Journal of Mathematical Modeling
Vol. 13, No. 3, 2025, pp. 593-608. Research Article

JMM

Non-classical sinc collocation method for approximating Hallen's integral equation

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Abstract. In this paper, we propose a novel numerical method for solving Hallens integral equation, based on the sinc collocation approximation. The key innovation of our approach lies in the incorporation of weight functions into the traditional sinc-expansion framework. By leveraging the properties of sinc collocation, we transform Hallens integral equation into a system of algebraic equations, which can be solved efficiently. Our method involves discretizing the singular kernel of Hallens integral equation and then applying the sinc approximation. Additionally, we provide a detailed analysis of the convergence and error estimation of the proposed method. Numerical results are presented for three distinct values of λ and l, as well as for three different weight functions: $w(t) = 1+\sin(\pi t)$, $w(t) = 1+\cos(\frac{\pi t}{2})$ and $w(t) = 1+t$.

Keywords: Non-classical sinc, collocation, thin wire Antennas, Fredholm integral equation, Hallens integral equations, reduced kernel.
AMS Subject Classification 2020: 45B05, 33E30, 41A05.

1 Introduction

In 1956, Erik Hallen formulated his famous integral equation to provide an exact treatment of antenna current wave reflection at the termination of a tube-shaped cylindrical antenna [11]. But, the initial investigations on this subject is back to 1938 [10]. This integral equation allowed Hallen to demonstrate that, for thin wires, the current distribution is approximately sinusoidal and propagates at nearly the speed of light. This integral equation is a first kind Fredholm integral equation and for the thin-wire cylindrical antenna with length d and radius r with $r \ll l$ is given as [2, 17]

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} K(t_1,t_2) I_{t_1}(t_2) dt_2 = \frac{j}{2a_0} V \sin(\alpha|t_1|) + C\cos(\alpha t_1), \quad \frac{-l}{2} < t_1 < \frac{l}{2}. \tag{1}$$

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Received: 17 November 2024 / Revised: 26 February 2025 / Accepted: 4 March 2025
DOI: 10.22124/jmm.2025.28972.2578

In Eq. (1), $j=\sqrt{-1}$ and the kernel $K(t_1,t_2)$ can be chosen in two different ways, which are the exact and the approximate or reduced kernel, in which the approximate kernel of the integral equation has no solution. However, usually the same approximate methods are used for both kinds of integral equations [9]. The approximate kernel that is used for our method is defined by

$$K(t_1,t_2)=\frac{1}{4\pi}\frac{e^{-j\alpha\sqrt{(t_1-t_2)^2+r^2}}}{\sqrt{(t_1-t_2)^2+r^2}}. \tag{2}$$

For Eqs. (1) and (2) we have $a_0=120\pi$, and the free wavenumber $\alpha=\frac{2\pi}{\lambda}$ where λ is called wavelength. Here, $I(t_2)$ is represents the current, V is the driven voltage and C is a constant that must be determined based on the given conditions $I(\frac{-l}{2})=I(\frac{l}{2})=0$. The extensive literature referenced in [25] demonstrates the significant interest in this equation.

In Eq. (2), the kernel $K(t_1,t_2)$ becomes highly peaked when $t_1=t_2$, particularly for small values of r. This behavior can lead to a poorly conditioned numerical formulation, posing challenges in developing a reliable numerical algorithm. Various methods have been proposed to estimate the solution of Hallens integral equation, including, sinc collocation method in [17], meshless radial basis functions in [12], radial basis function in [2], multiwavelet method in [19], comparison between solution of Pocklington's and Hallen's integral equations for thin wire antennas using method of moments and haar wavelet in [5] and finite element method in [16].

The sinc approximation has been used as a base for variaty of numerical methods. Sinc methods were founded by Frank Stenger [22, 23] and expanded upon him in [24]. Excellent overview of methods based on sinc function for solving differential equations (DE) and partial differential equations (PDE) and integral equations can be founded in [13, 24]. Sinc methods are widely used for solving linear and non-linear problems arising in different areas in science and engineering problems including fourth-order boundary value problems [8], Hallens integral equation [17], fractional derivatives for the sinc functions [18], solving a class of second-order non-linear BVPs [7], spectral poly-sinc collocation method [6].

In this paper, we use non-classical sinc method for solving (1). The concept of using non-classical weight functions was initially employed by Shizgal to solve the Boltzmann equation and related problems [20], and Shizgal and Chen later utilized these bases to determine the eigenvalues and eigenfunctions of the Schrdinger equation [21] and Alipanah applied the non-classical pseudospectral method to solve singular boundary value problems in physiology [1]. Also, Mohammadi et al. for the first time used non-classical sinc method for the singular boundary value issues of the second-order that occur in physiology [15] , the problems of the third-order with boundary conditions [3], systems of integro-differential equations of the second-order [14] and Alipanah et al. used the same method for getting numerical result of third order singular boundary value problems that are singularly disturbed [4].

