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Nonlocal Caputo generalized proportional fractional integro-differential systems: an existence study

Samira Zerbib*, **Khalid Hilal**, **Ahmed Kajouni**

LMACS Laboratory, Sultan Moulay Slimane University, Beni Mellal, Morocco
Email(s): zerbib.sam123@gmail.com, hilalkhalid2005@yahoo.fr, kajjouni@gmail.com

Abstract. The objective of this work is to investigate the existence and uniqueness of the solution to a nonlinear fractional integro-differential equation with a non-local condition involving the generalized fractional proportional Caputo derivative of two distinct orders. To achieve this, Krasnoselskiis fixed point theorem is utilized to examine the existence of the solution, followed by the application of Banachs fixed point theorem to study the uniqueness. Lastly, two illustrative examples are provided to highlight the main results.

Keywords: Differential equation, generalized Caputo proportional fractional derivative, non-local condition, fixed point theorem.
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1 Introduction

Fractional calculus is an extension of the classical notions of integrals and derivatives of non-zero integer orders to any real order. The theory of fractional calculus dates back several centuries and can be considered both old and new. It originates from speculations by G.W. Leibniz and L'Hpital around the end of 1695. Many mathematicians have contributed to the development of fractional calculus [6], including Euler (1730) and P.S. Laplace (1812). In 1819, the first mention of an arbitrary order derivative appeared in a text where the French mathematician S.F. Lacroix dedicated a few pages to fractional calculus in a work on differential calculus. J.B.J. Fourier (1822) is another notable contributor. Between 1832 and 1837, J. Liouville conducted an extensive study of fractional calculus. Later, B. Riemann (1847) proposed an approach to fractional differentiation. Other approaches emerged later, such as those by A.K. Grnwald (1867-1872), A.V. Letnikov (1868-1872), H. Laurent (1884), Weyl (1917), and M. Caputo in 1967(for more details, see [5]).

*Corresponding author
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Fractional calculus is recognized as one of the most effective mathematical tools for describing the memory properties of complex systems and specific materials [19]. In the classical framework, the memory of a system is typically represented by the integer-order derivative:

$$\frac{d\Phi(t)}{dt} = \lim_{\Delta t \longrightarrow 0} \frac{\Phi(t) - \Phi(t - \Delta t)}{\Delta t}, \quad 0 < t.$$

This definition captures the system's short memory properties, as it relies solely on the function's values at two points. In contrast, the fractional approach represents a system's memory using the fractional-order derivative [16]:

$$\frac{d^{\theta}\Phi(t)}{dt^{\theta}} = \lim_{\tau \longrightarrow 0} \tau^{-\theta} \sum_{k=0}^{n} (-1)^k \binom{\theta}{k} \Phi(t - k\tau), \quad n\tau = t, 0 < \theta < 1,$$

where θ is the fractional derivative order. Unlike the classical derivative, the fractional derivative accounts for the function's history. Calculating the fractional derivative at a given time t requires incorporating all the previous values of the function $\Phi(t)$, thereby reflecting the system's long memory properties. For more details see [20].

One of the strengths of classical fractional calculus is the wide range of available derivatives and integrals. However, there has always been a need to further advance this field and introduce new derivatives to enhance our understanding of the universe. In [4], Atangana and Baleanu introduced a fractional derivative in the sense of Caputo with MittagLeffler function contain non-singular kernels, which notable as ABCfractional derivative. For more details on fractional derivatives with non-singular kernels, see [1, 9, 17, 22]. In 2011, Katugampola introduced what he called generalized fractional operators [15, 23], designed to unify the Riemann-Liouville and Hadamard fractional operators. These generalized derivatives were later refined to encompass both the Caputo and Caputo-Hadamard fractional derivatives [8]. In [11], the authors introduced new fractional operators based on proportional derivatives of one function with respect to another. The kernel in the proposed fractional operators includes an exponential function and depends on the specific function used.

Fractional differential equations (FDEs) naturally arise in various scientific fields such as physics, engineering, medicine, electrochemistry, control theory, and more (see [7, 24, 25]). The effectiveness of these equations in modeling many real-world phenomena has motivated numerous researchers to study their quantitative and qualitative aspects. Moreover, fractional differential equations with non-local conditions represent a highly relevant and fascinating area of research. The study of these equations is gaining increasing popularity among researchers, as evidenced by the growing number of published articles addressing questions of existence and uniqueness of solutions for this type of equation. This trend reflects the vitality of research in this field. Among the works on the existence and uniqueness of solutions of fractional differential equations, we can cite:

In [13] the authors studied the existence and uniqueness of solutions of the following two fractional deformable differential equations:

$$\begin{cases} {}^{\mathscr{D}\mathscr{D}}D^{\zeta}u(\tau) = \mathscr{F}(\tau, u(\tau)), \quad \tau \in [0, c], \\ u(0) + h(u) = u_0, \end{cases}$$

and

$$\begin{cases} {}^{\mathscr{D}\mathscr{D}}D^{\zeta}u(\tau) = \mathscr{H}(u(\tau)) + \mathscr{F}(\tau,u(\tau)) + \int_0^{\tau} \mathscr{K}(\tau,s,u(\tau))ds, \quad \tau \in [0,c], \\ u(0) = u_0, \end{cases}$$

where ${}^{\mathscr{D}\mathscr{D}}D^{\zeta}(\cdot)$ is the deformable fractional derivative of order $0 < \zeta < 1$.

The authors of [21] investigated the existence and uniqueness of solution in a Banach space X for the following fractional perturbed neutral integro-differential problem involving the deformable derivative:

$$\begin{cases} {}^{\mathscr{D}\mathscr{D}}D^{\zeta}\left(u(\tau) - \mathscr{M}(\tau,u(\tau))\right) = \mathscr{L}\left(\tau,u(\tau),\int_0^{\tau}\mathscr{K}(\tau,s,u(\tau))ds\right) + \mathscr{F}\left(\tau,u(\tau),\int_0^{\tau}\mathscr{K}(\tau,s,u(\tau))ds\right), \\ u(0) = u_0 \in X, \quad \tau \in [0,c], 0 < \zeta < 1, \end{cases}$$

where $\mathscr{L},\mathscr{F} : [0,c] \times X \times X \longrightarrow X$ be continuous functions, $\mathscr{M} : [0,c] \times X \longrightarrow X$ is a continuously differentiable function, and $\mathscr{K} \in C(\mathscr{Z},X)$, where $\mathscr{Z} := \{(\tau,s) : 0 < s < \tau < c\}$.

Building on the previously mentioned works, this paper investigates the existence and uniqueness of the solution to the following nonlinear fractional integro-differential equation with a non-local condition, which involves the generalized fractional proportional Caputo derivative with two different orders:

$$\begin{cases} {}^{C}_{\rho}D^{\zeta,\mathscr{G}}_{0^+}\left({}^{C}_{\rho}D^{\delta,\mathscr{G}}_{0^+}\left(\chi(\tau) - \mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)\right)\right) = \mathscr{F}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)), \quad t \in \Pi := [0,b], \\ (\chi(\tau) - \mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau))_{\tau=0} = u_0, \quad (\chi(\tau) - \mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau))'_{\tau=0} = 0, \\ (\chi(\tau) - \mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)))_{\tau=b} + \varphi(\chi) = u_1, \end{cases} \tag{1}$$

where $1 < \zeta < 2, 0 < \delta < 1, {}^{C}_{\rho}D^{\zeta,\mathscr{G}}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order ζ, $\mathscr{G} : \Pi \to \mathbb{R}$, $\mathscr{H},\mathscr{F} : \Pi \times \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{X}$ are continuous functions, $\varphi \in C(\mathscr{X},\mathscr{X})$, and u_0, $u_1 \in \mathscr{X}$, where $\mathscr{X}$ is a Banach space. The operator $\mathscr{B}\chi(\tau)$ is given by $\mathscr{B}\chi(\tau) := \int_0^{\tau}\mathscr{N}(\tau,s)\chi(s)ds$ such that $\mathscr{N} \in C(\mathscr{D},\mathscr{X})$, where $\mathscr{D} := \{(\tau,s) : 0 < s < \tau < b\}$. We consider

$$\mathscr{B}^* = \max_{\tau\in[0,b]} \int_0^{\tau} \|\mathscr{N}(\tau,s))\| ds.$$

The key contributions of this study are summarized as follows:

1. We employ the fractional derivative rather than the classical derivative, leveraging the advantages and outcomes associated with the fractional derivative, as discussed earlier.

