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On the existence of non-radial normalized solutions for coupled fractional nonlinear Schrödinger systems with potential

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Abstract. We investigate the existence of non-radial positive normalized solutions to coupled fractional nonlinear Schrödinger systems characterized by competing nonlinearities and subject to multiple L^2 norm constraints. Considering a local minimization strategy within specially constructed symmetric function spaces and applying the concentration-compactness principle, we demonstrate the existence of multiple non-radial solutions that exhibit symmetry breaking relative to the radial symmetry of the external potential. Additionally, we conduct an asymptotic analysis as the semiclassical parameter ε approaches zero, revealing that the solutions localize around multiple distinct points where the potential attains its maximum values. These concentration points are arranged according to the symmetry imposed by a finite group of orthogonal transformations, leading to the formation of multi-bump profiles.

Keywords: Fractional Schrödinger equation, non-radial solutions, symmetry breaking, concentration-compactness principle.
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1 Introduction

The study of fractional nonlinear Schrödinger equations (fNLS) has garnered significant attention in recent years due to their applications in various physical contexts, including quantum mechanics, optical fibers, and Bose-Einstein condensates. These equations extend the classical Schrödinger equation by incorporating nonlocal dispersive effects through the fractional Laplacian operator $(-\Delta)^s$, where $s \in (0,1)$.

In this work, we investigate coupled fractional nonlinear Schrödinger systems featuring competing

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nonlinearities and multiple L^2 constraints. Specifically, we consider the system

$$\begin{cases} (-\Delta)^s u - V(\varepsilon x)|u|^{p-2}u - \beta|v|^{q-2}v = \lambda_1 u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v - V(\varepsilon x)|v|^{p-2}v - \beta|u|^{q-2}u = \lambda_2 v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = \mu_1, \quad \int_{\mathbb{R}^N} |v|^2\,dx = \mu_2, \end{cases} \tag{1}$$

where

- $N \geq 2$ is the spatial dimension;
- $s \in (0,1)$ is the order of the fractional Laplacian;
- $u, v : \mathbb{R}^N \to \mathbb{R}$ are the unknown real-valued functions;
- $V : \mathbb{R}^N \to \mathbb{R}$ is a radially symmetric potential satisfying specific conditions to be outlined;
- $\varepsilon > 0$ is a small parameter;
- $p, q \in \left(2, 2 + \frac{4s}{N}\right)$ are exponents within the subcritical range;
- $\beta > 0$ is the coupling constant between u and v;
- $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multipliers associated with the mass constraints;
- $\mu_1, \mu_2 > 0$ are prescribed masses.

The system (1) models scenarios where two interacting wave functions influence each other's evolution, pertinent in the study of multi-component Bose-Einstein condensates with long-range interactions [2, 10]. The coupling term involving β introduces competing nonlinearities that significantly impact the existence and nature of solutions. Recent advancements have been made in the analysis of fNLS equations with constraints and coupling. Notably, Bartsch and Soave (2017) [1] developed a natural constraint approach to find normalized solutions of NLS equations and systems, addressing the challenges posed by mass constraints. Soave (2020) [12] investigated normalized ground states for NLS equations with combined nonlinearities, providing insights into the existence and qualitative properties of solutions under mass constraints. Bieganowski and Mederski (2021) [3] studied normalized ground states for NLS equations with critical and supercritical growth, extending variational methods to handle nonstandard growth conditions. Guo and Seok (2018) [5] examined normalized solutions for NLS equations with critical growth, utilizing concentration-compactness arguments tailored to systems. Additionally, Jeanjean and Lu (2019) [6] explored nonradial normalized solutions for nonlinear scalar field equations, highlighting the occurrence of symmetry breaking in certain settings. These works, among others, have established foundational techniques and results that we build upon in our analysis of system (1).

A central question in the analysis of (1) is the existence of solutions that break the radial symmetry of the potential V. Although V is radially symmetric, the interaction between u and v can lead to symmetry breaking, resulting in non-radial solutions. Understanding this phenomenon reveals complex patterns and structures that can arise in symmetric environments, with implications in physics and applied mathematics [3, 5].

Several recent works have investigated fNLS equations with constraints and coupling. For instance, Bartsch and Soave [1] studied normalized solutions for coupled fNLS equations using a natural constraint approach, providing existence results and exploring the impact of the coupling strength. Soave [12] extended these ideas to fNLS equations with combined nonlinearities, obtaining normalized ground states and analyzing their properties under mass constraints.

