum or minimum and hence

$$\phi_1'(t)<0, \quad \phi_2'(t)>0, \quad \forall\, t\in G.$$

We aim to select the constants l and m in such a way that $y(t)$ belongs to the space $C^1(G)$. This involves imposing the condition:

$$y(d^-)=y(d^+) \quad\text{and}\quad z'(d^-)=z'(d^+).$$

The existence of the constants l and m necessitates that

$$\begin{vmatrix} \phi_2(d) & -\phi_1(d) \\ \phi_2'(d) & -\phi_1'(d) \end{vmatrix} \neq 0.$$

This is a consequence of $\phi_2'(d)\phi_1(d)-\phi_2(d)\phi_1'(d)>0$. □

2 Analysis of analytical solution

This section establishes the minimum principle for Eq. (1) . Additionally, it derives supplementary constraints for the solution, as well as the smooth and layer components along with their respective derivatives.

2.1 Minimum principle and stability result

The operator P_ε in Eq. (1) adheres to the following minimum principle over the closure of $\bar{G}$, substantiating the uniqueness of the solution for (1).

Theorem 2. *Let $\phi(t) \in C^0(\bar{G}) \cap C^2(G_- \cup G_+)$. If $P_0\phi(0) \geq 0$, $P_\varepsilon\phi(t) \leq 0 \forall x \in G_- \cup G_+, [\phi^{'}](d) \leq 0$ and $R_1\phi(1) \geq 0$, then $\phi(t) \geq 0$, $\forall t \in \bar{G}$.*

Proof. The proof follows the methodology in [9, 10].

□

Theorem 3. *If y_ε is the solution to (1), then*

$$|y_\varepsilon(t)| \leq C \max\left\{|P_0 y_\varepsilon(0)|, |R_1 y_\varepsilon(1)|, ||P_\varepsilon y_\varepsilon||_{G_- \cup G_+}\right\}, \ \ \forall t \in \bar{G}.$$

Proof. The proof is available in [9, 10]. □

2.2 Decomposition of continuous solution and their derivative bounds

To derive an error estimate, we require sharper bounds on the derivative of the solution $y_\varepsilon(t)$. We obtain these by decomposing the solution $y_\varepsilon(t)$ into a smooth component $v_\varepsilon(t)$ and a layer component $z_\varepsilon(t)$ using the following Shishkin decomposition:

$$y_\varepsilon(t) = v_\varepsilon(t) + z_\varepsilon(t).$$

As $a_\varepsilon(t)$ does not adhere to the bound $a_\varepsilon(t) \geq C > 0$ for all $t \in \bar{G}$, the smooth component problem is defined as following:

$$\begin{cases} P_* v_\varepsilon(t) = \varepsilon v_\varepsilon''(t) + a_0(t)v_\varepsilon'(t) - b(t)v_\varepsilon(t) = f(t), \ t \in G_- \cup G_+, \\ P_0 v_\varepsilon(0) = P_0 v_0(0) + \varepsilon P_0 v_1(0) + \varepsilon^2 P_0 v_2(0) + \varepsilon^3 P_0 v_3(0), \\ v_\varepsilon(d) = v_0(d) + \varepsilon v_1(d) + \varepsilon^2 v_2(d) + \varepsilon^3 v_3(d), \\ R_1 v_\varepsilon(1) = R_1 y_\varepsilon(1), \end{cases} \tag{3}$$

where $v_\varepsilon(t)$ can be written in the form $v_\varepsilon(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t)$ and the $C^0(D)$ functions $v_0(t)$, $v_1(t)$, $v_2(t)$ and $v_3(t)$ are defined respectively to be the solutions of the problems

$$\begin{aligned} a_0(t)v_0'(t) - b(t)v_0(t) &= f(t), \quad R_1 v_0(1) = B, \\ a_0(t)v_1'(t) - b(t)v_1(t) &= -v_0''(t), \ \ R_1 v_1(1) = 0, \\ a_0(t)v_2'(t) - b(t)v_2(t) &= -v_1''(t), \ \ R_1 v_2(1) = 0, \end{aligned}$$

and

$$\begin{cases} P_* v_3(t) = -v_2''(t), \\ \text{with boundary points and discontinuous point of } v_3(t) \text{ is } 0. \end{cases}$$

Note that in problem (3), the coefficient $a_\varepsilon(t)$, originally present in the first derivative term, has now been substituted with $a_0(t)$. To account for the error $(P_* - P_\varepsilon)v_\varepsilon$, it is incorporated into the layer component z_ε. The latter, defined as the solution of (2) with reference to the relationship (2), encapsulates this modification:

$$\begin{cases} P_\varepsilon z_\varepsilon(t) = d_\varepsilon(t) v_\varepsilon'(t), \\ P_0 z_\varepsilon(0) = P_0 y_\varepsilon(0) - P_0 v_\varepsilon(0), \; R_1 z_\varepsilon(1) = 0, \\ [z_\varepsilon'](d) = -[v_\varepsilon'](d). \end{cases} \tag{4}$$

Further we decompose $z_\varepsilon(t)$ as $z_\varepsilon(t) = z_0(t) + z_d(t)$, where $z_0(t)$ is an boundary layer function, belonging to the class $C^2(G)$, meets the condition

$$\begin{cases} P_\varepsilon z_0(t) = d_\varepsilon(t) v_\varepsilon'(t), \;\; t \in G, \\ P_0 z_0(0) = P_0 y_0(0) - P_0 y_0(0), \\ R_1 z_0(1) = 0, \end{cases} \tag{5}$$

and the interior layer function $z_d(t) \in C^0(G)$ satisfies

$$\begin{cases} P_\varepsilon z_d(t) = 0 \;\; t \in G_- \cup G_+, \\ P_0 z_d(0) = 0, \\ [z_d'](d) = -[v_\varepsilon'](d), \\ R_1 z_d(1) = 0. \end{cases} \tag{6}$$

Theorem 4. *Derivatives of the smooth and layer components solutions $v_\varepsilon(t)$, $z_0(t)$ and $z_d(t)$, respectively of the problems (3), (5) and (6) satisfy*

$$|v_\varepsilon^{(j)}(t)| \le C(1 + \varepsilon^{3-j}), \;\; t \in G_- \cup G_+, \;\; j \in \{1,2,3,4\}$$

$$|[v_\varepsilon](d)|, |[v_\varepsilon'](d)|, |[v_\varepsilon''](d)| \le C,$$

$$|z_0^{(j)}(t)| \le C\varepsilon^{-j} e^{-\theta t/2\varepsilon}, \; t \in \bar{G},$$

$$|z_d| \le C\varepsilon \;\; \text{and} \;\; |z_d^{(j)}(t)| \le \begin{cases} C(\varepsilon^{1-j} e^{-\theta t/2\varepsilon}), & t \in G_-, \\ C(\varepsilon^{1-j} e^{-\theta(t-d)/2\varepsilon}), & t \in G_+. \end{cases}$$

Proof. The proof for finding the bounds for the derivatives of smooth component and left layer component is the same as in [10, Lemma 2.3]. To find the bounds for $z_d(t)$ and its derivative, we use Theorem 2 and the barrier functions $\psi^\pm(t) = \phi(t) \pm z_d(t)$ where

$$\phi(t) = \frac{C\varepsilon}{\theta} \begin{cases} 1, & t \in G_- \cup \{0\}, \\ e^{-\theta(1-d)/2\varepsilon}(2-t), & t \in G_+ \cup \{1\}. \end{cases}$$

□

3 Analysis of numerical methods

3.1 Discretization of the mesh

The construction of a piecewise uniform mesh on G with M mesh intervals is done as follows. Let $[0,\sigma_1]$, $[\sigma_1,d]$, $[d,d+\sigma_2]$, and $[d+\sigma_2,1]$ be the four subintervals that make up the domain $\bar{G}$, where σ_1 and σ_2 transition points satisfying

$$\sigma_1 = \min\left\{\frac{d}{2}, \frac{4\varepsilon}{\theta}\ln M\right\} \text{ and } \sigma_2 = \min\left\{\frac{1-d}{2}, \frac{4\varepsilon}{\theta}\ln M\right\}.$$