The non-classical sinc collocation method is recognized as a more efficient approach for addressing singularity problems. This efficiency arises because the singularities in such problems typically occur at the endpoints of the interval; the sinc function does not require smoothness at these endpoints [15]. Furthermore, a distinguishing feature of this method is that the majority of the sinc grid points are concentrated near the interval's endpoints, which facilitates managing the singularities effectively.

Following is the layout of this paper: in Section 2 we discuss about some basic requirement information about sinc function, in Section 3 we discuss about the discretization of Halllen's integral equation, in Section 4 we discuss about error analysis. Finally, we show numerical results for our method.

2 Sinc functions

The sinc function is defined over the entire real line by

$$sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

With evenly spaced the translated sinc functions nodes for $h > 0$ are defined as:

$$S(i,h)(t) = sinc\left(\frac{t-ih}{h}\right), \quad i = 0, \pm 1, \pm 2, \ldots. \tag{3}$$

The sinc functions are defined on the infinite domain $D_S = \{q = u + iv : |v| < d_1 \leq \pi/2\}$, where d_1 is a distance. While Eq. (3) is defined on the finite interval $(0,1)$. To transform the finite interval $(0,1)$ into the infinite interval, we apply the following conformal map:

$$\phi(z) = \ln\left(\frac{z}{1-z}\right),$$

which maps the eye-shaped region

$$D_E = \left\{ z = t + iy : \left| arg\left(\frac{z}{1-z}\right) \right| < d_1 \leq \pi/2 \right\},$$

onto D_S (see Figure 1). So, we obtain the sinc functions on interval (0,1) as follows

$$S_i(z) = S(i,h)o\phi(z) = sinc\left(\frac{\phi(z) - ih}{h}\right), \tag{4}$$

for $z \in D_E$. The function

$$z = \phi^{-1}(q) = \frac{e^q}{1+e^q},$$

is the inverse of $q = \phi(z)$. In $(0,1)$, the sinc grid points are determined as follows

$$t_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{5}$$

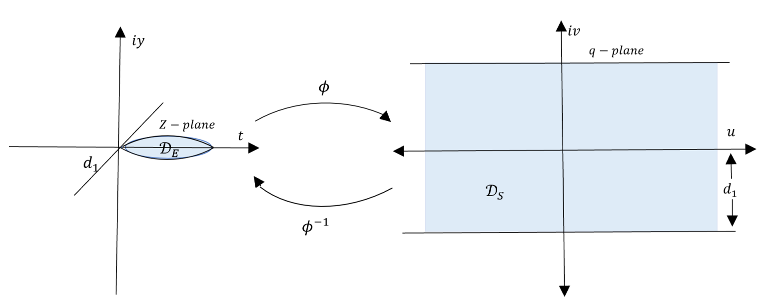

Figure 1: The transform between the regions D_E onto D_S.

Definition 1. *[13] Assume that $B(D_E)$ is the class of functions y that are analytic in D_E, and*

$$\int_{\phi(L+u)} |y(z)dz| \to 0, \quad u \to \pm\infty,$$

where $L = \{iv : |v| < d_1\}$ and on the boundary of D_E (denoted by ∂D_E). Then, $N(y, D_E)$ is defined as

$$N(y, D_E) \equiv \int_{\partial D_E} |y(z)dz| < \infty.$$

Theorem 1. *[14] Assume that $y \in B(D_E)$ and there exist positive constants β_1, β_2 and c such that*

$$\left|\frac{y(t)}{\phi'(t)}\right| \leq c \begin{cases} e^{-\beta_1|\phi(t)|}, & t \in \Gamma_a, \\ e^{-\beta_2|\phi(t)|}, & t \in \Gamma_b, \end{cases}$$

where

$$\Gamma_a = \{t \in \Gamma : \phi(t) \in (-\infty, 0)\}, \tag{6}$$

and

$$\Gamma_b = \{t \in \Gamma : \phi(t) \in [0, \infty)\}. \tag{7}$$

If $h = \sqrt{\frac{\pi d_1}{\beta_1 M}}$, then $\forall t \in \Gamma$ we have

$$\left|\int_0^1 y(t)dt - h\sum_{j=-M}^{N} \frac{y(t_j)}{\phi'(t_j)}\right| \leq Me^{-\sqrt{\pi d_1 \beta_1 M}}. \tag{8}$$

If we assume that $K(t_1, t_2) \in B(D_E)$ then by using (8) we get [17]

$$\int_0^1 K(t_1, t_2)dt_2 = h\sum_{i=-M}^{N} \frac{K(t_1, t_i)}{\phi'(t_i)} + O(e^{-\beta_2 Mh}) + O(e^{\frac{-2\pi d_1}{h}}). \tag{9}$$