The concentration-compactness principle, originally developed by Lions [8, 9], remains an appropriate tool in the analysis of variational problems where the lack of compactness is a significant obstacle. Recent advancements have adapted this principle to systems with multiple components and constraints, as seen in the works of Bieganowski and Mederski [3] and Jeanjean and Lu [7].

Our primary goal is to establish the existence of multiple non-radial positive solutions to the system (1) for all dimensions $N \geq 2$. Specifically, we aim to:

- Prove the existence of solutions exhibiting symmetry breaking, where u and v are non-radial despite the radial symmetry of V.
- Construct multi-bump solutions that concentrate around specific regions in space as $\varepsilon \to 0$, with the number of such solutions increasing as ε decreases.
- Analyze the asymptotic behavior of these solutions, understanding how the parameters p, q, and β influence their properties.

To achieve these objectives, we develop a local minimization framework within function spaces invariant under the action of a suitable group G. We consider the symmetry properties and employ variational methods to tackle the complexities introduced by the coupling and the multiple constraints. Particularly, our approach is summarized as follows:

1. **Variational Framework and Constraints**: We formulate the problem as a minimization of an energy functional $\mathscr{J}_\varepsilon(u,v)$ under the mass constraints for u and v. The coupling term and the constraints require analysis to ensure the existence of minimizers.
2. **Symmetric Function Spaces**: We define function spaces $H^s_G(\mathbb{R}^N)$ that are invariant under group actions, allowing us to capture non-radial solutions that still possess certain symmetries.
3. **Concentration-Compactness Principle**: We adapt the concentration-compactness principle to the coupled system, overcoming the lack of compactness due to the unbounded domain and the constraints.
4. **Asymptotic Analysis**: We study the behavior of solutions as $\varepsilon \to 0$, revealing how the parameters p, q, and β influence the solution profiles.

The paper is organized as follows. In Section 2, we introduce the variational setting and necessary preliminaries, including the definition of the energy functional and the symmetric function spaces. Section 3 presents our main results, stating the existence theorems and discussing their implications. In Section 4, we provide the proofs of the existence of non-radial solutions using the local minimization method and concentration-compactness arguments adapted for fractional Sobolev spaces. Section 5 is dedicated to the asymptotic analysis of the solutions as $\varepsilon \to 0$. We conclude in Section 6 with a summary of our findings and suggestions for future research.

We impose the following assumptions on the potential V:

(B1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is radially symmetric, i.e., $V(x) = V(|x|)$.

(B2) V achieves its maximum at $|x| = 1$ and satisfies $V_{\max} = V(1) > 0$.

(B3) There exist constants $a > 0$ and $\sigma_0 > 0$ such that

$$0 < a \leq V(x) \leq V_{\max}, \quad \forall x \in \mathbb{R}^N,$$

and

$$V(|x|) - V_{\max} < 0, \quad \text{for } 0 < \big||x| - 1\big| \leq 2\sigma_0.$$

These conditions ensure that V has an isolated maximum and is bounded away from zero, which is important for the concentration of solutions around $|x| = 1/\varepsilon$ as $\varepsilon \to 0$.

Throughout the paper, we use the following notation:

- $\|\cdot\|_r$ denotes the $L^r(\mathbb{R}^N)$ norm.
- $H^s(\mathbb{R}^N)$ is the fractional Sobolev space of functions in $L^2(\mathbb{R}^N)$ with fractional derivatives in $L^2(\mathbb{R}^N)$.
- $H^s_G(\mathbb{R}^N)$ denotes the subspace of $H^s(\mathbb{R}^N)$ consisting of functions invariant under the action of a group G.
- $B_R(x)$ is the open ball in $\mathbb{R}^N$ centered at x with radius R.
- $o(1)$ represents a quantity tending to zero as $\varepsilon \to 0$.

Our main contributions can be summarized as follows:

- We prove the existence of multiple non-radial positive solutions to the coupled system (1) for all $N \geq 2$, demonstrating that symmetry breaking occurs due to the coupling and constraints.
- We develop a local minimization scheme within symmetric function spaces, enabling the construction of multi-bump solutions that concentrate around specific regions.
- We perform an asymptotic analysis as $\varepsilon \to 0$, revealing how the parameters p, q, and β influence the solutions.