Each interval comprises $M/4$ mesh points. The interior mesh points are denoted by

$$G_\varepsilon^M = \{t_i : 1 \le i \le \frac{M}{2} - 1\} \cup \{t_i : \frac{M}{2} + 1 \le i \le M-1\}.$$

Clearly $t_{M/2} = d$ and $\bar{G}_\varepsilon^M = \{t_i\}_0^M$. The mesh widths are given by

$$h_i = t_i - t_{i-1} = \begin{cases} \Delta_1 = \frac{4\sigma_1}{M}, & i = 1,2,\cdots,M/4-1, \\ \Delta_2 = \frac{4(d-\sigma_1)}{M}, & i = M/4,\cdots,M/2, \\ \Delta_3 = \frac{4\sigma_2}{M}, & i = M/2+1\cdots,3M/4, \\ \Delta_4 = \frac{4(1-d-\sigma_2)}{M}, & i = 3M/4+1,\cdots,N. \end{cases}$$

3.2 Method I : Finite Difference Method

The problem (1) uses a finite difference approach:

$$\begin{cases} P_\varepsilon^N y_\varepsilon(t_i) = (\varepsilon\delta^2 + a_\varepsilon D^+ - c)y_\varepsilon(t_i) = f(t_i), \quad t_i \in G_\varepsilon^M, \\ P_0^N y_\varepsilon(t_0) = b_1 y_\varepsilon(t_0) - \varepsilon b_2 D^+ y_\varepsilon(t_0) = A, \\ P_d^N y_\varepsilon(t_{M/2}) = D^+ y_\varepsilon(t_{M/2}) - D^- y_\varepsilon(t_{M/2}) = 0, \\ R_1^N y_\varepsilon(t_N) = b_3 y_\varepsilon(t_N) + \varepsilon b_4 D^- y_\varepsilon(t_N) = B. \end{cases} \tag{7}$$

For any mesh function $V_i = V(t_i)$, we have

$$\delta^2 V_i = \frac{(D^+ - D^-)V_i}{(t_{i+1} - t_{i-1})/2}, \quad D^+ V_i = \frac{V_{i+1} - V_i}{(t_{i+1} - t_i)} \text{ and } D^- V_i = \frac{V_i - V_{i-1}}{(t_i - t_{i-1})}.$$

3.3 Method II : Hybrid Difference Method

Since M^{-1} gets exponentially tiny in relation to ε, let us suppose $\sigma_1 = \sigma_2 = \sigma = \frac{4\varepsilon}{\theta}\ln M$, where θ denotes the lower bound of $a_0(t)$. Using the midpoint technique in the fine mesh region, we discretize (1) using a central difference scheme, as per Method II.

$$\begin{aligned} P_c^N y_\varepsilon(t_i) \;=\; & \frac{2\varepsilon}{\Delta_i + \Delta_{i+1}}\left[\frac{y_\varepsilon(t_{i+1}) - y_\varepsilon(t_i)}{h_{i+1}} - \frac{y_\varepsilon(t_i) - y_\varepsilon(t_{i-1})}{h_i}\right] \\ & + a_\varepsilon(t_i)\left[\frac{y_\varepsilon(t_{i+1}) - y_\varepsilon(t_{i-1})}{h_i + h_{i+1}}\right] - c(t_i)y_\varepsilon(t_i) = f(t_i). \end{aligned} \tag{8}$$

Using the midpoint technique in the fine mesh region, we have

$$P_m^N y_\varepsilon(t_i) = \frac{2\varepsilon}{h_i+h_{i+1}}\left[\frac{y_\varepsilon(t_{i+1})-y_\varepsilon(t_i)}{h_{i+1}} - \frac{y_\varepsilon(t_i)-y_\varepsilon(t_{i-1})}{h_i}\right] + \bar{b}_\varepsilon(t_i)$$
$$\left[\frac{y_\varepsilon(t_{i+1})-y_\varepsilon(t_i)}{h_{i+1}}\right] - \left[\frac{c(t_i)y_\varepsilon(t_i)+c(t_{i+1})y_\varepsilon(t_{i+1})}{2}\right] = \bar{f}(t_i) \tag{9}$$

At the point $t_{M/2} = d$, we utilize the average operators for $b_\varepsilon(t_i)$ and $f(t_i)$, denoted as $\bar{b}\varepsilon(t_i)$ and $\bar{f}(t_i)$, respectively. Specifically, $\bar{b}\varepsilon(t_i)$ is defined as $(b_\varepsilon(t_i)+b_\varepsilon(t_{i+1}))/2$, and similarly for $\bar{f}(t_i)$. At this juncture, we employ the difference operator $P_t^N y_\varepsilon(t_{M/2})$:

$$P_t^N y_\varepsilon(t_{M/2}) \equiv \frac{-y_\varepsilon(t_{M/2+2})+4y_\varepsilon(t_{M/2+1})-3y_\varepsilon(t_{M/2})}{h_{M/2+1}+h_{M/2}}$$
$$-\left(\frac{y_\varepsilon(t_{M/2-2})-4y_\varepsilon(t_{M/2-1})+3y_\varepsilon(t_{M/2})}{h_{M/2}+h_{M/2-1}}\right) = 0. \tag{10}$$

There is no M-matrix linked with (8), (9), and (10). In order to solve this, we convert the Eqs. (8), (9), and (10) in a manner that guarantees the monotonicity property of the resulting equation. Eqs. (8) and (9) as a starting point give us:

$$y_\varepsilon(t_{M/2-2}) = \frac{\Delta_2}{\varepsilon}\left[\Delta_2\bar{f}(t_{M/2-1}) + \left(\frac{4\varepsilon+2\Delta_2\bar{b}_\varepsilon(t_{M/2-1})+\Delta_2^2 c(t_{M/2-1})}{2\Delta_2}\right)y_\varepsilon(t_{M/2-1})\right.$$
$$\left. -\left(\frac{2\varepsilon+2\Delta_2\bar{b}_\varepsilon(t_{M/2-1})-\Delta_2^2 c(t_{M/2})}{2\Delta_2}\right)y_\varepsilon(t_{M/2})\right], \tag{11}$$

$$y_\varepsilon(t_{M/2+2}) = \frac{2\Delta_3}{2\varepsilon+\Delta_3 b_\varepsilon(t_{M/2+1})}\left[\Delta_3 f(t_{M/2+1}) + \left(\frac{2\varepsilon+\Delta_3^2 c(t_{M/2+1})}{\Delta_3}\right)y_\varepsilon(t_{M/2+1})\right.$$
$$\left. -\left(\frac{2\varepsilon-\Delta_3 b_\varepsilon(t_{M/2+1})}{2\Delta_3}\right)y_\varepsilon(t_{M/2})\right]. \tag{12}$$