Lemma 1. *[13] Suppose that ϕ is a one-to-one conformal mapping from the simply connected domain D_E onto D_s, then*

$$\delta_{i,k}^{(0)} = [S(i,h)o\phi(t)]\Big|_{t=t_k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \tag{10}$$

$$\delta_{i,k}^{(1)} = h\frac{d}{d\phi}[S(i,h)o\phi(t)]\Big|_{t=t_k} = \begin{cases} 0, & i = k, \\ \frac{(-1)^{k-i}}{k-i}, & i \neq k, \end{cases} \tag{11}$$

$$\delta_{i,k}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(i,h)o\phi(t)]\Big|_{t=t_k} = \begin{cases} \frac{-\pi^2}{3}, & i = k, \\ \frac{-2(-1)^{k-i}}{(k-i)^2}, & i \neq k. \end{cases} \tag{12}$$

Definition 2. *[13] The approximation of the function $y(t)$ using non-classical sinc functions over the interval $(0,1)$ is defined as follows*

$$y(x) \approx y_N(t) = \sum_{i=-M}^{N} \frac{w(t)}{w(t_i)} y(ih) sinc\left(\frac{\phi(t) - ih}{h}\right), \tag{13}$$

where $w(t) > 0$, is the weight function. At the interpolation points $t_k = kh$ we have

$$y_N(kh) = y(kh), \quad k = 0, \pm 1, \pm 2, \ldots.$$

Theorem 2. *Assuming $\phi' y \in B(D_E)$ and the weight function are determined such away that $\forall t \in (0,1)$, we have $\frac{w(t)}{w(t_j)} < c_1$. Also suppose that constants β_1, β_2 are positive, and c such that*

$$|y(t)| \leq c \begin{cases} e^{-\beta_1|\phi(t)|}, & t \in \Gamma_a, \\ e^{-\beta_2|\phi(t)|}, & t \in \Gamma_b, \end{cases}$$

where $\Gamma_a = \{t \in \Gamma : \phi(t) \in (-\infty, 0)\}$, and $\Gamma_b = \{t \in \Gamma : \phi(t) \in [0, \infty)\}$. Then $\forall t \in \Gamma$ we have, [15]

$$\left| y(t) - \sum_{i=-M}^{N} y(ih) \frac{w(t)}{w(t_i)} sinc\left(\frac{\phi(t) - ih}{h}\right) \right| \leq c_1 k_1 M^{1/2} e^{-\sqrt{\pi d_1 \beta_1 M}}, \tag{14}$$

in which the mesh-size h is chosen as:

$$h = \sqrt{\frac{\pi d_1}{\beta_1 M}}, \quad and \quad N = \left[\left| \frac{\beta_1}{\beta_2} M \right| \right].$$

Proof. Proof can be found in [15]. □

3 Discretization of Halllen's Integral Equation

In order to use the non-classical sinc method in the interval $(0,1)$, firstly we normalize the Hallen's integral equation by changing the independent variable $t_1 = lz - \frac{l}{2}$ and $t_2 = lz' - \frac{l}{2}$. Then, the main equation and the condition $I(\frac{-l}{2}) = I(\frac{l}{2}) = 0$ can be rewritten as

$$l \int_0^1 K(lz, lz') I(z') dz' = g(z), \quad 0 < z < 1, \tag{15}$$

and $I(0) = I(1) = 0$, where

$$g(z) = \frac{j}{2a_0} V \sin(\alpha |lz - \frac{l}{2}|) + C \cos(\alpha (lz - \frac{l}{2})), \quad 0 < z < 1. \tag{16}$$

In equation (15) the kernel is sharply peaked especially for small value of r. Consequently, from a computational perspective, it is beneficial to isolate and extract the singularity from the kernel. This can be achieved by expressing $K(lz, lz')$ as

$$K(lz, lz') = K_n(lz, lz') + K_s(lz, lz'), \tag{17}$$

where $K_n(lz, lz')$ and $K_s(lz, lz')$ are called the nonsingular and singular parts of the Kernel respectively, and defined in [17] as

$$K_n(lz, lz') = \frac{1}{4\pi} \frac{e^{-j\alpha\sqrt{(lz-lz')^2 + r^2}} - 1}{\sqrt{(lz - lz')^2 + r^2}},$$

and

$$K_s(lz, lz') = \frac{1}{4\pi} \frac{1}{\sqrt{(lz - lz')^2 + r^2}}.$$

Now, by using Eq. (17) we can rewrite (15) by

$$l\int_0^1 K_n(lz,lz')I(z')dz' + l\int_0^1 K_s(lz,lz')I(z')dz' = g(z), \quad 0<z<1.$$

The first part of the above equation is well behaved as a result we can find the numerical solution for it, but the second part contain a singularity, so we evaluated as follows. Assume that

$$\int_0^1 K_s(lz,lz')I(z')dz' = C_1(z)+C_2(z),$$

where

$$C_1(z) = \int_0^1 K_s(lz,lz')(I(z')-I(z))dz', \tag{18}$$

and

$$C_2 = I(z)\int_0^1 K_s(lz,lz')dz'. \tag{19}$$

The integrand of Eq. (18), is well behaved, and the integral in Eq. (19), can be solved as

$$H(z) = \int_0^1 K_s(lz,lz')dz' = \frac{1}{4\pi l}\ln\frac{\sqrt{(lz-l)^2+r^2}+l-lz}{\sqrt{l^2z^2+r^2}-lz}. \tag{20}$$