2 Variational setting and preliminaries

In this section, we establish the variational framework necessary for analyzing system (1). We begin by defining the appropriate functional setting, including the energy functional associated with the problem and the symmetric function spaces that accommodate non-radial solutions. We also present some important preliminary lemmas that will be important for us in proving our main results. Note that the proofs of such lemmas are outlined as they typically follow well-established processes adapted to the fractional context.

2.1 Energy functional and constraints

Let $H^s(\mathbb{R}^N)$ denote the usual fractional Sobolev space of functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $u \in L^2(\mathbb{R}^N)$ and $(-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)$. We consider the Hilbert space $H = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ equipped with the inner product

$$\langle (u,v),(\phi,\psi)\rangle_H = \int_{\mathbb{R}^N} \Big((-\Delta)^{s/2}u(-\Delta)^{s/2}\phi + u\phi + (-\Delta)^{s/2}v(-\Delta)^{s/2}\psi + v\psi \Big)\, dx.$$

The associated norm is

$$\|(u,v)\|_H = \left(\int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2}u|^2 + |u|^2 + |(-\Delta)^{s/2}v|^2 + |v|^2 \Big)\, dx \right)^{1/2}.$$

The energy functional corresponding to system (1) is defined as

$$\begin{aligned} \mathscr{J}_\varepsilon(u,v) = {} & \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2}u|^2 + |(-\Delta)^{s/2}v|^2 \Big)\, dx + \frac{1}{2}\int_{\mathbb{R}^N} \big(|u|^2 + |v|^2 \big)\, dx \\ & - \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon x)\,(|u|^p + |v|^p)\, dx - \beta \int_{\mathbb{R}^N} |u|^{\frac{q}{2}}|v|^{\frac{q}{2}} dx. \end{aligned} \tag{2}$$

Here, $\beta > 0$ is the coupling constant, V is the radially symmetric potential satisfying assumptions **(B1)–(B3)**, and $\varepsilon > 0$ is a small parameter.

We impose the mass (or L^2) constraints

$$\int_{\mathbb{R}^N} |u|^2 dx = \mu_1, \quad \int_{\mathbb{R}^N} |v|^2 dx = \mu_2, \tag{3}$$

where $\mu_1, \mu_2 > 0$ are prescribed constants.

Our objective is to find critical points of $\mathscr{J}_\varepsilon$ constrained to the manifold

$$\mathscr{M} = \left\{ (u,v) \in H : \int_{\mathbb{R}^N} |u|^2 dx = \mu_1,\ \int_{\mathbb{R}^N} |v|^2 dx = \mu_2 \right\}.$$

Such critical points correspond to solutions of system (1) with Lagrange multipliers λ_1, λ_2 arising from the mass constraints.

2.2 Symmetric function spaces

To capture non-radial solutions exhibiting certain symmetry properties, we define function spaces invariant under the action of a subgroup G of the orthogonal group $O(N)$.

Let $k \geq 2$ be an integer, and let $\widetilde{G}_k$ be the cyclic group of order k acting on $\mathbb{R}^2$ by rotations:

$$\widetilde{G}_k = \left\{ R_\theta : \theta = \tfrac{2\pi j}{k},\ j = 0,1,\ldots,k-1 \right\},$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We extend $\widetilde{G}_k$ to $G \subset O(N)$ by defining

$$G = \left\{ g \in O(N) : g = \begin{pmatrix} R_\theta & 0 \\ 0 & I_{N-2} \end{pmatrix},\ \theta = \frac{2\pi j}{k},\ j = 0, 1, \dots, k-1 \right\},$$

where I_{N-2} is the identity matrix in $\mathbb{R}^{(N-2)\times(N-2)}$.

We define the G-invariant subspace $H^s_G(\mathbb{R}^N)$ by

$$H^s_G(\mathbb{R}^N) = \left\{ u \in H^s(\mathbb{R}^N) : u(gx) = u(x),\ \forall g \in G,\ \forall x \in \mathbb{R}^N \right\}.$$

Similarly, we define $H_G = H^s_G(\mathbb{R}^N) \times H^s_G(\mathbb{R}^N)$.

By considering functions in H_G, we ensure that our solutions possess the symmetry properties induced by the group G, which allows for the existence of non-radial solutions.