Inserting the values of $y_\varepsilon(t_{M/2-2})$ and $y_\varepsilon(t_{M/2+2})$ into (10) yields:

$$P_T^N y_\varepsilon(t_{M/2}) := \frac{1}{2\Delta_2}\left[4-\left(\frac{4\varepsilon+2\Delta_2\bar{b}_\varepsilon(t_{M/2-1})+\Delta_2^2 c(t_{M/2-1})}{2\varepsilon}\right)\right]y_\varepsilon(t_{M/2-1})$$
$$-\left[\frac{3}{2\Delta_3}+\frac{3}{2\Delta_2}-\frac{1}{2\Delta_3}\left(\frac{2\varepsilon-\Delta_3 b_\varepsilon(t_{M/2+1})}{2\varepsilon+\Delta_3 b_\varepsilon(t_{M/2+1})}\right)\right.$$
$$\left.-\left(\frac{2\varepsilon+2\Delta_2\bar{b}_\varepsilon(t_{M/2-1})-\Delta_2^2 c(t_{M/2})}{4\varepsilon\Delta_2}\right)\right]y_\varepsilon(t_{M/2})$$
$$+\frac{1}{2\Delta_3}\left[4-\left(\frac{4\varepsilon+2\Delta_3^2 c(t_{M/2+1})}{2\varepsilon+\Delta_3 b_\varepsilon(t_{M/2+1})}\right)\right]y_\varepsilon(t_{M/2+1})$$
$$= \frac{\Delta_3 f(t_{M/2+1})}{2\varepsilon+\Delta_3 b_\varepsilon(t_{M/2+1})} + \frac{\Delta_2\bar{f}(t_{M/2-1})}{2\varepsilon}. \tag{13}$$

Using the central difference operator, the derivative term $P_0 z_\varepsilon(0)$ in the left boundary condition is approximated, yielding:

$$y_\varepsilon(t_{-1}) = -\frac{2\Delta_1 b_1}{\varepsilon b_2} y_\varepsilon(t_0) + y_\varepsilon(t_1) + \frac{2\Delta_1 A}{\varepsilon b_2}. \tag{14}$$

Assuming that the difference Eqs. (8) hold for $i = 0$, it is possible to delete the value $y_\varepsilon(t_{-1})$. Eq. (8) may be obtained by substituting the value of $y_\varepsilon(t_{-1})$ for $i = 0$ in (14).

Assuming that the difference Eqs. (8) apply for $i = 0$, it is possible to delete the value $Z_\varepsilon(t_{-1})$. Upon substituting $Z_\varepsilon(t_{-1})$ for $i = 0$ in Eq. (8) from (14), then

$$P^N_{H0} y_\varepsilon(t_0) = \left[-\frac{2\varepsilon}{\Delta_1^2} - \frac{2b_1}{\Delta_1 b_2} + \frac{a_\varepsilon(t_0) b_1}{\varepsilon b_2} - c(t_0) \right] y_\varepsilon(t_0) + \frac{2\varepsilon}{\Delta_1^2} y_\varepsilon(t_1) = f(t_0) - \frac{2A}{\Delta_1 b_2} + \frac{a_\varepsilon(t_0) A}{\varepsilon b_2}. \tag{15}$$

Therefore, for the boundary turning point problem (1), the higher-order approach is as follows:

$$\begin{cases} P^N_H y_\varepsilon(t_i) = \begin{cases} P^N_c y_\varepsilon(t_i) = f(t_i), \text{ for } 1 \le i \le M/4 - 1 \text{ and } M/2 + 1 \le i \le 3M/4 - 1, \\ P^N_m y_\varepsilon(t_i) = \bar{f}(t_i), \text{ for } M/4 \le i \le M/2 - 1 \text{ and } 3M/4 \le i \le M - 1, \\ P^N_T y_\varepsilon(t_i) = \dfrac{\Delta_3 f(t_{i+1})}{2\varepsilon + \Delta_3 b_\varepsilon(t_{i+1})} + \dfrac{\Delta_2 \bar{f}(t_{i-1})}{2\varepsilon}, \text{ for } \ i = M/2, \end{cases} \\ P^N_{H0} y_\varepsilon(t_0) = f(t_0) - \dfrac{2A}{\Delta_1 b_2} + \dfrac{a_\varepsilon(t_0) A}{\varepsilon b_2} = A_0, \\ R^N_{H1} y_\varepsilon(t_N) = b_3 y_\varepsilon(t_N) + \varepsilon b_4 D^- y_\varepsilon(t_N) = B. \end{cases} \tag{16}$$

3.4 Error analysis for Method I

Error analysis for Method I focuses on evaluating its accuracy and convergence by comparing the numerical solution to the exact solution. It involves assessing the convergence rate and identifying sources of error, such as discretization or round-off errors, to determine the method's reliability and efficiency.

Theorem 5. *Let $\phi(t_i)$ represent any mesh function defined on $\bar{G}^M_\varepsilon$ that satisfies the conditions $P^N_0 \phi(t_0) \ge 0$, $P^N_H \phi(t_i) \le 0$, $t_i \in G^M_\varepsilon$, $P^N_d \phi(t_{M/2}) \le 0$ and $R^N_1 \phi(t_N) \ge 0$. Then $\phi(t_i) \ge 0$ for all $t_i \in \bar{G}^M_\varepsilon$.*

Proof. The proof is available in [9, 10]. □

Theorem 6. *The following bound is satisfied by the numerical solution y_ε of (7):*

$$|y_\varepsilon(t_i)| \le C \max \left\{ |P^N_0 y_\varepsilon(t_0)|, |R^N_1 y_\varepsilon(t_N)|, ||P^N_\varepsilon y_\varepsilon||_{G^M_\varepsilon} \right\}, \ \ \forall t_i \in \bar{G}^M_\varepsilon.$$

Proof. This theorem may be proved with Theorem 5 and the barrier functions, much like in the continuous case

$$\psi^\pm(t_i) = CK \pm y_\varepsilon(t_i) \ \ \forall t_i \in \bar{G}^M_\varepsilon,$$

where $K = \max \left\{ |P^N_0 y_\varepsilon(t_0)|, |R^N_1 y_\varepsilon(t_N)|, ||P^N_\varepsilon y_\varepsilon||_{G^M_\varepsilon} \right\}$. □

Define $V_\varepsilon(t_i)$, satisfies

$$\begin{cases} P^N_* V_\varepsilon(t_i) = f(t_i), \ t_i \in G^M_\varepsilon, \\ P^N_0 V_\varepsilon(t_0) = P_0 v_\varepsilon(0), \ V_\varepsilon(d) = v_\varepsilon(d) \ \text{ and} \\ R^N_1 V_\varepsilon(t_N) = R_1 v_\varepsilon(1), \end{cases} \tag{17}$$

and the discrete layer component $Z_\varepsilon(t_i)$ to be the solution of

$$\begin{cases} P_\varepsilon^N Z_\varepsilon(t_i) = (P_*^N - P_\varepsilon^N)V_\varepsilon(t_i),\ t_i \in G_\varepsilon^M, \\ P_d^M Z_\varepsilon(t_{M/2}) = -P_d^N V_\varepsilon(t_{M/2}), \\ P_0^N Z_\varepsilon(t_0) = P_0 y_\varepsilon(0) - P_0 v_\varepsilon(0) \ \text{ and} \\ R_1^N Z_\varepsilon(t_N) = R_1 z_\varepsilon(1). \end{cases} \tag{18}$$

Additionally, the discrete layer element $Z_\varepsilon(t_i)$ is broken down as $Z_\varepsilon(t_i) = Z_0(t_i) + Z_d(t_i)$ where it is satisfied by $Z_0(t_i)$

$$\begin{cases} P_\varepsilon^N Z_0(t_i) = (P_*^N - P_\varepsilon^N)V_\varepsilon(t_i) \ \ t_i \in G_\varepsilon^M \cup \{d\}, \\ P_0^N Z_0(t_0) = P_0 y_\varepsilon(0) - P_0 v_\varepsilon(0) \ \text{ and} \\ R_1^N Z_0(t_N) = 0, \end{cases} \tag{19}$$

and $Z_d(t_i)$ satisfies

$$\begin{cases} P_\varepsilon^N Z_d(t_i) = 0, \ \ t_i \in G_\varepsilon^M, \\ P_0^N Z_d(t_0) = 0, \\ P_d^M Z_d(t_{M/2}) = -(P_d^M Z_0(t_{M/2}) + P_d^N V_\varepsilon(t_{M/2})) \text{ and} \\ R_1^N Z_d(t_N) = 0. \end{cases} \tag{20}$$