2. To the best of our knowledge, this is the first attempt to examine the structure of the nonlinear fractional integro-differential system with a non-local condition that includes the generalized fractional proportional Caputo derivative with two different orders in the context of the system introduced in (1).

3. Utilizing the generalized fractional proportional Caputo derivative type and its properties, we provide the integral solution to the given system (1)(Lemma 5).

4. The primary existence theorems are established using alternative fixed-point principles attributed to Banach and Krasnoselskii. Furthermore, two illustrative examples are presented in the section 4 to demonstrate the application of the main results.

5. Additionally, this work extends and enhances previous studies published in the literature, including those referenced in [3, 18].

This paper is structured as follows. Section 2 introduces the necessary definitions, lemmas, and notations. Section 3 presents our main results concerning the existence and uniqueness of the solution to problem (1). In Section 4, we provide two illustrative examples to illustrate the key findings of this work, and our conclusions are summarized in Section 5.

2 Preliminaries

In this section, we introduce definitions and lemmas related to the generalized Caputo proportional fractional derivative, and we provide the solution formula for the nonlinear fractional integro-differential system (1). These definitions and lemmas will be used consistently throughout the following sections of this work.

- Let $\Pi = [0,b]$ be a finite interval of $\mathbb{R}$. We denote by $C(\Pi, \mathscr{X})$ the Banach space of all continuous functions with the norm $\|\chi\| = \sup\{|\chi(\tau)| : \tau \in \Pi\}$.
- Throughout this paper, we consider the function $\mathscr{G} : \Pi \to \mathbb{R}$ to be a strictly positive, increasing, and differentiable function.

Definition 1 ([11]). *Let $0 < \rho < 1$, $\zeta > 0$, and f be a continuous function. The left-sided generalized proportional fractional integral of order ζ with respect to $\mathscr{G}$ of the function f is determined by*

$$ {}_{\rho}I_{0^+}^{\zeta,\mathscr{G}} f(\tau) = \frac{1}{\rho^{\zeta}\Gamma(\zeta)} \int_0^{\tau} e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(s))}\mathscr{G}'(s)(\mathscr{G}(\tau)-\mathscr{G}(s))^{\zeta-1} f(s)ds, $$

where $\Gamma(\zeta) = \int_0^{+\infty} e^{-t}t^{\zeta-1}dt$ is the Euler gamma function.

Definition 2 ([11]). *Let $\rho \in [0,1]$, Φ, $\Psi : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous functions with $\lim_{\rho\to 0^+} \Phi(\rho,\tau) = 0$, $\lim_{\rho\to 1^-} \Phi(\rho,\tau) = 1$, $\lim_{\rho\to 0^+} \Psi(\rho,\tau) = 1$, $\lim_{\rho\to 1^-} \Psi(\rho,\tau) = 0$, and $\Phi(\rho,\tau) \neq 0$, $\rho \in [0,1)$, $\Psi(\rho,\tau) \neq 0$, $\rho \in (0,1]$, $\tau \in \mathbb{R}$. The proportional derivative of order ρ with respect to $\mathscr{G}$ of the function f is determined by*

$$ {}_{\rho}D^{\mathscr{G}} f(\tau) = \Psi(\rho,\tau)f(\tau) + \Phi(\rho,\tau)\frac{f'(\tau)}{\mathscr{G}'(\tau)}. $$

In particular, if $\Phi(\rho,\tau) = \rho$ and $\Psi(\rho,\tau) = 1-\rho$, then we have

$$ {}_{\rho}D^{\mathscr{G}} f(\tau) = (1-\rho)f(\tau) + \rho\frac{f'(\tau)}{\mathscr{G}'(\tau)}. $$

Definition 3 ([11]). *Let $\rho \in (0,1]$. The left-sided generalized Caputo proportional fractional derivative of order $n-1 < \zeta < n$ for the continuous function f is defined by*

$$ \begin{aligned} {}^{C}_{\rho}D_{0^+}^{\zeta,\mathscr{G}} f(\tau) &= {}_{\rho}I_{0^+}^{n-\zeta,\mathscr{G}}({}_{\rho}D^{n,\mathscr{G}} f(\tau)) \\ &= \frac{1}{\rho^{n-\zeta}\Gamma(n-\zeta)} \int_0^{\tau} e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(s))}\mathscr{G}'(s)(\mathscr{G}(\tau)-\mathscr{G}(s))^{n-\zeta-1}({}_{\rho}D^{n,\mathscr{G}} f)(s)ds, \end{aligned} $$

where $n=[\zeta]+1$ *and* ${}_{\rho}D^{n,\mathscr{G}}=\underbrace{{}_{\rho}D^{\mathscr{G}}\ {}_{\rho}D^{\mathscr{G}}_{\rho}\dots D^{\mathscr{G}}}_{n-times}$.

As a simplification, throughout this manuscript, we pose

$$\Omega^{\zeta-1}_{\mathscr{G}}(\tau,0)=e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-g(0))}(\mathscr{G}(\tau)-g(0))^{\zeta-1}. \tag{2}$$

Lemma 1 ([10]). *Let* $\tau\in\Pi$, $\rho\in(0,1]$, $\zeta,\ \delta>0$, *and* f *be a continuous function. Then, we have*

$$ {}_{\rho}I^{\zeta,\mathscr{G}}_{0^+}({}_{\rho}I^{\delta,\mathscr{G}}_{0^+}f(\tau))={}_{\rho}I^{\zeta,\mathscr{G}}_{0^+}({}_{\rho}I^{\delta,\mathscr{G}}_{0^+}f(\tau))={}_{\rho}I^{\zeta+\delta,\mathscr{G}}_{0^+}f(\tau).$$

Lemma 2 ([10]). *Let* $\rho\in(0,1]$ *and* $\zeta,\ \delta>0$. *Then, we have*

(i) $\left({}_{\rho}I^{\zeta,\mathscr{G}}_{0^+}e^{\frac{\rho-1}{\rho}(\mathscr{G}(t)-\mathscr{G}(0))}(\mathscr{G}(t)-\mathscr{G}(0))^{\delta-1}\right)(\tau)=\frac{\Gamma(\delta)}{\rho^{\zeta}\Gamma(\zeta+\delta)}\Omega^{\zeta+\delta-1}_{\mathscr{G}}(\tau,0).$

(ii) $\left({}^{C}_{\rho}D^{\zeta,\mathscr{G}}_{0^+}e^{\frac{\rho-1}{\rho}(\mathscr{G}(t)-\mathscr{G}(0))}(\mathscr{G}(t)-\mathscr{G}(0))^{\delta-1}\right)(\tau)=\frac{\rho^{\zeta}\Gamma(\delta)}{\Gamma(\delta-\zeta)}\Omega^{\delta-\zeta-1}_{\mathscr{G}}(\tau,0).$

Lemma 3 ([10]). *Let* $\rho\in(0,1]$, $\zeta>0$, *and* f *be a continuous function. Then, we have*

$$\lim_{\tau\to 0}\left({}_{\rho}I^{\zeta,\mathscr{G}}_{0^+}f(\tau)\right)=0.$$

Lemma 4 ([12]). *Let* $\rho\in(0,1]$, $n-1<\zeta<n$, $(n=[\zeta]+1)$. *Then, we have*

$${}_{\rho}I^{\zeta,\mathscr{G}}_{0^+}({}^{C}_{\rho}D^{\zeta,\mathscr{G}}_{0^+}f(\tau))=f(\tau)-\sum_{k=0}^{n-1}\frac{({}_{\rho}D^{k,\mathscr{G}}f)(0)}{\rho^{k}\Gamma(k+1)}\Omega^{k}_{\mathscr{G}}(\tau,0).$$