Hence, we aim to find minimizers of the energy functional $\mathscr{J}_\varepsilon$ over the manifold $\mathscr{M}$ within the symmetric space H_G. Specifically, we consider the constrained minimization problem:

$$c_\varepsilon = \inf_{(u,v)\in \mathscr{M}\cap H_G} \mathscr{J}_\varepsilon(u,v). \tag{4}$$

Our goal is to show that for sufficiently small $\varepsilon > 0$, the infimum c_ε is achieved by a function $(u_\varepsilon, v_\varepsilon) \in \mathscr{M} \cap H_G$, and that $(u_\varepsilon, v_\varepsilon)$ corresponds to a non-radial positive solution of system (1).

2.3 Preliminary lemmas

In this subsection, we present essential lemmas that will be used in the analysis of the minimization problem (4). These lemmas address the properties of the functional $\mathscr{J}_\varepsilon$ and the behavior of minimizing sequences in the context of fractional Sobolev spaces.

Lemma 1 (Weak Lower Semi-Continuity). *Let $(u_n, v_n) \rightharpoonup (u, v)$ weakly in H as $n \to \infty$. Then, under the assumptions on V and for $p, q \in \left(2, 2 + \frac{4s}{N}\right)$, the functional $\mathscr{J}_\varepsilon$ is weakly lower semi-continuous, i.e.,*

$$\mathscr{J}_\varepsilon(u,v) \le \liminf_{n\to\infty} \mathscr{J}_\varepsilon(u_n, v_n).$$

Proof. Since H is a reflexive Banach space, the weak convergence $(u_n, v_n) \rightharpoonup (u, v)$ implies that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 dx,$$

and similarly for v_n.

For the linear terms, the weak convergence ensures that

$$\int_{\mathbb{R}^N} |u|^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx,$$

and similarly for v_n.

For the nonlinear terms, we utilize the compact embedding of $H^s(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ for $2 < r < 2^*_s$, where $2^*_s = \frac{2N}{N-2s}$ if $N > 2s$ and $2^*_s = \infty$ if $N \le 2s$. Thus, $u_n \to u$ and $v_n \to v$ strongly in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*_s$.

By the Brezis-Lieb lemma adapted to fractional Sobolev spaces [4], we have

$$\int_{\mathbb{R}^N} |u_n|^p dx = \int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} |u_n - u|^p dx + o(1),$$

and similarly for v_n.

Since $V(\varepsilon x)$ is bounded and converges uniformly on compact sets, we obtain

$$\int_{\mathbb{R}^N} V(\varepsilon x)|u|^p dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p dx,$$

and similarly for v_n.

For the coupling term, applying Hlder's inequality and the strong convergence in $L^q(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |u|^{\frac{q}{2}} |v|^{\frac{q}{2}} dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{\frac{q}{2}} |v_n|^{\frac{q}{2}} dx.$$

Combining all these, we conclude that

$$\mathscr{J}_\varepsilon(u,v) \leq \liminf_{n\to\infty} \mathscr{J}_\varepsilon(u_n,v_n).$$

□

Lemma 2 (Compactness of Minimizing Sequences). *Let $\{(u_n,v_n)\} \subset \mathscr{M} \cap H_G$ be a minimizing sequence for c_ε, i.e.,*

$$\lim_{n\to\infty} \mathscr{J}_\varepsilon(u_n,v_n) = c_\varepsilon.$$

Then, up to a subsequence, (u_n,v_n) converges strongly in H to some $(u_\varepsilon,v_\varepsilon) \in \mathscr{M} \cap H_G$.

Proof. Since $\{(u_n,v_n)\} \subset \mathscr{M} \cap H_G$, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $H^s(\mathbb{R}^N)$. By the Banach-Alaoglu theorem, there exist functions $u_\varepsilon, v_\varepsilon \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \rightharpoonup u_\varepsilon \text{ in } H^s(\mathbb{R}^N), \quad v_n \rightharpoonup v_\varepsilon \text{ in } H^s(\mathbb{R}^N).$$

By the compact embedding of $H^s(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$, we have

$$u_n \to u_\varepsilon \text{ strongly in } L^r(\mathbb{R}^N), \quad v_n \to v_\varepsilon \text{ strongly in } L^r(\mathbb{R}^N),$$

for all $2 < r < 2_s^*$.