Theorem 7. *Let v_ε and V_ε be the solutions of (3) and (17), respectively. Then*

$$|(V_\varepsilon - v_\varepsilon)(t_i)| \le \begin{cases} CM^{-1}(1-t_i),\ 0 \le t_i < d, \\ CM^{-1}(2-t_i),\ d < t_i \le 1. \end{cases}$$

Proof. Utilizing $t_i - t_{i-1} = M^{-1}$, $t_{i+1} - t_{i-1} = 2M^{-1}$, and Theorem 4, then

$$|P_\varepsilon^N (V_\varepsilon - v_\varepsilon)(t_i)| \le \frac{\varepsilon}{3}(t_{i+1} - t_{i-1})|v_\varepsilon^{(3)}| + \frac{a_\varepsilon(t_i)}{2}(t_{i+1} - t_i)|v_\varepsilon^{(2)}| \le CM^{-1}, \quad \forall t_i \in G_\varepsilon^N.$$

Now

$$\begin{aligned} P_0^N(V_\varepsilon - v_\varepsilon)(t_0) &= b_1 v_\varepsilon(t_0) - \varepsilon b_2 v_\varepsilon'(t_0) - b_1 v_\varepsilon(t_0) + \varepsilon b_2 D^+ v_\varepsilon(t_0) \\ \left|P_0^N(V_\varepsilon - v_\varepsilon)(t_0)\right| &\le \frac{\varepsilon b_2}{2}(t_1 - t_0)|v_\varepsilon^{(2)}| \le CM^{-1}. \end{aligned}$$

Similarly $\left|R_1^N(V_\varepsilon - v_\varepsilon)(t_N)\right| \le CM^{-1}$.

Consider $\Phi^\pm(t_i) = \phi(t_i) \pm (V_\varepsilon - v_\varepsilon)(t_i)$, where

$$\phi(t_i) = \begin{cases} CM^{-1}\dfrac{(1-t_i)}{1-d}, \ \ 0 \le t_i \le d, \\ CM^{-1}\dfrac{(2-t_i)}{2-d}, \ \ d \le t_i \le 1. \end{cases}$$

Using $\Phi^\pm(t_i)$ as the mesh functions, Theorem 5 may be used to obtain

$$\Phi^\pm(t_i) \ge 0,\ \forall t_i \in \bar{G}_\varepsilon^M.$$

Therefore, we achieve the desired result. □

Theorem 8. *Let Z_0 and z_0 be the solutions of the problems (19) and (5), respectively. Then,*

$$|(Z_0 - z_0)(t_i)| \leq CM^{-1},\ t_i \in \bar{G}_{\varepsilon}^{M}.$$

Proof. The singular component error estimate is contingent upon the mesh parameters σ_1 and σ_2. Based on the values of σ_1 and σ_2, we examine two cases:
Case (i): $\sigma_1 = \frac{d}{2}$ and $\sigma_2 = \frac{1-d}{2}$. The mesh is uniform in this instance, and $\varepsilon^{-1} \leq C \ln M$. The following results are obtained by using the constraints on the derivatives of z_0 and the knowledge that $t_{i+1} - t_i = M^{-1}$ and $t_{i+1} - t_{i-1} = 2M^{-1}$:

$$\begin{aligned} P_0^N (Z_0 - z_0)(t_0) &= P_0^N Z_0(t_0) - P_0^N z_0(t_0), \\ \left| P_0^N (Z_0 - z_0)(t_0) \right| &\leq \frac{\varepsilon b_2}{2}(t_1 - t_0)|z_0^{(2)}| \leq CM^{-1}(\ln M). \end{aligned}$$

Similarly, $|R_1^N (Z_0 - z_0)(t_N)| \leq CM^{-1}(\ln M)$. Referring to [2, Lemma 3.14], we have

$$|P_{\varepsilon}^N (Z_0 - z_0)(t_i)| \leq CM^{-1}.$$

By applying Theorem 6, we obtain the desired result.
Case(ii): $\sigma_1 = \sigma_2 = \sigma = \dfrac{4\varepsilon \ln M}{\theta}$. In this case, the mesh is piecewise uniform. The mesh spacing in the subintervals $(0, \sigma)$ and $(\sigma, 1)$ is $\Delta_1 = 2\sigma/M$ and $\Delta_2 = 2(1-\sigma)/M$, respectively. Given $t_i \in [\sigma, 1)$, we obtain using triangle inequality

$$|(Z_0 - z_0)(t_i)| \leq |Z_0(t_i)| + |z_0(t_i)|.$$

Using $z_{\varepsilon}(t)$ as the bound, we obtain

$$|z_0(t_i)| \leq Ce^{-\theta\sigma/2\varepsilon} \leq CM^{-1}.$$

Following the procedure given in [2, Lemma 3.5] we can established the bound $|Z_0(t_i)| \leq CM^{-1}$. Consequently, we have for each $t_i \in (\sigma, 1)$

$$|(Z_0 - z_0)(t_i)| \leq CM^{-1}.$$

In the event that $t_i \in (0, \sigma)$, we have

$$|P_{\varepsilon}^N (Z_0 - z_0)(t_i)| \leq CM^{-1}.$$

Next, using Theorem 6, we obtain $|(Z_0 - z_0)(t_i)| \leq CM^{-1}$, $t_i \in \bar{D}_{\varepsilon}^N$. □

Lemma 1. *According to [3, Lemma 6 and 7], let $Z_d(t_i)$ be the solution of (20). Then it satisfies $|P_d^N Z_d(d)| \leq C(1 + \varepsilon^{-1}M^{-1})$ and $|Z_d(t_i)| \leq C\varepsilon |P_d^N Z_d(d)|$.*

Remarks: It is easy to infer from the previous lemma for $\varepsilon \leq M^{-1}$ that

$$|Z_d(t_i) - z_d(t_i)| \leq C(\varepsilon + M^{-1}) \leq CM^{-1}.$$

Theorem 9. *Let the solutions to (20) and (6) be Z_d and z_d. Subsequently, the internal layer component's error fulfils*

$$|(Z_d - z_d)(t_i)| \leq CM^{-1}(\ln M), \; t_i \in \bar{G}_\varepsilon^M. \tag{21}$$

Proof. Using the constraints on the derivatives of z_d and the usual truncation errors, we have

$$|P_0^N(Z_d - z_d)(t_0)| \leq CM^{-1}(\ln M), \quad |R_1^N(Z_d - z_d)(t_N)| \leq CM^{-1}(\ln M)$$

and

$$\begin{aligned}|P_\varepsilon^N(Z_d - z_d)(t_i)| &\leq \frac{\varepsilon}{3}(t_{i+1} - t_{i-1})|z_d^{(3)}| + \frac{a_\varepsilon(t_i)}{2}(t_{i+1} - t_i)|z_d^{(2)}| \leq CM^{-1}\ln M, \text{ for } t_i \in (0, \sigma_1),\\ |P_\varepsilon^N(Z_d - z_d)(t_i)| &\leq C||\varepsilon z_d''||_{(t_{i+1}-t_{i-1})} + C||z_d'||_{[t_{i+1}-t_1]}, \leq Ce^{-\theta\sigma_1/\varepsilon} \leq CM^{-2}, \text{ for } t_i \in (\sigma_1, d).\end{aligned}$$

Similarly, we have

$$\begin{aligned}|P_\varepsilon^N(Z_d - z_d)(t_i)| &\leq CM^{-1}\ln M, \text{ for } t_i \in (d, d + \sigma_2),\\ |P_\varepsilon^N(Z_d - z_d)(t_i)| &\leq CM^{-2}, \text{ for } t_i \in (d + \sigma_2, 1).\end{aligned}$$