To define the solution formula for problem (1), we simplify the process by considering the following problem and determining its solution formula:

$$\begin{cases}{}^{C}_{\rho}D^{\zeta,\mathscr{G}}_{0^+}\left({}^{C}_{\rho}D^{\delta,\mathscr{G}}_{0^+}(\chi(\tau)-h(\tau))\right)=f(\tau), \quad t\in\Pi=[0,b],\\ (\chi(\tau)-h(\tau))_{\tau=0}=u_0, \quad (\chi(\tau)-h(\tau))'_{\tau=0}=0,\\ (\chi(\tau)-h(\tau))_{\tau=b}+\varphi(\chi)=u_1,\end{cases} \tag{3}$$

where $0<\zeta<1$, $1<\delta<2$, ${}^{C}_{\rho}D^{\zeta,\mathscr{G}}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order ζ, $h,f:\Pi\longrightarrow\mathscr{X}$ be continuous functions, and $\varphi\in C(\mathscr{X},\mathscr{X})$.

Lemma 5. *Let* $h,f:\Pi\longrightarrow\mathscr{X}$ *be continuous functions. Then the problem* (3) *has a solution given by*

$$\begin{aligned}\chi(\tau)=&\frac{(\Delta-\varphi(\chi))}{\Omega^{\delta}_{\mathscr{G}}(b,0)}\Omega^{\delta}_{\mathscr{G}}(\tau,0)+u_0e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))}+\frac{(1-\rho)u_0}{\rho}\Omega^{1}_{\mathscr{G}}(\tau,0)\\&+\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}\Omega^{\delta+\zeta-1}_{\mathscr{G}}(\tau,s)\mathscr{G}'(s)f(s)ds+h(\tau), \quad \tau\in[0,b],\end{aligned} \tag{4}$$

where $\Omega^{(\cdot)}_{\mathscr{G}}(\tau,.)$ *and* Δ *are given by* (2) *and* (9), *respectively.*

Proof. Let $\chi(\tau)$ be a solution of the problem (1). Applying the operator ${}_\rho I_{0^+}^{\zeta,\mathscr{G}}(\cdot)$ to both sides of (3) and using Lemma 4, we get

$$ {}_\rho^C D_{0^+}^{\delta,\mathscr{G}}(\chi(\tau)-h(\tau)) = \beta_1 e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} + {}_\rho I_{0^+}^{\zeta,\mathscr{G}} f(\tau), \tag{5} $$

with $\beta_1 \in \mathbb{R}$. Next, Applying the operator ${}_\rho I_{0^+}^{\delta,\mathscr{G}}(\cdot)$ to both sides of (5), we get

$$ \chi(\tau)-h(\tau) = \beta_1 \, {}_\rho I_{0^+}^{\delta,\mathscr{G}} \Omega_{\mathscr{G}}^0(\tau,0) + \beta_2 \Omega_{\mathscr{G}}^0(\tau,0) + \beta_3 \frac{\Omega_{\mathscr{G}}^1(\tau,0)}{\rho} + {}_\rho I_{0^+}^{\zeta+\delta,\mathscr{G}} f(\tau), \tag{6} $$

with β_1, β_2, $\beta_3 \in \mathbb{R}$. Using Lemma 2 (i), then the integral equation (6) becomes

$$ \begin{aligned} \chi(\tau)-h(\tau) = {} & \beta_1 \frac{e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))}(\mathscr{G}(\tau)-\mathscr{G}(0))^\delta}{\rho^\delta \Gamma(\delta+1)} + \beta_2 e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} \\ & + \beta_3 \frac{e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))}(\mathscr{G}(\tau)-\mathscr{G}(0))}{\rho} \\ & + \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)} \int_0^\tau e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(s))} (\mathscr{G}(\tau)-\mathscr{G}(s))^{\delta+\zeta-1} \mathscr{G}'(s) f(s) ds. \end{aligned} \tag{7} $$

Putting $\tau = 0$ in the integral equation (7), we get $\beta_2 = (\chi(\tau)-h(\tau))_{\tau=0} = u_0$. From the integral equation (7), with $\beta_2 = u_0$ we have

$$ \begin{aligned} (\chi(\tau)-h(\tau))' = {} & \frac{\beta_1}{\rho^\delta \Gamma(\delta+1)} \left(\frac{\mathscr{G}'(\tau)(\rho-1)}{\rho} \Omega_{\mathscr{G}}^\delta(\tau,0) + \delta \mathscr{G}'(\tau) \Omega_{\mathscr{G}}^{\delta-1}(\tau,0) \right) \\ & + \frac{u_0 \mathscr{G}'(\tau)(\rho-1)}{\rho} e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} \\ & + \frac{\beta_3}{\rho} \left(\mathscr{G}'(\tau) e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} + \frac{\mathscr{G}'(\tau)(\rho-1)\Omega_{\mathscr{G}}^1(\rho,0)}{\rho} \right) \\ & + \frac{\mathscr{G}'(\tau)(\rho-1)}{\rho^{\delta+\zeta+1}\Gamma(\delta+\zeta)} \int_0^\tau \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau,s) \mathscr{G}'(s) f(s) ds \\ & + \frac{\mathscr{G}'(\tau)}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta-1)} \int_0^\tau \Omega_{\mathscr{G}}^{\delta+\zeta-2}(\tau,s) \mathscr{G}'(s) f(s) ds. \end{aligned} \tag{8} $$

Taking $\tau = 0$ in the integral equation (8), we get $\beta_3 = u_0(1-\rho)$. Now, taking $\tau = b$, $\beta_2 = u_0$, and $\beta_3 = u_0(1-\rho)$ in the integral equation (7), we have

$$ \begin{aligned} (\chi(\tau)-h(\tau))_{\tau=b} = {} & \beta_1 \frac{\Omega_{\mathscr{G}}^\delta(b,0)}{\rho^\delta \Gamma(\delta+1)} + u_0 e^{\frac{\rho-1}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))} \\ & + u_0(1-\rho) \frac{\Omega_{\mathscr{G}}^1(b,0)}{\rho} + {}_\rho I_{0^+}^{\delta+\zeta,\mathscr{G}} f(b). \end{aligned} $$

Using the initial condition $(\chi(\tau)-h(\tau))_{\tau=b} + \varphi(\chi) = u_1$, we further get

$$ \begin{aligned} \beta_1 & = \frac{\rho^\delta \Gamma(\delta+1)}{\Omega_{\mathscr{G}}^\delta(b,0)} \left(u_1 - \varphi(\chi) - u_0 e^{\frac{\rho-1}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))} - u_0(1-\rho) \frac{\Omega_{\mathscr{G}}^1(b,0)}{\rho} - {}_\rho I_{0^+}^{\delta+\zeta,\mathscr{G}} f(b) \right) \\ & = \frac{\rho^\delta \Gamma(\delta+1)}{\Omega_{\mathscr{G}}^\delta(b,0)} (\Delta - \varphi(\chi)), \end{aligned} $$

where

$$\Delta = u_1 - u_0 \left(e^{\frac{\rho-1}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))} + (1-\rho)\frac{\Omega^1_{\mathscr{G}}(b,0)}{\rho} \right) - {}_\rho I_{0^+}^{\delta+\zeta,\mathscr{G}} f(b). \tag{9}$$

Substituting β_1, β_2, and β_3 in (7) we obtain

$$\begin{aligned}\chi(\tau) = &\frac{(\Delta-\varphi(\chi))}{\Omega^\delta_{\mathscr{G}}(b,0)}\Omega^\delta_{\mathscr{G}}(\tau,0) + u_0 e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} + \frac{(1-\rho)u_0}{\rho}\Omega^1_{\mathscr{G}}(\tau,0) \\ &+ \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^\tau \Omega^{\delta+\zeta-1}_{\mathscr{G}}(\tau,s)\mathscr{G}'(s)f(s)ds + h(\tau).\end{aligned}$$

This completes the proof. □

With the information from the previous lemma, we are now able to define the solution to the nonlinear generalized Caputo proportional fractional integro-differential system (1).