Using Eqs. (17)-(20) we can express Eq. (15) by

$$l\int_0^1 K_n(lz,lz')I(z')dz' + l\int_0^1 K_s(lz,lz')\Big(I(z')-I(z)\Big)dz' + lI(z)H(z) = g(z), \quad 0<z<1. \tag{21}$$

Now, we approximate Eq. (21) by using non-classical sinc collocation method. For this reason, since $I(0)=I(1)=0$, we can approximate $I(z')$ by using Eq. (13), for $M=N$ as

$$I_N(z') = \sum_{i=-N}^{N} c_i \frac{w(z')}{w(z_i')} sinc\left(\frac{\phi(z')-ih}{h}\right). \tag{22}$$

Now, we substitute Eq. (22) in (21), and for obtaining c_j for $j=-N,\dots,N$ in Eq. (21), by collocating at the sinc points

$$z_k = \frac{e^{kh}}{1+e^{kh}}, \quad k=-N,\dots,N,N+1. \tag{23}$$

We get

$$l\sum_{i=-N}^{N}\frac{c_i}{w(z_i')}\int_0^1 K_n(lz_k,lz')S(i,h)o\phi(z')w(z')dz' + l\sum_{i=-N}^{N}\frac{c_i}{w(z_i')}\int_0^1 K_s(lz_k,lz')S(i,h)o\phi(z')W(z')dz'$$
$$-lck\int_0^1 K_s(lz_k,lz')dz' + lc_kH(z_k) = g(z_k), \quad k=-N,\dots,N,N+1.$$

Here we have assumed that $c_{N+1} = 0$. Moreover, because the integrals in the last equation is well behaved so by using the quadrature formula given in Eq. (9), we approximate the integrals in the above equation, to obtain

$$lh \sum_{i=-N}^{N} \frac{K_n(lz_k, lz_i)}{\phi'(z_i)} I(z_i) + lh \sum_{i=-N}^{N} (I(z_j) - c_k) \frac{K_s(lz_k, lz_i)}{\phi'(z_i)} + \frac{c_k}{4\pi} \ln \left(\frac{\sqrt{(lz_k - l)^2 + r^2} + l - lz_k}{\sqrt{l^2 z_k^2 + r^2}} \right) = g(z_k), \qquad k = -N, \dots, N. \tag{24}$$

And the last equation in the matrix form becomes

$$\mathbf{B}\mathbf{c} = \mathbf{b}, \tag{25}$$

where $\mathbf{B}$ is $2N+1 \times 2N+1$ matrix of the form

$$\mathbf{B}_{ik} = \begin{cases} S_1(z_i, z_i) + S_2(z_i, z_i) - \sum_{i=-N}^{N} [S_2(z_i, z_i) + lH(z_i)\delta_{ik}], & i = k, \\ S_1(z_k, z_i) + S_2(z_k, z_i), & i \neq k, \end{cases}$$

where

$$S_1(z_k, z_i) = \frac{lhK_n(lz_k, lz_i)}{\phi'(z_i)}, \quad S_2(z_k, z_i) = \frac{lhK_s(lz_k, lz_i)}{\phi'(z_i)},$$

and $\mathbf{c} = [c_{-N}, \dots, c_N]^T$, $\mathbf{b} = [g(t_{-N}), \dots, g(t_N)]^T$. To obtain the unknowns $c_i, i = -N, \dots, N$, we solve the linear system (25) by iterative methods.

4 Error analysis

Theorem 3. *Let $I(z)$ be the exact solution of Eq. (22) and $I(z)$ and $K(z, z')$ be in $B(D_E)$. Also suppose that*

$$U_N(z) = l \left[\int_0^1 K_n(lz, lz') I_N(z') dz' + \int_0^1 K_s(lz, lz') I_N(z') dz' \right] - g(z),$$

then

$$||U_N(z_i)||_\infty \leq Ac_1 lk_1 hN^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 N}}.$$

Proof.

$$\left| U_N(z_i) \right| = \left| l \left[\int_0^1 K_n(lz_i, lz')(I_N(z') - I(z') + I(z')) dz' + \int_0^1 K_s(lz_i, lz')(I_N(z') - I(z') + I(z')) dz' \right] - g(z_i) \right|$$
$$= l \left[\left| \int_0^1 K_n(lz_i, lz')(I_N(z') - I(z')) dz' + \int_0^1 K_s(lz_i, lz')(I_N(z') - I(z')) dz' \right] + l \left[\int_0^1 K_n(lz_i, lz') I(z') dz' + \int_0^1 K_s(lz_i, lz') I(z') dz' - g(z_i) \right| \right],$$

since the last part of the above equation equal to zero, then

$$\leq l\left[\left|\int_0^1 K_n(lz_i,lz')(I_N(z')-I(z'))dz'\right|+\left|\int_0^1 K_s(lz_i,lz')(I_N(z')-I(z'))dz'\right|\right]$$
$$\leq l\left[\int_0^1 \left|K_n(lz_i,lz')(I(z')-I_N(z'))\right|dz'+\int_0^1 \left|K_s(lz_i,lz')(I(z')-I_N(z'))\right|dz'\right]$$

then by using (14), we have

$$\leq c_1 l k_1 N^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 N}}\left(\int_0^1 \left|K_n(lz_j,lz')\right|dz'+\int_0^1 \left|K_s(lz_j,lz')\right|dz'\right).$$