Next, we look at the truncation error magnitude at $x = d$, the discontinuity point. Remember that $[z_d'](d) + [v_\varepsilon'](d) = 0$. As a result

$$P_d^N(Z_d - z_d)(d) = P_d^N Z_d(d) - P_d^N z_d(d) = [v_\varepsilon'](d) - P_d^N V_\varepsilon(d) + [z_d'] - P_d^N z_d(d) - P_d^N Z_0(d). \tag{22}$$

Consider

$$[v_\varepsilon'](d) - P_d^N V_\varepsilon(d) = v_\varepsilon'(d^+) - D^+ v_\varepsilon(d) + D^- v_\varepsilon(d) - v_\varepsilon'(d^-) + P_d^N(V_\varepsilon - v_\varepsilon)(d).$$

Based on [3, Lemma 6], we can deduce

$$|P_d^N(V_\varepsilon - v_\varepsilon)(d)| \leq C\varepsilon^{-1}M^{-1}$$

and

$$|[v_\varepsilon'](d) - P_d^N v_\varepsilon(d)| \leq CM^{-1}||v_\varepsilon^{(2)}||_{G_- \cup G_+} \leq CM^{-1}.$$

Hence

$$|[v_\varepsilon'](d) - P_d^N V_\varepsilon(d)| \leq C\varepsilon^{-1}M^{-1}.$$

Similarly,

$$|[z_d'](d) - P_d^N z_d(d)| \leq C\Delta_3||z_d^{(2)}||_{[d, d+\Delta_3]} + C\Delta_2||z_d^{(2)}||_{[d-\Delta_2, d]} \leq CM^{-1}(\ln M).$$

Since $e^{-\theta(d-\Delta_2)/\varepsilon} \leq e^{-\theta d/2\varepsilon}$, we have

$$|P_d^N z_0(d)| \leq C(\Delta_2 + \Delta_3)\varepsilon^{-2}e^{-\theta(d-\Delta_2)/\varepsilon} \leq CM^{-1}.$$

Thus

$$|P_d^N(Z_d - z_d)(d)| \leq C(\varepsilon M)^{-1}\ln M.$$

Consider the following barrier function

$$\psi^{\pm}(t_i) = CM^{-1}(\ln M)\begin{cases}1, & t_i \le d\\ \phi, & t_i \ge d\end{cases} + CM^{-1}\ln M(1-t_i) \pm (Z_d - z_d)(t_i), \tag{23}$$

where ϕ is the solution of

$$\varepsilon\delta^2\phi + \theta D^+\phi = 0, \ \ \phi(d) = 1, \ R_1^N\phi(t_N) = 0.$$

For $t_i \ge d$, $D^+\phi(t_i) < 0$, as demonstrated via a straightforward contradiction argument. The use of (23) and Theorem 5 completes the proof. □

Theorem 10. *Let y_ε, y_ε be the solutions of (1), (7), respectively . Then, $\forall\ M \ge 4$, we have*

$$\sup_{0<\varepsilon\le 1} \|y_\varepsilon - y_\varepsilon\|_{\bar{G}_\varepsilon^N} \le CM^{-1}(\ln M).$$

Proof. The error estimates of the form

$$(y_\varepsilon - y_\varepsilon)(t_i) = ((V_\varepsilon - v_\varepsilon) + (Z_0 - z_0) + (Z_d - z_d))(t_i), \ \ \forall t_i \in \bar{G}_\varepsilon^M. \tag{24}$$

The necessary outcome is then obtained from Theorems 7, 8, and 9. □

3.5 Error analysis for Method II

3.5.1 Truncation errors for the problem (16)

We have

$$\begin{aligned} P_{H0}^N(y_\varepsilon - y_\varepsilon)(t_0) \ =\ & \varepsilon y_\varepsilon''(t_0) + a_\varepsilon(t_0)y_\varepsilon'(t_0) - c(t_0)y_\varepsilon(t_0) + \left(\frac{a_\varepsilon(t_0)}{\varepsilon b_2} - \frac{2}{\Delta_1\alpha_2}\right)\\ & [b_1 y_\varepsilon(0) - \varepsilon\alpha_2 y_\varepsilon'(0)] - \left[\left(-\frac{2\varepsilon}{\Delta_1^2} - \frac{2b_1}{\Delta_1\alpha_2} + \frac{a_\varepsilon(t_0)b_1}{\varepsilon b_2} - c(t_0)\right)y_\varepsilon(t_0)\right.\\ & \left. + \left(\frac{2\varepsilon}{\Delta_1^2}\right)\left(y_\varepsilon(t_0) + \Delta_1 y_\varepsilon'(t_0) + \frac{\Delta_1^2}{2}y_\varepsilon''(t_0) + \frac{\Delta_1^3}{6}y_\varepsilon'''(t_0) + \cdots\right)\right]. \end{aligned}$$

Thus

$$|P_{H0}^N(y_\varepsilon - y_\varepsilon)(t_0)| \le \ C\varepsilon\Delta_1|y_\varepsilon^{(3)}(t_0)| \le C\Delta_1^2|y_\varepsilon^{(3)}(t_0)| \tag{25}$$

Moreover, from [6], we have

$$|P_H^N(y_\varepsilon - y_\varepsilon)(t_i)| \le \begin{cases}\varepsilon h_i^2\|y_\varepsilon^{(4)}\| + h_i^2\|a_\varepsilon\|\|y_\varepsilon^{(3)}\|, \ 1 \le i \le M/4-1, \ M/2+1 \le i \le 3M/4-1,\\ \varepsilon h_i\|y_\varepsilon^{(3)}\| + C_{(\|a_\varepsilon\|,\|a_\varepsilon'\|)}h_i^2(\|y_\varepsilon^{(3)}\| + \|y_\varepsilon^{(2)}\|), \ M/4 \le i \le M/2-1, \ 3M/4 \le i \le M-1.\end{cases} \tag{26}$$

For $i = M/2$, we have

$$\begin{aligned} |P_H^N(y_\varepsilon - y_\varepsilon)(t_i)| \ \le\ & \left|\frac{y_\varepsilon(t_{i-2}) - 4y_\varepsilon(t_{i-1}) + 3y_\varepsilon(t_i)}{h_i + h_{i-1}} - y_\varepsilon'(t_i)\right| + \left|\frac{-y_\varepsilon(t_{i+2}) + 4y_\varepsilon(t_{i+1}) - 3y_\varepsilon(t_i)}{h_{i+1} + h_i} - y_\varepsilon'(t_i)\right|\\ & + C|P_c^N y_\varepsilon(t_{i+1}) - P_\varepsilon y_\varepsilon(t_{i+1})| + C|P_m^N y_\varepsilon(t_{i-1}) - P_\varepsilon y_\varepsilon(t_{i-1})|. \end{aligned} \tag{27}$$

Further we have,

$$\left|R_{H1}^N(y_\varepsilon - y_\varepsilon)(t_N)\right| \le \frac{\varepsilon b_4}{2}(t_M - t_{M-1})|y_\varepsilon^{(2)}| \le C\varepsilon 4|y_\varepsilon^{(2)}|. \tag{28}$$

Note that $||c||h_i < 2\theta$ in $(d-\sigma_1, d)$ and $(d+\sigma_2, 1)$. We place the following modest assumption on the minimal number of mesh points in order to ensure the monotonicity property of the difference operator P_H^N:

$$\frac{M}{\ln\, M} \ge 2\frac{\|a_\varepsilon\|}{\theta}. \tag{29}$$

3.5.2 Error estimate

Theorem 11. *[15]. Let $\phi(t_i)$ be any mesh function on $\bar{G}_\varepsilon^M$ satisfying $L_{H0}^N\phi(t_0) \ge 0$, $P_H^N\phi(t_i) \le 0$, $t_i \in G_\varepsilon^M$, $P_T^N\phi(t_{M/2}) \le 0$ and $R_{H1}^N\phi(t_N) \ge 0$. Then $\phi(t_i) \ge 0$, $\forall t_i \in \bar{G}_\varepsilon^M$.*