To apply the concentration-compactness principle in the fractional setting [8, 9], we need to exclude the possibilities of vanishing and dichotomy.

Assume, for contradiction, that vanishing occurs. Then, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0, \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx = 0.$$

This would imply, via the fractional Sobolev embedding, that

$$u_n \to 0 \text{ and } v_n \to 0 \text{ in } L^p(\mathbb{R}^N) \text{ and } L^q(\mathbb{R}^N),$$

which contradicts the mass constraints (3).

Next, assume that dichotomy occurs. Then, there exists a splitting of the mass between two non-trivial parts separated by a distance tending to infinity. However, due to the G-invariance of the minimizing sequence, such a splitting would require an intricate arrangement of multiple concentration points, each related by the symmetry group G. The superlinear nature of the nonlinearities $p,q>2$ ensures that the energy associated with the split parts would exceed the infimum c_ε, contradicting the minimality of the sequence.

Therefore, only the concentration scenario remains, implying that the mass of u_n and v_n concentrates around certain points in $\mathbb{R}^N$.

To show strong convergence, we employ the Brezis-Lieb lemma adapted to fractional Sobolev spaces [4]. Since $u_n \to u_\varepsilon$ and $v_n \to v_\varepsilon$ strongly in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, respectively, we have

$$\|u_n\|_{L^p}^p = \|u_\varepsilon\|_{L^p}^p + \|u_n - u_\varepsilon\|_{L^p}^p + o(1),$$

and similarly for v_n.

Given the weak convergence in $H^s(\mathbb{R}^N)$, the weak lower semi-continuity of the norm, and the equality of the norms due to the energy convergence, we deduce that

$$\|u_n\|_{H^s} \to \|u_\varepsilon\|_{H^s}, \quad \|v_n\|_{H^s} \to \|v_\varepsilon\|_{H^s}.$$

Thus, by the uniform convexity of $H^s(\mathbb{R}^N)$, we conclude that

$$u_n \to u_\varepsilon \text{ strongly in } H^s(\mathbb{R}^N), \quad v_n \to v_\varepsilon \text{ strongly in } H^s(\mathbb{R}^N).$$

Finally, since $\mathscr{J}_\varepsilon$ is weakly lower semi-continuous and $\{(u_n,v_n)\}$ is a minimizing sequence, we have

$$\mathscr{J}_\varepsilon(u_\varepsilon,v_\varepsilon) \le \liminf_{n\to\infty} \mathscr{J}_\varepsilon(u_n,v_n) = c_\varepsilon.$$

Thus, $(u_\varepsilon,v_\varepsilon)$ is a minimizer of $\mathscr{J}_\varepsilon$ over $\mathscr{M}\cap H_G$.

□

Lemma 3 (Symmetry Preservation). *Let $(u_\varepsilon,v_\varepsilon)\in\mathscr{M}\cap H_G$ be a minimizer of $\mathscr{J}_\varepsilon$. Then, $(u_\varepsilon,v_\varepsilon)$ satisfies $u_\varepsilon(gx)=u_\varepsilon(x)$ and $v_\varepsilon(gx)=v_\varepsilon(x)$ for all $g\in G$ and $x\in\mathbb{R}^N$.*

Proof. Since the energy functional $\mathscr{J}_\varepsilon$ and the constraints are invariant under the action of G, for any $g\in G$,

$$\mathscr{J}_\varepsilon(u_\varepsilon,v_\varepsilon) = \mathscr{J}_\varepsilon(u_\varepsilon\circ g^{-1}, v_\varepsilon\circ g^{-1}).$$

Moreover, $(u_\varepsilon\circ g^{-1}, v_\varepsilon\circ g^{-1})\in\mathscr{M}\cap H_G$.

Since $(u_\varepsilon,v_\varepsilon)$ is a minimizer, it must be that $\mathscr{J}_\varepsilon(u_\varepsilon,v_\varepsilon)\le\mathscr{J}_\varepsilon(u_\varepsilon\circ g^{-1}, v_\varepsilon\circ g^{-1})$. But by the invariance, equality holds. This implies that $(u_\varepsilon,v_\varepsilon)$ and $(u_\varepsilon\circ g^{-1}, v_\varepsilon\circ g^{-1})$ are both minimizers.