Now, by using (9) we get

$$\leq c_1 l k_1 N^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 N}} h\left(\sum_{k=-N}^{N}\frac{K_n(z_i,z_k')}{\phi'(x_k)}+\frac{K_s(z_i,z_k')}{\phi'(x_k)}\right)$$
$$\leq c_1 l k_1 N^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 N}} h\left(A_1+A_2\right),$$

where A_1 and A_2 are constants and define by $\frac{K_n(z_i,z_k')}{\phi'(x_k)}\leq A_1$ and $\frac{K_s(z_i,z_k')}{\phi'(x_k)}\leq A_2$. Therefore,

$$\|U_N(z_i)\|_\infty \leq c_1 l k_1 N^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 N}} hA,$$

where $A=\max\{A_1+A_2\}$. □

5 Numerical results

In this section, we present a numerical example for two selected wire lengths so that they include special cases of practical interest, e.g. $l=\lambda$, $l=\frac{\lambda}{2}$ and $l=\frac{\lambda}{3}$.

Example 1. In this example, we consider the Hallen's integral equation for thin wire with radius $r=0.0005\lambda$ for $l=\frac{\lambda}{2}$ and $r=1/518$ for $l=\lambda$ and $l=\frac{\lambda}{3}$. Also, we let $V=1$, for the cases:
(i) $l=\lambda$,
(ii) $l=\frac{\lambda}{2}$,
(iii) $l=\frac{\lambda}{3}$.
For each case, we choose $\lambda=2$, $d_1=\frac{\pi}{16}$ and $\beta_1=1$. Then, we applied our method for three different weights $w(t)=1+t$, $w(t)=1+\cos(\frac{\pi t}{2})$ and $w(t)=1+\sin(\pi t)$. In addition, we showed current approximation $|I_N(z')|$ for $w(t)=1+\cos(\frac{\pi t}{2})$ for $N=10,20,40,80$ and $N=160$ for the above cases in tables 1 and 2, respectively. Furthermore, we present the magnitude of the current in Figs. 2, 3, 4 for $l=\lambda$, Figs. 5, 6 and 7 for $l=\frac{\lambda}{2}$ and Figs. 8, 9 and 10 for $l=\frac{\lambda}{3}$. It can be seen that by increasing the values of N the solution converge rapidly.

Table 1: Approximation solution for Hallen's integral equation for $l = 2$ and $w(t) = 1 + \cos(\frac{\pi t}{2})$.

t_k	$\lvert I_{10}(t_k)\rvert$	$\lvert I_{20}(t_k)\rvert$	$\lvert I_{40}(t_k)\rvert$	$\lvert I_{80}(t_k)\rvert$	$\lvert I_{160}(t_k)\rvert$
0.0	0.00000253	0.00005690	0.00009102	0.00012275	0.00013348
0.1	0.00082881	0.00088320	0.00093651	0.00095375	0.00096521
0.2	0.00134755	0.00137076	0.00137533	0.00136121	0.00137124
0.3	0.00124132	0.00130378	0.00124533	0.00124532	0.00125646
0.4	0.00079128	0.00079185	0.00068566	0.00067399	0.00066835
0.5	0.00033726	0.00038993	0.00043190	0.00046806	0.00047280
0.6	0.00098345	0.00089348	0.00080788	0.00083634	0.00081792
0.7	0.00129355	0.00136065	0.00133305	0.00132676	0.00131684
0.8	0.00124473	0.00130535	0.00130960	0.00132944	0.00132631
0.9	0.00065630	0.00076356	0.00079682	0.00081734	0.00082658

Table 2: Approximation solution for Hallen's integral equation for $l = 1$ and $w(t) = 1 + \cos(\frac{\pi t}{2})$.

t_k	$\lvert I_{10}(t_k)\rvert$	$\lvert I_{20}(t_k)\rvert$	$\lvert I_{40}(t_k)\rvert$	$\lvert I_{80}(t_k)\rvert$	$\lvert I_{160}(t_k)\rvert$
0.0	0.00003402	0.00192167	0.00162869	0.00104294	0.00074495
0.1	0.00223897	0.00236232	0.00298945	0.00382361	0.00413875
0.2	0.00259117	0.00287875	0.00415600	0.00602299	0.00678537
0.3	0.00263505	0.00308968	0.00488306	0.00756039	0.00867729
0.4	0.00249041	0.00300429	0.00510900	0.00831769	0.00968157
0.5	0.00203020	0.00252191	0.00478414	0.00821940	0.00970516
0.6	0.00261237	0.00304849	0.00507584	0.00825441	0.00956139
0.7	0.00259856	0.00309731	0.00478700	0.00731909	0.00836137
0.8	0.00249206	0.00280594	0.00393561	0.00564271	0.00630450
0.9	0.00206060	0.00228705	0.00272464	0.00329854	0.00352357

Table 3: Time for the approximation solution for Hallen's integral equation for $l = 2$ and $w(t) = 1 + \cos(\frac{\pi t}{2})$.