Theorem 12. *[15]. We have $\|y_\varepsilon\|_{\bar{G}_\varepsilon^M} \le C\ \max\left\{|P_{H0}^N y_\varepsilon(t_0)|,\ |R_{H1}^N y_\varepsilon(t_N)|, \|P_H^N y_\varepsilon\|_{G_\varepsilon^M}\right\}$.*

The discrete problem given below is satisfied by the regular component $V_\varepsilon(t_i)$:

$$\begin{cases} P_{*H}^N V_\varepsilon(t_i) = f_H(t_i),\ t_i \in G_\varepsilon^M, \\ P_{H0}^N V_\varepsilon(t_0) = P_0 v_\varepsilon(0),\ V_\varepsilon(t_{M/2}) = v_\varepsilon(d) \ \text{ and }\ R_{H1}^N V_\varepsilon(t_N) = R_1 v_\varepsilon(1). \end{cases}$$

The mesh function $Z_\varepsilon(t_i)$ is defined as the solution to

$$\begin{cases} P_H^N Z_\varepsilon(t_i) = (P_{*H}^N - P_H^N)V_\varepsilon(t_i),\ t_i \in G_\varepsilon^M, L_{H0}^N Z_\varepsilon(t_0) = P_0 y_\varepsilon(0) - P_0 v_\varepsilon(0), \\ P_T^M Z_\varepsilon(t_{M/2}) = -P_T^N V_\varepsilon(t_{M/2}) \ \text{ and }\ L_{H1}^N Z_\varepsilon(t_N) = R_1 z_\varepsilon(1). \end{cases} \tag{30}$$

Additionally, the discrete layer component $Z_\varepsilon(t_i)$ is broken down into the discrete interior layer function $Z_d(t_i)$ and discrete boundary layer function $Z_0(t_i)$, where $Z_0(t_i)$ fulfils the following

$$\begin{cases} P_H^N Z_0(t_i) = (P_{*H}^N - P_H^N)V_\varepsilon(t_i),\ \ t_i \in G_\varepsilon^M \cup \{d\}, \\ L_{H0}^N Z_0(t_0) = P_0 y_\varepsilon(0) - P_0 v_\varepsilon(0) \ \text{ and }\ R_{H1}^N Z_0(t_N) = 0, \end{cases} \tag{31}$$

and $Z_d(t_i)$ is the solution of

$$\begin{cases} P_H^N Z_d(t_i) = 0,\ \ t_i \in G_\varepsilon^M, \\ P_T^M Z_d(t_{M/2}) = -P_T^M Z_0(t_{M/2}) - P_T^N V_\varepsilon(t_{M/2}), \\ L_{H0}^N Z_d(t_0) = 0 \ \text{ and }\ R_{H1}^N Z_d(t_N) = 0. \end{cases} \tag{32}$$

Theorem 13. *Let $v_\varepsilon(t)$ and $V_\varepsilon(t_i)$ represent the numerical and analytical solutions to (30), (3), respectively. Then the smooth component's error for N satisfies*

$$|(V_\varepsilon - v_\varepsilon)(t_i)| \quad \le \quad CM^{-2},\ t_i \in \bar{G}_\varepsilon^M.$$

Proof. Using constraints on the derivatives of v_ε and $\varepsilon \le CM^{-1}$, as well as the typical truncation error bounds provided in Subsection 3.5.1, we have

$$|L_{H0}^N(V_\varepsilon - v_\varepsilon)(t_0)| \le CM^{-2}, \quad \left|R_{H1}^N(V_\varepsilon - v_\varepsilon)(t_N)\right| \le CM^{-2}$$

and

$$|P_H^N(V_\varepsilon - v_\varepsilon)(t_i)| \le \begin{cases} CM^{-2}, \ \ 1 \le i \le M/4-1, \ M/2+1 \le i \le 3M/4-1, \\ CM^{-1}(\varepsilon + M^{-1}), \ M/4 \le i \le M/2-1, \ 3M/4 \le i \le M-1. \end{cases}$$
$$\le CM^{-2}, \ t_i \in G_\varepsilon^M.$$

Using Theorem 12, we get $|V_\varepsilon(t_i) - v_\varepsilon(t_i)| \le CM^{-2}, \ t_i \in \bar{G}_\varepsilon^M$. □

Theorem 14. *Let Z_0 and z_0 represent the corresponding solutions to the problems (31) and (5). Subsequently, the left layer component's error satisfies*

$$|(Z_0 - z_0)(t_i)| \le CM^{-2}(\ln M)^2, \ t_i \in \bar{G}_\varepsilon^M.$$

Proof. The mesh is piecewise uniform and $\sigma = (4\varepsilon \ln M)/\theta$. For $t_i, \ M/4 \le i \le M$, using triangle inequality, we have

$$|(Z_0 - z_0)(t_i)| \le |Z_0(t_i)| + |z_0(t_i)|.$$

By using Theorem 4, we obtain

$$|(Z_0 - z_0)(t_i)| \le CM^{-2}, \ \text{for } t_i \in [\sigma, 1].$$

In this case, the mesh spacing for $t_i \in [0, \sigma)$ is $h = 2\sigma/M$. Applying the derivative constraints on z_0 and usual truncation errors, we have

$$|L_{H0}^N(Z_0 - z_0)(t_0)| \le CM^{-2}(\ln M)^2 \ \text{ and } \ \left|L_{H1}^N(Z_0 - z_0)(t_{M/4})\right| \le CM^{-2}.$$

Now

$$P_H^N(Z_0 - z_0)(t_i) = P_H^N Z_0(t_i) - P_H^N z_0(t_i) = d_\varepsilon(t_i) D^0(V_\varepsilon - v_\varepsilon).$$

From [14, Lemma 5.2], we have

$$|P_H^N(Z_0 - z_0)(t_i)| \le CM^{-2}.$$

Using this in Theorem 12, we obtain $|(Z_0 - z_0)(t_i)| \le CM^{-2}(\ln M)^2$ for $t_i \in \bar{G}_\varepsilon^M$. □

Theorem 15. *Let the equivalent solutions to (32) and (6) be Z_d and z_d. Therefore, the error of the interior layer component fulfils*

$$|(Z_d - z_d)(t_i)| \le CM^{-2}(\ln M)^2, \ t_i \in \bar{G}_\varepsilon^M.$$

Proof. Applying Theorem 4 and using the truncation errors given in (26), we get

$$|L_{H0}^N(Z_d - z_d)(t_0)| \le C\Delta_1^2 |z_d^{(3)}| \le CM^{-2}(\ln M)^2.$$

Similarly $\left|R_{H1}^{N}(Z_d - z_d)(t_M)\right| \le C\varepsilon\Delta_4 |z_d^{(2)}| \le CM^{-2}$. Also

$$\begin{aligned}|P_H^N(Z_d - z_d)(t_i)| &\le \varepsilon h_i^2 \|z_d^{(4)}\| + h_i^2 \|a_\varepsilon\| \|z_d^{(3)}\| \qquad 1 \le i \le \frac{M}{4} - 1,\ \frac{M}{2} + 1 \le i \le \frac{3M}{4} - 1 \\ &\le CM^{-2}(\ln M)^2\end{aligned}$$

and

$$|P_H^N(Z_d - z_d)(t_i)| \le \varepsilon h_i \|v_\varepsilon^{(3)}\| + C_{(\|a_\varepsilon\|, \|a'_\varepsilon\|)} h_i^2 (\|v_\varepsilon^{(3)}\| + \|v_\varepsilon^{(2)}\|) \le CM^{-2}, \quad M/4 \le i \le M/2 - 1.$$

For $i = M/2$, we have

$$|P_H^N(Z_d - z_d)(t_i)| \;=\; |P_T^N Z_d(t_i) - \frac{h_{i+1} f_{i+1}}{2\varepsilon + h_{i+1} b_\varepsilon(t_{i+1})} - \frac{h_{i-1}\bar{f}_{i-1}}{2\varepsilon}| \le CM^{-2}(\ln M)^2. \tag{33}$$

With the mesh function $(Z_d - z_d)(t_i)$ and Theorem 12, we obtain $|(Z_d - z_d)(t_i)| \le CM^{-2}(\ln M)^2$, for $t_i \in \bar{G}_\varepsilon^M$.