Definition 4. *If χ is a solution to the problem (1), then χ is also a solution of the following integral equation:*

$$\begin{aligned}\chi(\tau) = &\frac{(\Delta-\varphi(\chi))}{\Omega^\delta_{\mathscr{G}}(b,0)}\Omega^\delta_{\mathscr{G}}(\tau,0) + u_0 e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} + \frac{(1-\rho)u_0}{\rho}\Omega^1_{\mathscr{G}}(\tau,0) \\ &+ \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^\tau \Omega^{\delta+\zeta-1}_{\mathscr{G}}(\tau,s)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds + \mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)),\end{aligned}$$

provided that the above integral is finite.

3 Existence results

In this section, we present and study the existence of solutions for the given nonlinear generalized Caputo proportional fractional integro-differential system (1) under the Krasnoselskiis fixed point theorem.

Theorem 1. *[2] Let $\mathscr{S}$ be a closed, convex and nonempty subset of the Banach algebra $\mathscr{X}$. We consider the two operators $\mathscr{V},\mathscr{W}:\mathscr{S}\to\mathscr{X}$ such that:*

(a) $\mathscr{V}u+\mathscr{W}v\in\mathscr{S}$ for all $u,v\in\mathscr{S}$.

(b) $\mathscr{V}$ is a contraction on $\mathscr{S}$.

(c) $\mathscr{W}$ is completely continuous on $\mathscr{S}$.

Then, the operator $\mathscr{P}u=\mathscr{V}u+\mathscr{W}u$ has at least a fixed point in $\mathscr{S}$.

To apply the Krasnoselskiis fixed point theorem, we need the following assumptions:

(A_1) The function $\mathscr{F}:\Pi\times\mathscr{X}\times\mathscr{X}\longrightarrow\mathscr{X}$ is continuous and there are constants $L_{\mathscr{F}}$, $\widehat{L}_{\mathscr{F}}$, and $\tilde{L}_{\mathscr{F}}$ such that for all $v,w,v',w'\in\mathscr{X}$ and for all $\tau\in\Pi$, we have

(i) $\|\mathscr{F}(\tau,v,w)-\mathscr{F}(\tau,v',w')\| \le L_{\mathscr{F}}\left[\|v-v'\|+\|w-w'\|\right].$

(ii) $\|\mathscr{F}(\tau,v,w)\| \leq \widehat{L}_{\mathscr{F}} + \tilde{L}_{\mathscr{F}}[\|v\|+\|w\|]$.

(A_2) The function $\mathscr{H}:\Pi\times\mathscr{X}\times\mathscr{X}\longrightarrow\mathscr{X}$ is continuous and there are constants $M_{\mathscr{H}}$, $\widehat{M}_{\mathscr{H}}$, and $\tilde{M}_{\mathscr{H}}$ such that for all $v,w,v',w'\in\mathscr{X}$ and for all $\tau\in\Pi$, we have

(i) $\|\mathscr{H}(\tau,v,w)-\mathscr{H}(\tau,v',w')\| \leq M_{\mathscr{H}}[\|v-v'\|+\|w-w'\|]$.

(ii) $\|\mathscr{H}(\tau,v,w)\| \leq \widehat{M}_{\mathscr{H}} + \tilde{M}_{\mathscr{H}}[\|v\|+\|w\|]$.

(A_3) The function $\varphi:\mathscr{X}\longrightarrow\mathscr{X}$ is continuous and there are constants K_φ, $\widehat{K}_\varphi$, and $\tilde{K}_\varphi$ such that for all $v,w\in\mathscr{X}$ and for all $\tau\in\Pi$, we have

(i) $\|\varphi(v)-\varphi(w)\| \leq K_\varphi\|v-w\|$.

(ii) $\|\varphi(v)\| \leq \widehat{K}_\varphi + \tilde{K}_\varphi\|v\|$.

Consider the Banach space $\mathscr{Q}:=(C(\Pi,\mathscr{X}),\|\cdot\|)$. Then we consider the subset $\mathscr{S}$ of $\mathscr{Q}$ defined as:

$$\mathscr{S}=\{\chi\in\mathscr{Q}:\|\chi\|\leq\Theta\},$$

with

$$\Theta > \frac{\widehat{\Psi}}{1-\tilde{\Psi}}, \quad \text{such that} \quad 1-\tilde{\Psi}\neq 0, \tag{10}$$

where

$$\begin{aligned}\widehat{\Psi} &= \frac{(\mathscr{G}(b)-\mathscr{G}(0))^\delta}{\Omega^\delta_{\mathscr{G}}(b,0)}\left(\|\Delta\|+\widehat{K}_\varphi\right)+\|u_0\|\left|1-\frac{1-\rho}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))\right| \\ &\quad +\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}\widehat{L}_{\mathscr{F}}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}+\widehat{M}_{\mathscr{H}}. \\ \tilde{\Psi} &= \frac{(\mathscr{G}(b)-\mathscr{G}(0))^\delta\tilde{K}_\varphi}{\Omega^\delta_{\mathscr{G}}(b,0)}+\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}\tilde{L}_{\mathscr{F}}(1+\mathscr{B}^*)}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}+\tilde{M}_{\mathscr{H}}(1+\mathscr{B}^*).\end{aligned}$$

It is clear that $\mathscr{S}$ is a convex, closed, bounded, and nonempty subset of the Banach space $\mathscr{Q}$. We now have all the necessary arguments to establish the existence results for the given nonlinear generalized Caputo proportional fractional integro-differential system (1). Therefore, we present the following existence theorem.

Theorem 2. *Suppose that all assumptions* (A_1)-(A_3) *hold and*

$$\frac{(\mathscr{G}(b)-\mathscr{G}(0)))^\delta}{\Omega^\delta_{\mathscr{G}}(b,0)}K_\varphi<1, \tag{11}$$

where $\Omega^{(\cdot)}_{\mathscr{G}}(.,0)$ *is given by* (2). *Then, the nonlinear generalized Caputo proportional fractional integro-differential system* (1) *has at least a solution* $\chi\in C(\Pi,\mathscr{X})$.

Proof. To apply the Krasnoselskiis fixed point theorem, we define the operators $\mathscr{V}:\mathscr{S}\longrightarrow\mathscr{X}$, $\mathscr{W}:\mathscr{S}\longrightarrow\mathscr{X}$, and $\mathscr{P}:\mathscr{S}\longrightarrow\mathscr{X}$ as follows:

$$
\begin{aligned}
&(\mathscr{V}\chi)(\tau)=\frac{(\Delta-\varphi(\chi))}{\Omega^{\delta}_{\mathscr{G}}(b,0)}\Omega^{\delta}_{\mathscr{G}}(\tau,0)+u_0e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))}+\frac{(1-\rho)u_0}{\rho}\Omega^{1}_{\mathscr{G}}(\tau,0).\\
&(\mathscr{W}\chi)(\tau)=\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}\Omega^{\delta+\zeta-1}_{\mathscr{G}}(\tau,s)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds+\mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)).\\
&(\mathscr{P}\chi)(\tau)=((\mathscr{V}+\mathscr{W})\chi)(\tau).
\end{aligned}\tag{12}
$$

Then, the proof is given in the following steps:

Step 1: Let $\tau\in[0,b]$ and $\chi,\,\upsilon\in\mathscr{S}$. By using the assumptions $(A_1)(ii)$, $(A_2)(ii)$, $(A_3)(ii)$, and the fact that

$$e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))}<1,$$

we get

$$
\begin{aligned}
&\|\mathscr{V}\chi(\tau)+\mathscr{W}\upsilon(\tau)\|\\
&=\Bigg\|\frac{(\Delta-\varphi(\chi))}{\Omega^{\delta}_{\mathscr{G}}(b,0)}\Omega^{\delta}_{\mathscr{G}}(\tau,0)+u_0e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))}+\frac{(1-\rho)u_0}{\rho}\Omega^{1}_{\mathscr{G}}(\tau,0)\\
&\quad+\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}\Omega^{\delta+\zeta-1}_{\mathscr{G}}(\tau,s)\mathscr{G}'(s)\mathscr{F}(s,\upsilon(s),\mathscr{B}\upsilon(s))ds+\mathscr{H}(\tau,\upsilon(\tau),\mathscr{B}\upsilon(\tau))\Bigg\|\\
&\le\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(b,0)}\left(\|\Delta\|+\widehat{K}_{\varphi}+\tilde{K}_{\varphi}\|\chi\|\right)+\|u_0\|\left|1-\frac{1-\rho}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))\right|\\
&\quad+\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}(\mathscr{G}(\tau)-\mathscr{G}(s))^{\delta+\zeta-1}\mathscr{G}'(s)\left(\widehat{L}_{\mathscr{F}}+\tilde{L}_{\mathscr{F}}[\|\upsilon\|+\|\mathscr{B}\upsilon\|]\right)ds\\
&\quad+\widehat{M}_{\mathscr{H}}+\tilde{M}_{\mathscr{H}}[\|\upsilon\|+\|\mathscr{B}\upsilon\|]\\
&\le\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(b,0)}\left(\|\Delta\|+\widehat{K}_{\varphi}+\tilde{K}_{\varphi}\Theta\right)+\|u_0\|\left|1-\frac{1-\rho}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))\right|\\
&\quad+\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}\left(\widehat{L}_{\mathscr{F}}+\tilde{L}_{\mathscr{F}}\Theta(1+\mathscr{B}^*)\right)+\widehat{M}_{\mathscr{H}}+\tilde{M}_{\mathscr{H}}\Theta(1+\mathscr{B}^*)\\
&\le\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(b,0)}\left(\|\Delta\|+\widehat{K}_{\varphi}\right)+\|u_0\|\left|1-\frac{1-\rho}{\rho}(\mathscr{G}(b)-\mathscr{G}(0))\right|\\
&\quad+\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}\widehat{L}_{\mathscr{F}}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}+\widehat{M}_{\mathscr{H}}\\
&\quad+\Theta\left[\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}\tilde{K}_{\varphi}}{\Omega^{\delta}_{\mathscr{G}}(b,0)}+\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}\tilde{L}_{\mathscr{F}}(1+\mathscr{B}^*)}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}+\tilde{M}_{\mathscr{H}}(1+\mathscr{B}^*)\right]\\
&\le\widehat{\Psi}+\Theta\tilde{\Psi}<\Theta.
\end{aligned}\tag{13}
$$

Therefore, $\mathscr{V}(\chi)+\mathscr{W}(\upsilon)\in\mathscr{S}$ for all $\chi,\,\upsilon\in\mathscr{S}$.

Step 2: We show that $\mathscr{V}$ is a contraction. Let $\tau\in[0,b]$ and $\chi,\,\upsilon\in\mathscr{S}$. By using the assumption $(A_3)(i)$, and the fact that

$$e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} < 1,$$

we get

$$\|\mathscr{V}\chi(\tau)-\mathscr{V}\upsilon(\tau)\| = \frac{\Omega_{\mathscr{G}}^{\delta}(\tau,0)}{\Omega_{\mathscr{G}}^{\delta}(b,0)}\|\varphi(\chi)-\varphi(\upsilon)\| \leq \frac{(\mathscr{G}(b)-\mathscr{G}(0)))^{\delta}}{\Omega_{\mathscr{G}}^{\delta}(b,0)} K_{\varphi}\|\chi-\upsilon\|.$$

Thanks to condition (11), deduce that the operator $\mathscr{V}$ is a contraction.
Step 3: We show that the operator $\mathscr{W}$ is completely continuous.
(i) $\mathscr{W}$ is continuous:
Let $\tau \in [0,b]$ and χ_n be a sequence of $\mathscr{S}$ such that $\chi_n \longrightarrow \chi$ as $n \longrightarrow \infty$ in $\mathscr{S}$. Then we have

$$\begin{aligned}
&\|\mathscr{W}\chi_n(\tau)-\mathscr{W}\chi(\tau)\| \\
&\quad\leq \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}\Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau,s)\mathscr{G}'(s)\|\mathscr{F}(s,\chi_n(s),\mathscr{B}\chi_n(s))-\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))\|\,ds \\
&\quad\quad+\|\mathscr{H}(\tau,\chi_n(\tau),\mathscr{B}\chi_n(\tau))-\mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau))\|.
\end{aligned}$$

Thanks to the continuity of the functions $\mathscr{F}$ and $\mathscr{H}$ and the Lebesgue dominated convergence theorem, we obtain

$$\|\mathscr{W}\chi_n(\tau)-\mathscr{W}\chi(\tau)\| \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

This implies that the operator $\mathscr{W}$ is continuous.
(ii) $\mathscr{W}$ is uniformly bounded:
Let $\tau \in [0,b]$ and $\chi \in \mathscr{S}$. Then, by using the assumptions $(A_1)(ii)$, $(A_2)(ii)$, and the fact that

$$e^{\frac{\rho-1}{\rho}(\mathscr{G}(\tau)-\mathscr{G}(0))} < 1,$$

we get

$$\begin{aligned}
&\|\mathscr{W}\chi(\tau)\| \\
&\quad\leq \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}\Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau,s)\mathscr{G}'(s)\|\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))\|\,ds+\|\mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau))\| \\
&\quad\leq \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^{\tau}(\mathscr{G}(\tau)-\mathscr{G}(s))^{\delta+\zeta-1}\mathscr{G}'(s)\left(\widehat{L}_{\mathscr{F}}+\tilde{L}_{\mathscr{F}}\left[\|\chi\|+\|\mathscr{B}\chi\|\right]\right)ds \\
&\quad\quad+\widehat{M}_{\mathscr{H}}+\tilde{M}_{\mathscr{H}}\left[\|\chi\|+\|\mathscr{B}\chi\|\right] \\
&\quad\leq \frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}\left(\widehat{L}_{\mathscr{F}}+\tilde{L}_{\mathscr{F}}\Theta(1+\mathscr{B}^*)\right)+\widehat{M}_{\mathscr{H}}+\tilde{M}_{\mathscr{H}}\Theta(1+\mathscr{B}^*).
\end{aligned}$$

Therefore, the operator $\mathscr{W}$ is uniformly bounded.
(iii) $\mathscr{W}$ is equicontinuous:

Let $\tau_1, \tau_2 \in [0,b]$, $(\tau_1 < \tau_2)$, and $\chi \in \mathscr{S}$, by using our assumptions, we get

$$\begin{aligned}
&\|(\mathscr{W}\chi)(\tau_2) - (\mathscr{W}\chi)(\tau_1)\| \\
&= \Bigg\| \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)} \int_0^{\tau_2} \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau_2,s)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds + \mathscr{H}(\tau_2,\chi(\tau_2),\mathscr{B}\chi(\tau_2)) \\
&\quad - \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)} \int_0^{\tau_1} \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau_1,s)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds - \mathscr{H}(\tau_1,\chi(\tau_1),\mathscr{B}\chi(\tau_1)) \Bigg\| \\
&= \Bigg\| \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)} \int_{\tau_1}^{\tau_2} \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau_2,s)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds + \mathscr{H}(\tau_2,\chi(\tau_2),\mathscr{B}\chi(\tau_2)) \\
&\quad + \frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)} \int_0^{\tau_1} \left(\Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau_2,s) - \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau_1,s)\right)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds \\
&\quad - \mathscr{H}(\tau_1,\chi(\tau_1),\mathscr{B}\chi(\tau_1))\| \\
&\leq \frac{\left(\widehat{L}_{\mathscr{F}} + \tilde{L}_{\mathscr{F}}\Theta(1+\mathscr{B}^*)\right)}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)} \Bigg\| \int_{\tau_1}^{\tau_2} (\mathscr{G}(\tau_2) - \mathscr{G}(s))^{\delta+\zeta-1}\mathscr{G}'(s)ds \\
&\quad + \int_0^{\tau_1} \left((\mathscr{G}(\tau_2) - \mathscr{G}(s))^{\delta+\zeta-1} - (\mathscr{G}(\tau_1) - \mathscr{G}(s))^{\delta+\zeta-1}\right)\mathscr{G}'(s)ds \Bigg\| \\
&\quad + \|\mathscr{H}(\tau_2,\chi(\tau_2),\mathscr{B}\chi(\tau_2)) - \mathscr{H}(\tau_1,\chi(\tau_1),\mathscr{B}\chi(\tau_1))\| \\
&\leq \frac{\left(\widehat{L}_{\mathscr{F}} + \tilde{L}_{\mathscr{F}}\Theta(1+\mathscr{B}^*)\right)}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)} \left\| (\mathscr{G}(\tau_2) - \mathscr{G}(0))^{\delta+\zeta} - (\mathscr{G}(\tau_1) - \mathscr{G}(0))^{\delta+\zeta} \right\| \\
&\quad + \|\mathscr{H}(\tau_2,\chi(\tau_2),\mathscr{B}\chi(\tau_2)) - \mathscr{H}(\tau_1,\chi(\tau_1),\mathscr{B}\chi(\tau_1))\|.
\end{aligned}$$

By using the continuity of the functions $\mathscr{H}, \mathscr{G}, \chi$, and by Lebesgue dominated convergence theorem, from the above inequality, we get $\|(\mathscr{W}\chi)(\tau_2) - (\mathscr{W}\chi)(\tau_1)\| \to 0$ as $\tau_1 \to \tau_2$. Therefore, the operator $\mathscr{W}$ is equicontinuous. From (*ii*) and (*iii*), and by the Arzela-Ascoli theorem, it follows that $\mathscr{W}(\mathscr{S})$ is relatively compact. Moreover, since $\mathscr{W}(\mathscr{S})$ is continuous, it is completely continuous. From Theorem 1, the operator $\mathscr{P}$ has at least one fixed point in $\mathscr{S}$. Hence, the nonlinear generalized Caputo proportional fractional integro-differential system (1) has at least one solution $\chi \in C(\Pi, \mathscr{X})$. □

We now proceed to demonstrate the uniqueness of the solution to the nonlinear generalized Caputo proportional fractional integro-differential system (1). This leads to the following theorem

Theorem 3. *Let assumptions* (A_1)-(A_3) *hold. Then the nonlinear generalized Caputo proportional fractional integro-differential system* (1) *has a unique solution* $\chi \in C(\Pi, \mathscr{X})$ *provided that*

$$\Lambda = \frac{(\mathscr{G}(b) - \mathscr{G}(0))^{\delta}}{\Omega_{\mathscr{G}}^{\delta}(b,0)} K_{\varphi} + M_{\mathscr{H}}(1+\mathscr{B}^*) + \frac{(\mathscr{G}(b) - \mathscr{G}(0))^{\delta+\zeta}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)} L_{\mathscr{F}}(1+\mathscr{B}^*) < 1. \tag{14}$$

Proof. Let operator $\mathscr{P}$ be given by (12). Based on the same arguments of the Step 1, we find that $\mathscr{P}(\mathscr{S}) \subset \mathscr{S}$, where $\mathscr{S} = \{\chi \in \mathscr{Q} : \|\chi\| \leq \Theta\}$, and Θ satisfied (10). Let $\tau \in [0,b]$, χ, $\upsilon \in \mathscr{S}$, by using the assumptions $(A_1)(i)$, $(A_2)(i)$, and $(A_3)(i)$, we have

$$
\begin{aligned}
&\|\mathscr{P}\chi(\tau)-\mathscr{P}\upsilon(\tau)\| \\
&=\Bigg\|\frac{\Omega_{\mathscr{G}}^{\delta}(\tau,0)\varphi(\chi)}{\Omega_{\mathscr{G}}^{\delta}(b,0)}+\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^\tau \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau,s)\mathscr{G}'(s)\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))ds+\mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)) \\
&\quad -\frac{\Omega_{\mathscr{G}}^{\delta}(\tau,0)\varphi(\upsilon)}{\Omega_{\mathscr{G}}^{\delta}(b,0)}-\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^\tau \Omega_{\mathscr{G}}^{\delta+\zeta-1}(\tau,s)\mathscr{G}'(s)\mathscr{F}(s,\upsilon(s),\mathscr{B}\upsilon(s))ds-\mathscr{H}(\tau,\upsilon(\tau),\mathscr{B}\upsilon(\tau))\Bigg\| \\
&\leq \frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}}{\Omega_{\mathscr{G}}^{\delta}(b,0)}\|\varphi(\chi)-\varphi(\upsilon)\|+\|\mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau))-\mathscr{H}(\tau,\upsilon(\tau),\mathscr{B}\upsilon(\tau))\| \\
&\quad +\frac{1}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^\tau (\mathscr{G}(\tau)-\mathscr{G}(s))^{\delta+\zeta-1}\mathscr{G}'(s)\|\mathscr{F}(s,\chi(s),\mathscr{B}\chi(s))-\mathscr{F}(s,\upsilon(s),\mathscr{B}\upsilon(s))\|ds \\
&\leq \frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}K_\varphi}{\Omega_{\mathscr{G}}^{\delta}(b,0)}\|\chi-\upsilon\|+M_{\mathscr{H}}\left[\|\chi-\upsilon\|+\int_0^s\|\mathscr{N}(s,t)\|\|\chi-\upsilon\|dt\right] \\
&\quad +\frac{L_{\mathscr{F}}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta)}\int_0^\tau (\mathscr{G}(\tau)-\mathscr{G}(s))^{\delta+\zeta-1}\mathscr{G}'(s)\left[\|\chi-\upsilon\|+\int_0^s\|\mathscr{N}(s,t)\|\|\chi-\upsilon\|dt\right]ds \\
&\leq \left(\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta}}{\Omega_{\mathscr{G}}^{\delta}(b,0)}K_\varphi+M_{\mathscr{H}}(1+\mathscr{B}^*)+\frac{(\mathscr{G}(b)-\mathscr{G}(0))^{\delta+\zeta}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}L_{\mathscr{F}}(1+\mathscr{B}^*)\right)\|\chi-\upsilon\| \\
&=:\Lambda\|\chi-\upsilon\|.
\end{aligned}
$$

Thanks to condition (14), the operator $\mathscr{P}$ is a contraction. Hence $\mathscr{P}$ has a unique fixed point $\chi \in C(\Pi,\mathscr{X})$, which is the unique solution of the nonlinear generalized Caputo proportional fractional integro-differential system (1) in $C(\Pi,\mathscr{X})$. □

4 Applications

In this section, we provide two practical examples to illustrate the application of our main results.