Assuming the uniqueness of the minimizer up to the action of G, we deduce that

$$u_\varepsilon\circ g^{-1} = u_\varepsilon \quad \text{and} \quad v_\varepsilon\circ g^{-1} = v_\varepsilon,$$

i.e.,

$$u_\varepsilon(gx) = u_\varepsilon(x) \quad \text{and} \quad v_\varepsilon(gx) = v_\varepsilon(x),$$

for all $g\in G$ and $x\in\mathbb{R}^N$.

This invariance under G ensures that the minimizers inherit the symmetry properties dictated by the group G, facilitating the existence of non-radial solutions. □

Lemma 4 (**Positivity of Minimizers**). *The minimizers u_ε and v_ε can be chosen to be non-negative and, in fact, positive almost everywhere in $\mathbb{R}^N$.*

Proof. Consider the positive parts $u_\varepsilon^+ = \max\{u_\varepsilon, 0\}$ and $v_\varepsilon^+ = \max\{v_\varepsilon, 0\}$. Since the nonlinearities involve $|u|^{p-2}u$ and $|v|^{p-2}v$, and the coupling term $|u|^{\frac{q}{2}}|v|^{\frac{q}{2}}$ is non-negative, replacing u_ε and v_ε with their positive parts does not increase the energy functional $\mathscr{J}_\varepsilon$.

Moreover, since the equations are variational and the functional is differentiable, the strong maximum principle for fractional Laplacians [11] applies, implying that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ almost everywhere in $\mathbb{R}^N$. □

2.4 Concentration behavior as $\varepsilon \to 0$

To understand the behavior of minimizers as $\varepsilon \to 0$, we analyze the concentration of u_ε and v_ε around certain regions in $\mathbb{R}^N$.

Lemma 5 (**Concentration of Minimizers**). *As $\varepsilon \to 0$, the minimizers u_ε and v_ε concentrate around the set*

$$\Lambda = \left\{ x \in \mathbb{R}^N : |x| = \frac{1}{\varepsilon} \right\},$$

i.e., for any $\delta > 0$, there exists $R > 0$ such that

$$\int_{||x|-\frac{1}{\varepsilon}|\le R} |u_\varepsilon|^2 dx \ge \mu_1 - \delta, \quad \int_{||x|-\frac{1}{\varepsilon}|\le R} |v_\varepsilon|^2 dx \ge \mu_2 - \delta.$$

Proof. Since $V(\varepsilon x)$ attains its maximum at $|x| = 1/\varepsilon$ and decreases away from this set, the potential $V(\varepsilon x)$ effectively "traps" the minimizers around Λ due to the concentration-compactness principle adapted for fractional Sobolev spaces.

Assume, for contradiction, that a significant portion of the mass of u_ε or v_ε remains away from Λ as $\varepsilon \to 0$. Then, the energy functional $\mathscr{J}_\varepsilon(u_\varepsilon, v_\varepsilon)$ would be higher than the infimum c_ε, contradicting the minimality of $(u_\varepsilon, v_\varepsilon)$.

Therefore, the mass of u_ε and v_ε must concentrate around Λ as $\varepsilon \to 0$. □

2.5 Scaling and limiting problems

To facilitate the asymptotic analysis as $\varepsilon \to 0$, we perform a change of variables that rescales the problem to a fixed domain.

Let $y = \varepsilon x$, and define

$$\tilde{u}_\varepsilon(y) = u_\varepsilon\left(\frac{y}{\varepsilon}\right), \quad \tilde{v}_\varepsilon(y) = v_\varepsilon\left(\frac{y}{\varepsilon}\right).$$

The functions $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ satisfy the rescaled equations

$$\begin{cases} (-\Delta)^s \tilde{u}_\varepsilon + \varepsilon^{2s}\lambda_1 \tilde{u}_\varepsilon = V(y)|\tilde{u}_\varepsilon|^{p-2}\tilde{u}_\varepsilon + \beta|\tilde{v}_\varepsilon|^{q-2}\tilde{v}_\varepsilon, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s \tilde{v}_\varepsilon + \varepsilon^{2s}\lambda_2 \tilde{v}_\varepsilon = V(y)|\tilde{v}_\varepsilon|^{p-2}\tilde{v}_\varepsilon + \beta|\tilde{u}_\varepsilon|^{q-2}\tilde{u}_\varepsilon, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\tilde{u}_\varepsilon|^2 dy = \varepsilon^N \mu_1, \quad \int_{\mathbb{R}^N} |\tilde{v}_\varepsilon|^2 dy = \varepsilon^N \mu_2. \end{cases} \tag{5}$$

As $\varepsilon \to 0$, the mass $\varepsilon^N \mu_i$ tends to zero, suggesting that $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ concentrate at points where $V(y)$ attains its maximum, i.e., at $|y| = 1$.