N	Time(s)
64	23.843
128	199.641
150	320.906

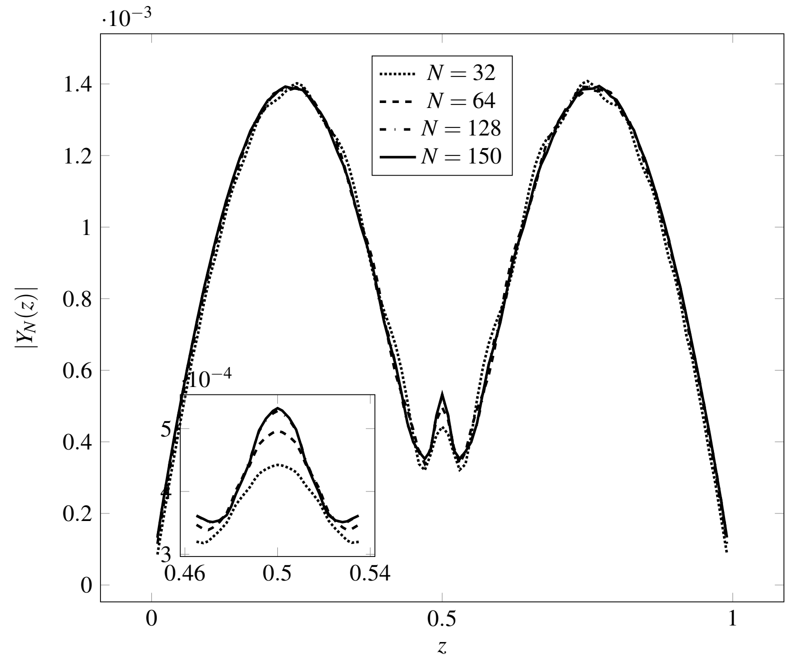

Figure 2: Approximation solution for Hallen's integral equation for four different N and $l = 2$ with $w(t) = 1+t$.

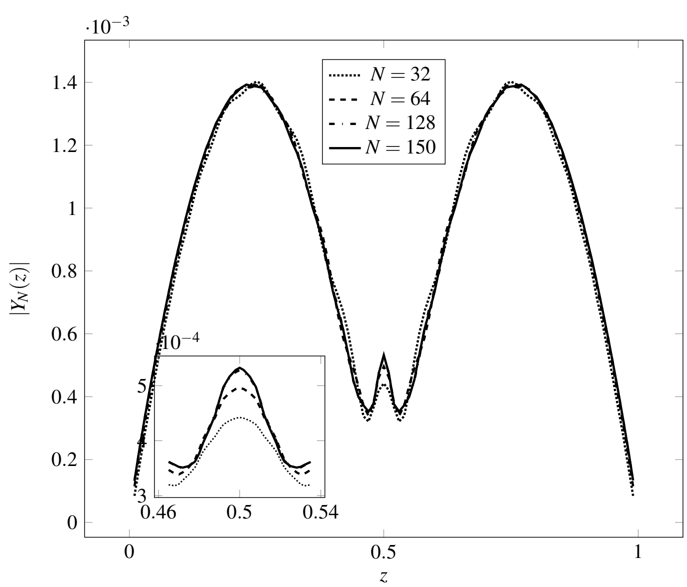

Figure 3: Approximation solution for Hallen's integral equation for four different N and $l = 2$ with $w(t) = 1+\sin(\pi t)$.

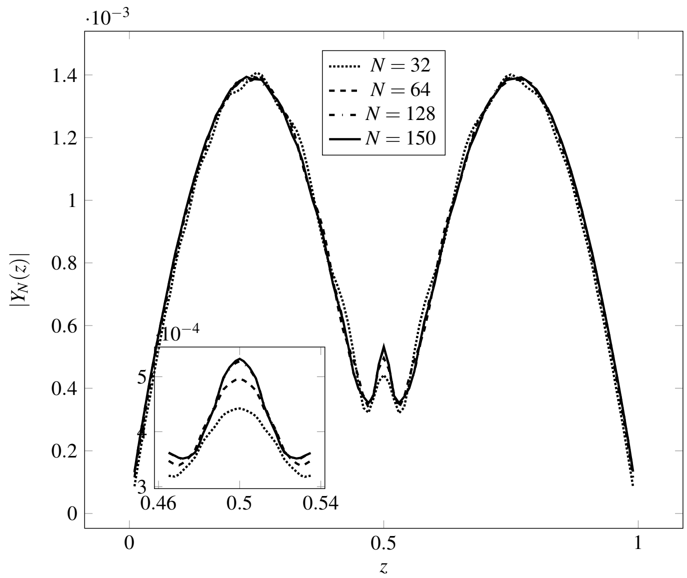

Figure 4: Approximation solution for Hallen's integral equation for four different N and $l = 2$ with $w(t) = 1 + \cos(\frac{\pi t}{2})$.