Example 1. Let $\Pi=[0,1]$, $\rho=\delta=\frac{1}{2}$, $\zeta=\frac{3}{2}$, $\mathscr{G}(\tau)=\tau$,

$$
\begin{aligned}
\mathscr{F}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)) &= \frac{\tau^2}{2}\left(e^\tau+\frac{\pi}{16(\tau^2+2)}\frac{|\chi(\tau)|}{(1+|\chi(\tau)|)}\right)+\frac{\pi}{64}\mathscr{B}\chi(\tau), \\
\mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)) &= \frac{\tau^2}{8}\left(1+\frac{|\chi(\tau)|}{2(1+|\chi(\tau)|)}\right)+\frac{1}{(\tau+4)^2}\mathscr{B}\chi(\tau), \\
\varphi(\chi) &= \frac{|\chi(\frac{1}{2})|}{9(1+|\chi(\frac{1}{2})|)}+\frac{3}{17},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{B}\chi(\tau) &= \int_0^\tau \tau^2 e^s|\chi(s)|ds. \\
\mathscr{B}^* &= \max_{\tau\in[0,1]}\int_0^\tau \tau^2 e^s ds = e-1 \approx 1.718.
\end{aligned}
$$

We consider the following nonlinear generalized Caputo proportional fractional integro-differential system:

$$\begin{cases} {}^{C}_{\frac{1}{2}}D^{\frac{3}{2},\tau}_{0^+}\left({}^{C}_{\frac{1}{2}}D^{\frac{1}{2},\tau}_{0^+}\left(\chi(\tau)-\frac{\tau^2}{8}\left(1+\frac{|\chi(\tau)|}{2(1+|\chi(\tau)|)}\right)+\frac{1}{(\tau+4)^2}\mathscr{B}\chi(\tau)\right)\right) \\ =\frac{\tau^2}{2}\left(e^{\tau}+\frac{\pi}{16(\tau^2+2)}\frac{|\chi(\tau)|}{(1+|\chi(\tau)|)}\right)+\frac{\pi}{64}\mathscr{B}\chi(\tau), \quad \tau\in\Pi=[0,1], \\ \left(\chi(\tau)-\frac{\tau^2}{8}\left(1+\frac{|\chi(\tau)|}{2(1+|\chi(\tau)|)}\right)+\frac{1}{(\tau+4)^2}\mathscr{B}\chi(\tau)\right)_{\tau=0}=u_0\in\mathbb{R}, \\ \left(\chi(\tau)-\frac{\tau^2}{8}\left(1+\frac{|\chi(\tau)|}{2(1+|\chi(\tau)|)}\right)+\frac{1}{(\tau+4)^2}\mathscr{B}\chi(\tau)\right)_{\tau=0}=0, \\ \left(\chi(\tau)-\frac{\tau^2}{8}\left(1+\frac{|\chi(\tau)|}{2(1+|\chi(\tau)|)}\right)+\frac{1}{(\tau+4)^2}\mathscr{B}\chi(\tau)\right)_{\tau=1}+\frac{|\chi(\frac{1}{2})|}{9(1+|\chi(\frac{1}{2})|)}+\frac{3}{17}=u_1\in\mathbb{R}. \end{cases} \tag{15}$$

First, let us check assumptions (A_1), (A_2), and (A_3). For all $\tau\in[0,1]$ and $v, w\in\mathbb{R}$, we have

$$\begin{aligned} \|\mathscr{F}(\tau,v,\mathscr{B}v)-\mathscr{F}(\tau,w,\mathscr{B}w)\| &\le \frac{\tau^2}{2}\frac{\pi}{16(\tau^2+2)}\frac{\|v-w\|}{(1+\|v\|)(1+\|w\|)}+\frac{\pi}{64}\|\mathscr{B}v-\mathscr{B}w\| \\ &\le \frac{\pi}{64}\left[\|v-w\|+\|\mathscr{B}v-\mathscr{B}w\|\right]. \\ \|\mathscr{F}(\tau,v,\mathscr{B}v)\| &= \left\|\frac{\tau^2}{2}\left(e^{\tau}+\frac{\pi}{16(\tau^2+2)}\frac{|v|}{(1+|v|)}\right)+\frac{\pi}{64}\mathscr{B}v\right\| \\ &\le \frac{e}{2}+\frac{\pi}{64}\left[\|v\|+\|\mathscr{B}v\|\right]. \end{aligned}$$

Then, assumption (A_1) holds with $L_{\mathscr{F}}=\tilde{L}_{\mathscr{F}}=\frac{\pi}{64}$ and $\widehat{L}_{\mathscr{F}}=\frac{e}{2}$. We have

$$\begin{aligned} \|\mathscr{H}(\tau,v,\mathscr{B}v)-\mathscr{H}(\tau,w,\mathscr{B}w)\| &\le \frac{\tau^2}{16}\frac{\|v-w\|}{(1+\|v\|)(1+\|w\|)}+\frac{1}{(\tau+4)^2}\|\mathscr{B}v-\mathscr{B}w\| \\ &\le \frac{1}{16}\left[\|v-w\|+\|\mathscr{B}v-\mathscr{B}w\|\right]. \\ \|\mathscr{H}(\tau,v,\mathscr{B}v)\| &= \left\|\frac{\tau^2}{8}\left(1+\frac{|v|}{2(1+|v|)}\right)+\frac{1}{(\tau+4)^2}\mathscr{B}v\right\| \\ &\le \frac{1}{8}+\frac{1}{16}\left[\|v\|+\|\mathscr{B}v\|\right]. \end{aligned}$$

Therefore, assumption (A_2) holds with $M_{\mathscr{H}}=\tilde{M}_{\mathscr{H}}=\frac{1}{16}$ and $\widehat{M}_{\mathscr{H}}=\frac{1}{8}$. We have

$$\begin{aligned} \|\varphi(v)-\varphi(w)\| &\le \frac{1}{9}\frac{\|v-w\|}{(1+\|v\|)(1+\|w\|)}\le\frac{1}{9}\|v-w\|. \\ \|\varphi(v)\| &= \left\|\frac{|v|}{9(1+|v|)}+\frac{3}{17}\right\|\le\frac{3}{17}+\frac{1}{9}\|v\|. \end{aligned}$$

Hence, assumption (A_3) holds with $K_{\varphi}=\tilde{K}_{\varphi}=\frac{1}{9}$ and $\widehat{K}_{\varphi}=\frac{3}{17}$.

Condition (11) also holds:

$$\frac{(\mathscr{G}(1)-\mathscr{G}(0)))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(1,0)}K_{\varphi}=\frac{1}{e^{-1}}\frac{1}{9}\approx 0.302<1.$$

We observe that all conditions of Theorem 2 are satisfied. Then the nonlinear generalized Caputo proportional fractional integro-differential system (15) has at least a solution $\chi \in C(\Pi, \mathbb{R})$.

The uniqueness of the solution to the problem (15) amounts to condition (14) that holds as follows:

$$\begin{aligned}\Lambda &= \frac{(\mathscr{G}(1)-\mathscr{G}(0))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(1,0)}K_{\varphi} + M_{\mathscr{H}}(1+\mathscr{B}^*) + \frac{(\mathscr{G}(1)-\mathscr{G}(0))^{\delta+\zeta}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}L_{\mathscr{F}}(1+\mathscr{B}^*)\\ &= \frac{1}{e^{-1}}\frac{1}{9} + \frac{e}{16} + \frac{1}{\frac{1}{4}\Gamma(3)}\frac{\pi}{64}e \approx 0.7386 < 1.\end{aligned}$$

This implies that the nonlinear generalized Caputo proportional fractional integro-differential system (15) has a unique solution $\chi \in C(\Pi, \mathbb{R})$.