□

Theorem 16. *Let y_ε, y_ε be the solutions of (1), (16), respectively. Then for sufficiently large N, the maximum pointwise error satisfies* $\sup\limits_{0<\varepsilon\le 1} \|y_\varepsilon - y_\varepsilon\|_{\bar{G}_\varepsilon^N} \le CM^{-2}(\ln M)^2$.

Proof. This is deduced from the equation

$$(y_\varepsilon - y_\varepsilon)(t_i) = ((V_\varepsilon - v_\varepsilon) + (Z_0 - z_0) + (Z_d - z_d))(t_i), \ \ \forall t_i \in \bar{G}_\varepsilon^N$$

with reference to Theorems 13, 14, and 15. □

4 Numerical results

To validate the analytical results obtained in previous sections, two examples are given. The double mesh principle is used to estimate the error in the absence of analytical solutions for the test problems.

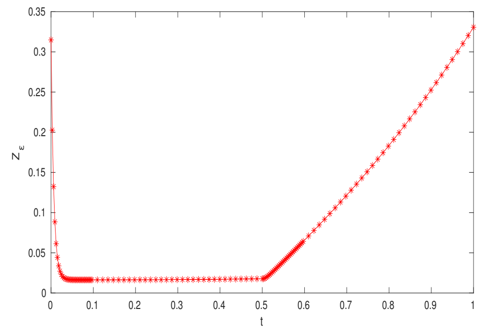
Example 1. Consider

$$\begin{cases} \varepsilon y''_\varepsilon(t) + 2\tanh\left(\dfrac{4t}{\varepsilon}\right) y'_\varepsilon(t) - (1 - \cos(3t)) y_\varepsilon(t) = f(t),\ t \in G_- \cup G_+, \\ 2y_\varepsilon(0) - \varepsilon y'_\varepsilon(0) = 1, \\ 3y_\varepsilon(1) + \varepsilon y'_\varepsilon(1) = 1, \end{cases}$$

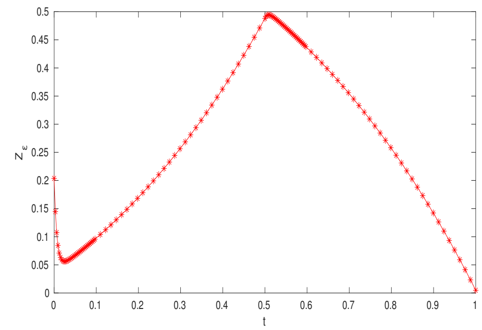
where $f(t) = \begin{cases} 0, & t \le d, \\ 1, & t > d. \end{cases}$

Example 2. Consider

$$\begin{cases} \varepsilon y''_\varepsilon(t) + 2(1 - e^{-4t/\varepsilon}) y'_\varepsilon(t) - (1 + t) y_\varepsilon(t) = f(t),\ t \in G_- \cup G_+, \\ 3y_\varepsilon(0) - 2\varepsilon y'_\varepsilon(0) = 1, \\ 3y_\varepsilon(1) + \varepsilon y'_\varepsilon(1) = 0, \end{cases}$$

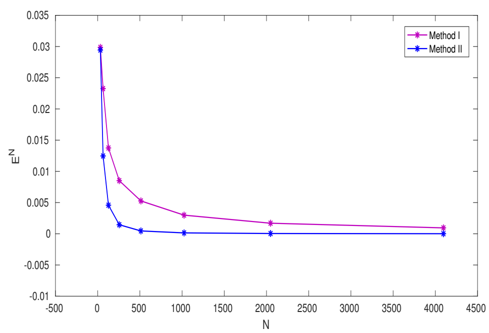


(a) Numerical graph for Example 1, $\varepsilon = 10^{-2}$, $M = 2^7$, $d = 0.5$.

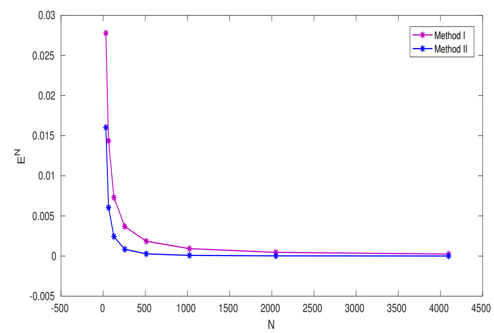


(b) Numerical graph for Example 2, $\varepsilon = 10^{-2}$, $M = 2^7$, $d = 0.5$.

Figure 1: Numerical graphs for Examples 1 and 2.



(a) Error plot for Example 1, $\varepsilon = 10^{-8}$.



(b) Error plot for Example 2, $\varepsilon = 10^{-8}$.

Figure 2: Error plots for Examples 1 and 2.

where $f(t) = \begin{cases} 2t+1, & t \leq d, \\ -(2t+1), & t > d. \end{cases}$

Figures 1a and 1b show numerical solutions for Examples 1 and 2, with errors and convergence rates listed in Tables 1 and 2. Method II reduces errors more effectively than Method I, aligning with theoretical rates validated by error plots for $\varepsilon = 10^{-8}$.

5 Conclusion

We have suggested finite difference and hybrid difference methods for a class of SPBTPPs of mixed type boundary conditions with discontinuous source term. The the solution exhibits a boundary layer in the neighbourhood of turning point $x = 0$ and a weak interior layer in the neighbourhood of $x = d$, the point of discontinuity. The error estimates for both Method I and Method II have been derived in the maximum norm. Finally, our analytical results have been verified and compared for Method I and II through two examples. We conclude that Method II gives a better approximation than the Method I.

Table 1: Maximum pointwise errors E^M and order of convergence P^M generated by Method I and II applied to problem 1 for various values of ε and N.

ε/ M		64	128	256	512	1024	2048	4096	8192
10^{-2}	M I	2.326e-2	1.374e-2	8.511e-3	5.264e-3	2.984e-3	1.693e-3	9.458e-4	5.178e-4
	M II	1.247e-2	4.578e-3	1.448e-3	4.563e-4	1.415e-4	4.309e-5	1.290e-5	3.808e-6
10^{-4}	M I	2.319e-2	1.367e-2	8.389e-3	5.194e-3	2.945e-3	1.671e-3	9.332e-4	5.108e-4
	M II	1.244e-2	4.561e-3	1.439e-3	4.526e-4	1.402e-4	4.265e-5	1.275e-5	3.761e-6
10^{-6}	M I	2.319e-2	1.367e-2	8.388e-3	5.193e-3	2.945e-3	1.670e-3	9.331e-4	5.107e-4
	M II	1.244e-2	4.561e-3	1.439e-3	4.526e-4	1.402e-4	4.264e-5	1.275e-5	3.760e-6
10^{-8}	M I	2.319e-2	1.367e-2	8.388e-3	5.193e-3	2.945e-3	1.670e-3	9.344e-4	5.066e-4
	M II	1.244e-2	4.561e-3	1.439e-3	4.526e-4	1.402e-4	4.264e-5	1.275e-5	3.761e-6
E^M	M I	2.326e-2	1.374e-2	8.511e-3	5.264e-3	2.984e-3	1.693e-3	9.458e-4	5.178e-4
	M II	1.247e-2	4.578e-3	1.448e-3	4.563e-4	1.415e-4	4.309e-5	1.290e-5	3.808e-6
P^M	M I	0.758	0.691	0.692	0.818	0.817	0.840	0.868	-
	M II	1.445	1.660	1.666	1.689	1.715	1.739	1.760	-

Table 2: Maximum pointwise errors E^M and order of convergence P^M generated by Method I and II applied to problem 2 for different values of ε and M.