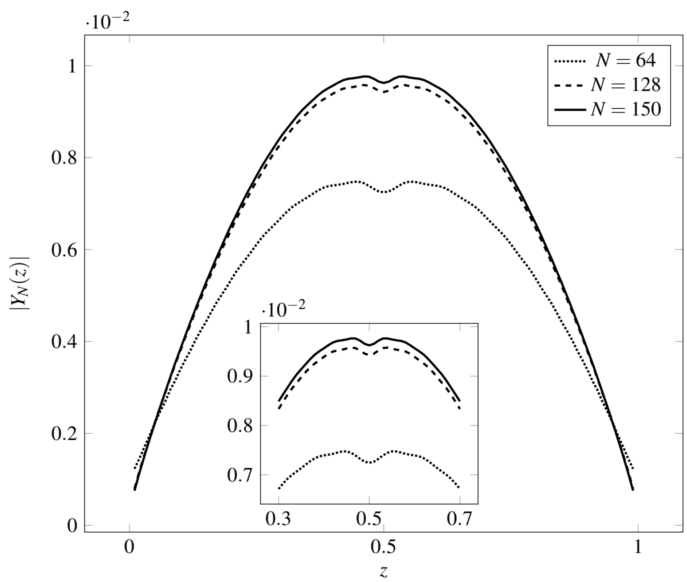

Figure 5: Approximation solution for Hallen's integral equation for three different N and $l = 1$ with $w(t) = 1 + \sin(\pi t)$.

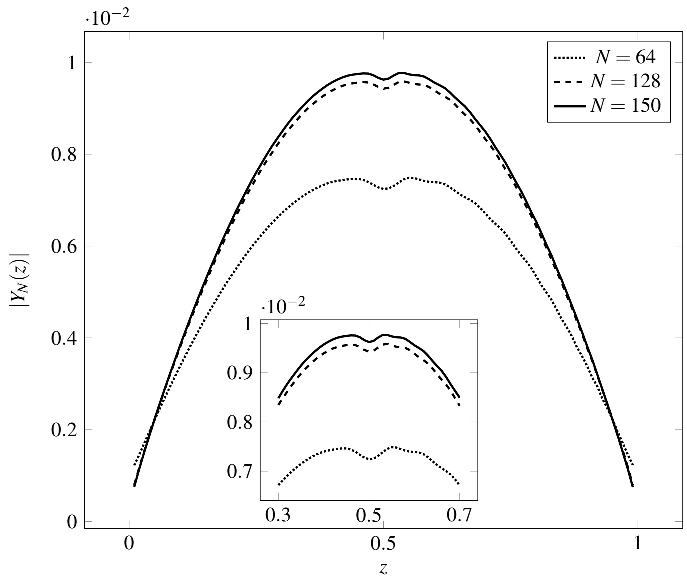

Figure 6: Approximation solution for Hallen's integral equation for three different N and $l = 1$ with $w(t) = 1 + \cos(\frac{\pi t}{2})$.

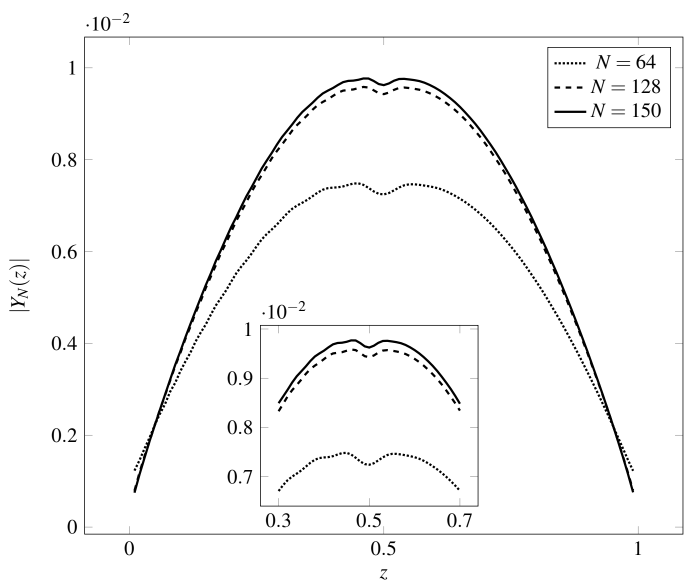

Figure 7: Approximation solution for Hallen's integral equation for three different N and $l = 1$ with $w(t) = 1 + t$.