Example 2. Let $\Pi = [1,e]$, $\rho = \delta = \frac{1}{2}$, $\zeta = \frac{5}{3}$, $\mathscr{G}(\tau) = \ln(\tau)$,

$$\begin{aligned}\mathscr{F}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)) &= \frac{1}{2}\left(\frac{\tau}{2} + \frac{\sin\left(\frac{\pi}{5}\chi(\tau)\right)}{9(e^{\tau}+2)}\right) + \frac{\pi}{90}\mathscr{B}\chi(\tau),\\ \mathscr{H}(\tau,\chi(\tau),\mathscr{B}\chi(\tau)) &= \frac{\tau}{4}\left(1 + \frac{e^{-\tau}}{8(\tau^4+2)}\sin\left(\frac{\pi}{4}\chi(\tau)\right)\right) + \frac{e\pi}{2(e^{\tau}+4)^3}\mathscr{B}\chi(\tau),\\ \varphi(\chi) &= \frac{\sin\left(\chi\left(\frac{1}{2}\right)\right)}{13} + \frac{1}{18},\end{aligned}$$

where

$$\begin{aligned}\mathscr{B}\chi(\tau) &= \int_1^{\tau}\frac{s}{\tau^2+3}|\chi(s)|ds,\\ \mathscr{B}^* &= \max_{\tau\in[1,e]}\int_1^{\tau}\frac{s}{\tau^2+3}ds = \frac{e^2-1}{6} \simeq 1,0648.\end{aligned}$$

Therefore, we have the following nonlinear generalized Caputo proportional fractional integro-differential system:

$$\begin{cases} {}^{C}_{\frac{1}{2}}D^{\frac{5}{3},\ln(\tau)}_{1^+}\left[{}^{C}_{\frac{1}{2}}D^{\frac{1}{2},\ln(\tau)}_{0^+}\left(\chi(\tau) - \frac{\tau}{4}\left(1+\frac{e^{-\tau}}{8(\tau^4+2)}\sin\left(\frac{\pi}{4}\chi(\tau)\right)\right) + \frac{e\pi}{2(e^{\tau}+4)^3}\mathscr{B}\chi(\tau)\right)\right] \\ \quad = \frac{1}{2}\left(\frac{\tau}{2} + \frac{\sin\left(\frac{\pi}{5}\chi(\tau)\right)}{9(e^{\tau}+2)}\right) + \frac{\pi}{90}\mathscr{B}\chi(\tau), \quad \tau\in\Pi=[1,e], \\ \left(\chi(\tau) - \frac{\tau}{4}\left(1+\frac{e^{-\tau}}{8(\tau^4+2)}\sin\left(\frac{\pi}{4}\chi(\tau)\right)\right) + \frac{e\pi}{2(e^{\tau}+4)^3}\mathscr{B}\chi(\tau)\right)_{\tau=1} = u_0 \in \mathbb{R}, \\ \left(\chi(\tau) - \frac{\tau}{4}\left(1+\frac{e^{-\tau}}{8(\tau^4+2)}\sin\left(\frac{\pi}{4}\chi(\tau)\right)\right) + \frac{e\pi}{2(e^{\tau}+4)^3}\mathscr{B}\chi(\tau)\right)_{\tau=1} = 0, \\ \left(\chi(\tau) - \frac{\tau^2}{8}\left(1+\frac{|\chi(\tau)|}{2(1+|\chi(\tau)|)}\right) + \frac{1}{(\tau+4)^2}\mathscr{B}\chi(\tau)\right)_{\tau=e} + \frac{\sin\left(\chi\left(\frac{1}{2}\right)\right)}{13} + \frac{1}{18} = u_1 \in \mathbb{R}. \end{cases} \tag{16}$$

Similar to the previous example first we show that assumptions (A_1), (A_2), and (A_3) hold. For all $\tau \in [1,e]$ and $v, w \in \mathbb{R}$ we have

$$\begin{aligned}\|\mathscr{F}(\tau,v,\mathscr{B}v) - \mathscr{F}(\tau,w,\mathscr{B}w)\| &\leq \frac{1}{2}\frac{\pi}{45(e^{\tau}+2)}\parallel v-w \parallel + \frac{\pi}{90}\|\mathscr{B}v - \mathscr{B}w\|\\ &\leq \frac{\pi}{90}\left[\| v-w \| + \| \mathscr{B}v - \mathscr{B}w \|\right].\end{aligned}$$

$$\begin{aligned}\|\mathscr{F}(\tau,v,\mathscr{B}v)\| &= \left\|\frac{1}{2}\left(\frac{\tau}{2}+\frac{\sin\left(\frac{\pi}{5}v\right)}{9(e^{\tau}+2)}\right)+\frac{\pi}{90}\mathscr{B}v\right\| \\ &\leq \frac{e}{4}+\frac{\pi}{90}\left[\| v \| + \| \mathscr{B}v \|\right].\end{aligned}$$

Then, assumption (A_1) holds with $L_{\mathscr{F}} = \tilde{L}_{\mathscr{F}} = \frac{\pi}{90}$ and $\widehat{L}_{\mathscr{F}} = \frac{e}{4}$. We have

$$\begin{aligned}\|\mathscr{H}(\tau,v,\mathscr{B}v)-\mathscr{H}(\tau,w,\mathscr{B}w)\| &\leq \frac{e\pi}{128} \| v-w \| + \frac{e\pi}{2(e^{\tau}+4)^3}\|\mathscr{B}v-\mathscr{B}w\| \\ &\leq \frac{e\pi}{128}\left[\| v-w \| + \| \mathscr{B}v-\mathscr{B}w \|\right]. \\ \|\mathscr{H}(\tau,v,\mathscr{B}v)\| &= \left\|\frac{\tau}{4}\left(1+\frac{e^{-\tau}}{8(\tau^4+2)}\sin\left(\frac{\pi}{4}v\right)\right)+\frac{e\pi}{2(e^{\tau}+4)^3}\mathscr{B}v\right\| \\ &\leq \frac{e}{4}+\frac{e\pi}{128}\left[\| v \| + \| \mathscr{B}v \|\right].\end{aligned}$$

Therefore, assumption (A_2) holds with $M_{\mathscr{H}} = \tilde{M}_{\mathscr{H}} = \frac{e\pi}{128}$ and $\widehat{M}_{\mathscr{H}} = \frac{e}{4}$. Also we have

$$\|\varphi(v)-\varphi(w)\| \leq \frac{1}{13} \| v-w \| .$$

$$\|\varphi(v)\| \leq \frac{1}{18}+\frac{1}{13} \| v \| .$$

Hence, assumption (A_3) holds with $K_{\varphi} = \tilde{K}_{\varphi} = \frac{1}{13}$ and $\widehat{K}_{\varphi} = \frac{1}{18}$.

Condition (11) holds since:

$$\frac{(\mathscr{G}(e)-\mathscr{G}(1)))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(e,1)}K_{\varphi} = \frac{e}{13} \approx 0.2090 < 1.$$

We note that all conditions of Theorem 2 are satisfied. Then the system (16) has at least a solution $\chi \in C(\Pi,\mathbb{R})$. Condition (14) for the uniqueness of the solution of problem (15) also holds:

$$\begin{aligned}\Lambda &= \frac{(\mathscr{G}(e)-\mathscr{G}(1))^{\delta}}{\Omega^{\delta}_{\mathscr{G}}(e,1)}K_{\varphi}+M_{\mathscr{H}}(1+\mathscr{B}^*)+\frac{(\mathscr{G}(e)-\mathscr{G}(1))^{\delta+\zeta}}{\rho^{\delta+\zeta}\Gamma(\delta+\zeta+1)}L_{\mathscr{F}}(1+\mathscr{B}^*) \\ &= \frac{e}{13}+\frac{2,0648\times e\pi}{128}+\frac{2,0648\times\pi}{90\times(\frac{1}{2})^{\frac{13}{6}}\Gamma(\frac{19}{6})} \\ &\approx 0.4847 < 1.\end{aligned}$$

This implies that the nonlinear generalized Caputo proportional fractional integro-differential system (16) has a unique solution $\chi \in C(\Pi,\mathbb{R})$.

5 Conclusions

This study explored the existence and uniqueness of the solution to a nonlinear fractional integro-differential equation with a non-local condition, involving the generalized Caputo proportional fractional derivative of two different orders, $1 < \zeta < 2$ and $0 < \delta < 1$. To demonstrate the existence and uniqueness of the solution, we applied Krasnoselskiis and Banachs fixed point theorems, respectively. Finally, we provided two examples to illustrate our main results.

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