ε/ M		64	128	256	512	1024	2048	4096	8192
10^{-2}	M I	1.239e-2	7.158e-3	3.787e-3	2.282e-3	1.288e-3	7.253e-4	4.003e-4	2.181e-4
	M II	5.611e-3	2.248e-3	7.772e-4	2.498e-4	7.732e-5	2.333e-5	6.893e-6	1.995e-6
10^{-4}	M I	1.298e-2	7.616e-3	4.477e-3	2.709e-3	1.516e-3	8.524e-4	4.719e-4	2.575e-4
	M II	6.047e-3	2.427e-3	8.463e-4	2.748e-4	8.587e-5	2.616e-5	7.826e-6	2.305e-6
10^{-6}	M I	1.298e-2	7.621e-3	4.484e-3	2.714e-3	1.518e-3	8.539e-4	4.726e-4	2.578e-4
	M II	6.052e-3	2.429e-3	8.470e-4	2.750e-4	8.596e-5	2.619e-5	7.836e-6	2.309e-6
10^{-8}	M I	1.298e-2	7.621e-3	4.485e-3	2.714e-3	1.519e-3	8.538e-4	4.710e-4	2.481e-4
	M II	6.052e-3	2.429e-3	8.470e-4	2.750e-4	8.596e-5	2.620e-5	7.843e-6	2.296e-6
E^M	M I	1.298e-2	7.621e-3	4.485e-3	2.714e-3	1.519e-3	8.539e-4	4.726e-4	2.578e-4
	M II	6.052e-3	2.429e-3	8.470e-4	2.750e-4	8.596e-5	2.620e-5	7.843e-6	2.309e-6
P^M	M I	0.769	0.764	0.724	0.837	0.830	0.853	0.874	-
	M II	1.316	1.520	1.622	1.678	1.714	1.739	1.764	-

References

[1] R.K. Dunne, E. O'Riordan, G.I. Shishkin, *A fitted mesh method for a class of singularly perturbed parabolic problems with a boundary turning point*, Comput. Methods Appl. Math. **3** (2003) 361–372.

[2] P.A. Farrell, A.F. Hegarty, J.M. Miller, E. O'Riordan, G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman and hall, CRC, 2000.

[3] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Singularly perturbed convection–diffusion problems with boundary and weak interior layers*, J. Comput. Appl. Math. **166** (2004) 133–151.

[4] L. Govindarao, J. Mohapatra, *A numerical scheme to solve mixed parabolic–elliptic problem involving singular perturbation*, Int. J. Comput. Math. **99** (2022) 2069–2090.

[5] L. Govindarao, H. Ramos, E. Sekar, *Numerical scheme for singularly perturbed Fredholm integro–differential equations with non–local boundary conditions*, Comput. Appl. Math. **43** (2024) 126.

[6] J.L. Gracia, E. O'Riordan, M.L. Pickett, *A parameter robust second order numerical method for a singularly perturbed two–parameter problem*, Appl. Numer. Math. **56** (2006) 962–980.

[7] T.C. Hanks, *Model relating heat-flow values near, and vertical velocities of mass transport beneath, oceanic rises*, J. Geophys. Res. **76** (1971) 537–544.

[8] G.J. Jayalakshmi, A. Tamilselvan, *Hybrid Difference Scheme for Singularly Perturbed Convection Diffusion Boundary Turning Point Problems with Discontinuous Source Term*, Int. J. Appl. Comput. Math. **5** (2019) 88.

[9] G.J. Jayalakshmi, A. Tamilselvan, *An ε–uniform method for a class of singularly perturbed parabolic problems with Robin boundary conditions having boundary turning point*, Asian–Eur. J. Math. **13** (2020) 2050025.

[10] G.J. Jayalakshmi, A. Tamilselvan, *Comparative study on difference schemes for singularly perturbed boundary turning point problems with Robin boundary conditions*, J. Appl. Math. Comput. **62** (2020) 341–360.

[11] G.J. Jayalakshmi, A. Tamilselvan, *Second order difference scheme for singularly perturbed boundary turning point problems*, J. Math. Model. **9** (2021) 633–643.

[12] J. Mohapatra, S. Priyadarshana, N. Raji Reddy, *Uniformly convergent computational method for singularly perturbed time delayed parabolic differential–difference equations*, Eng. Comput. **40** (2023) 694–717.

[13] P. Mushahary, S.R. Sahu, J. Mohapatra, *A Parameter Uniform Numerical Scheme for Singularly Perturbed Differential–difference Equations with Mixed Shifts*, J. Appl. Comput. Mech. **6** (2020) 344–356.

[14] R. Mythili Priyadharshini, N. Ramanujam, *A hybrid difference scheme for singularly perturbed second order ordinary differential equations with discontinuous convection coefficient and mixed type boundary conditions*, Int. J. Comput. Methods **5** (2008) 575–593.

[15] R. Mythili Priyadharshini, N. Ramanujam, *High–order method for a singularly perturbed second–order ordinary differential equation with discontinuous source term subject to mixed type boundary conditions*, International Conference on Mathematical Modelling and Scientific Computation, **283** (2012) 351–359.

[16] E. O'Riordan, J. Quinn, *Parameter–uniform numerical methods for some linear and nonlinear singularly perturbed convection diffusion boundary turning point problems*, BIT Numer. Math. **51** (2011) 317–337.

[17] E. O'Riordan, Jason Quinn, *A singularly perturbed convection diffusion turning point problem with an interior layer*, Comput. Methods Appl. Math. **12** (2012) 206–220.

[18] S. Priyadarshana, S.R. Sahu, J. Mohapatra, *Asymptotic and numerical methods for solving singularly perturbed differential difference equations with mixed shifts*, Iran. J. Numer. Anal. Optim. **12** (2022) 55–72.

[19] S. Priyadarshana, J. Mohapatra, *An efficient computational technique for time dependent semilinear parabolic problems involving two small parameters*, J. Appl. Math. Comput **(69)** (2023) 3721–3754.

[20] S. Priyadarshana, J. Mohapatra, *An efficient numerical approximation for mixed singularly perturbed parabolic problems involving large time–lag*, Indian J. Pure Appl. Math **(55)** (2024) 1389–1408.

[21] P. Rai, K.K. Sharma, *Singularly Perturbed Parabolic Differential Equations with Turning Point and Retarded Arguments*, IAENG Int. J. Appl. Math. **45** (2015) 4.

[22] E. Sekar, *Second order singularly perturbed delay differential equations with nonlocal boundary condition*, J. Comput. Appl. Math. **417** (2023) 114498.

[23] E. Sekar, L. Govindarao, J. Mohapatra, R. Vadivel, Nien–Tsu Hu, *Numerical analysis for second order differential equation of reaction–diffusion problems in viscoelasticity*, Alexandria Eng. J. **92** (2024) 92–101.

[24] E. Sekar, L. Govindarao, R. Vadivel, *A comparative study on numerical methods for Fredholm integro–differential equations of convection–diffusion problem with integral boundary conditions*, Appl. Numer. Math. **207** (2025) 323–338.

[25] E. Sekar, A. Tamilselvan, *Singularly perturbed delay differential equations of convection–diffusion type with integral boundary condition*, J. Appl. Math. Comput. **59** (2019) 701–722.

[26] K.K. Sharma, P. Rai, K.C. Patidar, *A review on singularly perturbed differential equations with turning points and interior layers*, Appl. Math. Comput. **219** (2013) 10575–10609.

[27] R. Vulanovic, P.A. Farrell, *Continuous and numerical analysis of a multiple boundary turning point problem*, SIAM J. Numer. Anal. **30** (1993) 1400–1418.