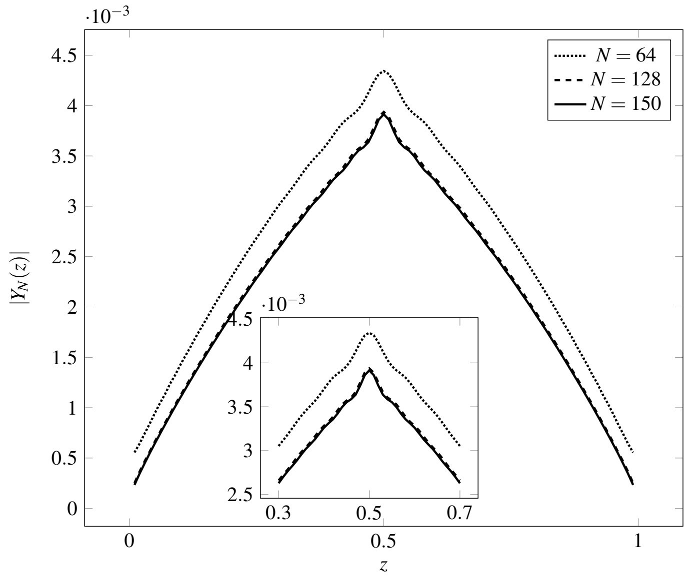

Figure 8: Approximation solution for Hallen's integral equation for three different N and $l = \frac{2}{3}$ with $w(t) = 1 + \sin(\pi t)$.

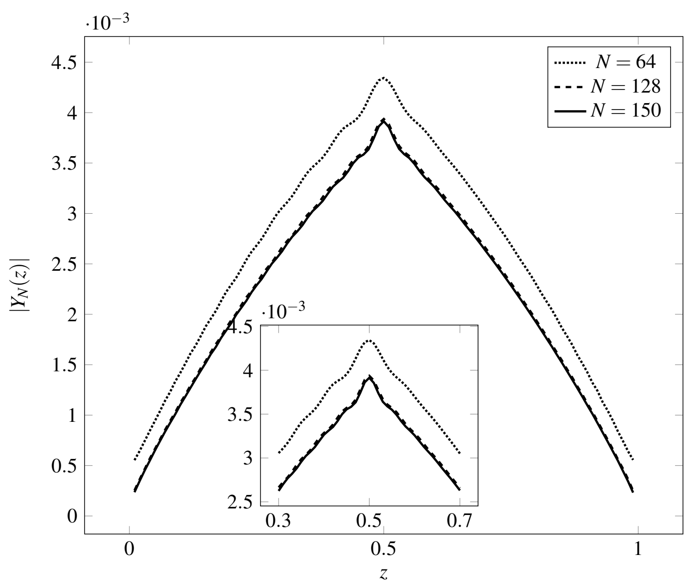

Figure 9: Approximation solution for Hallen's integral equation for three different N and $l = \frac{2}{3}$ with $w(t) = 1 + \cos(\frac{\pi t}{2})$.

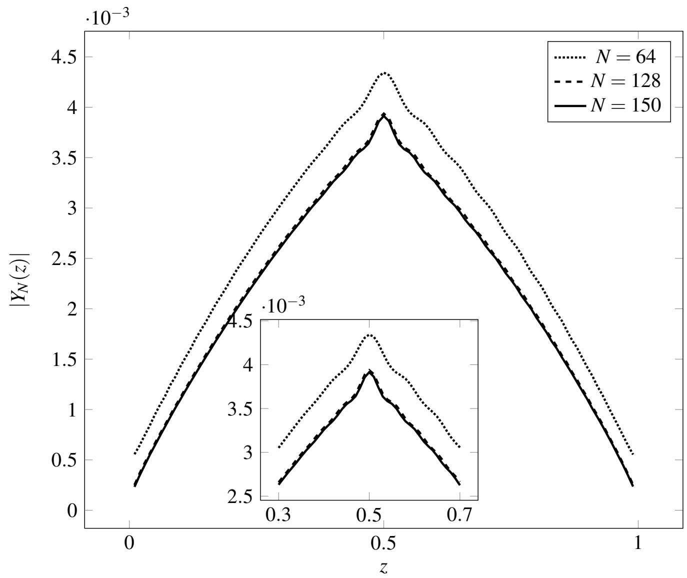

Figure 10: Approximation solution for Hallen's integral equation for three different N and $l = \frac{2}{3}$ with $w(t) = 1 + t$.

6 Conclusion

In this paper, we investigated the non-classical sinc method for approximating Hallen's integral equation. The properties of the sinc collocation method are utilized to transform Hallen's integral equation into a system of linear equations. We discussed the convergence and error estimation of our approach and presented numerical results for $\lambda = 2$ and three different values of l and three different weights which are $w(t) = 1 + \sin(\pi t)$, $w(t) = 1 + \cos(\frac{\pi t}{2})$ and $w(t) = 1 + t$. We concluded that the method is computationally efficient.

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