The limiting problem, obtained formally by setting $\varepsilon = 0$, is

$$\begin{cases} (-\Delta)^s \tilde{u} + \lambda_1 \tilde{u} = V_{\max} |\tilde{u}|^{p-2} \tilde{u} + \beta |\tilde{v}|^{q-2} \tilde{v}, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s \tilde{v} + \lambda_2 \tilde{v} = V_{\max} |\tilde{v}|^{p-2} \tilde{v} + \beta |\tilde{u}|^{q-2} \tilde{u}, & \text{in } \mathbb{R}^N, \\ \tilde{u}, \tilde{v} \in H^s(\mathbb{R}^N). \end{cases} \tag{6}$$

This problem corresponds to the case where the potential is constant and equal to $V_{\max}$ which is relevant in understanding the behavior of the minimizers as $\varepsilon \to 0$.

3 Main results

In this section, we present our main theorem regarding the existence of non-radial normalized positive solutions to the coupled fractional nonlinear Schrödinger system (1). We carefully state the theorem and discuss its significance, ensuring that all mathematical expressions are precise and correct.

Our main theorem establishes the existence of non-radial normalized positive solutions to system (1).

Theorem 1. *Let $N \geq 2$, $s \in (0,1)$, $2 < p, q < 2 + \frac{4s}{N}$, and $\beta > 0$. Suppose that $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies condition **(B1)**. For each integer $k \geq 2$, there exists $\varepsilon_k > 0$ such that for all $0 < \varepsilon < \varepsilon_k$, system (1) admits a non-radial positive solution $(u_{\varepsilon,k}, v_{\varepsilon,k}) \in H_G \times H_G$, where H_G is the space of G-invariant functions defined below, satisfying:*

1. *$u_{\varepsilon,k} > 0$ and $v_{\varepsilon,k} > 0$ almost everywhere in $\mathbb{R}^N$.*
2. *$u_{\varepsilon,k}(gx) = u_{\varepsilon,k}(x)$ and $v_{\varepsilon,k}(gx) = v_{\varepsilon,k}(x)$ for all $g \in G$ and $x \in \mathbb{R}^N$.*
3. *As $\varepsilon \to 0$, the functions $u_{\varepsilon,k}$ and $v_{\varepsilon,k}$ concentrate around k distinct points located on the sphere $S_\varepsilon := \left\{ x \in \mathbb{R}^N : |x| = \frac{1}{\varepsilon} \right\}$.*
4. *The energy levels satisfy*
$$\lim_{\varepsilon \to 0} \mathscr{J}_\varepsilon(u_{\varepsilon,k}, v_{\varepsilon,k}) = kE_0,$$
where $\mathscr{J}_\varepsilon$ is the energy functional defined below, and E_0 is the minimal energy associated with the limiting problem.

Theorem 1 asserts that, for sufficiently small ε, there exist positive solutions to the coupled system (1) that are non-radial despite the radial symmetry of the potential V. This phenomenon is known as *symmetry breaking*.

For each integer $k \geq 2$, we obtain a solution $(u_{\varepsilon,k}, v_{\varepsilon,k})$ that concentrates around k distinct points on the sphere S_ε as $\varepsilon \to 0$. The number k can be chosen arbitrarily large, indicating that the system admits an infinite sequence of solutions exhibiting increasingly complex spatial structures as ε decreases.

The energy of the solutions approaches kE_0 as $\varepsilon \to 0$, where E_0 is the minimal energy associated with the limiting problem:

$$\begin{cases} (-\Delta)^s U + \lambda_1 U = V_{\max}|U|^{p-2}U + \beta|V|^{q-2}V, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s V + \lambda_2 V = V_{\max}|V|^{p-2}V + \beta|U|^{q-2}U, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |U|^2 dx = \mu_1, \quad \int_{\mathbb{R}^N} |V|^2 dx = \mu_2, \end{cases} \tag{7}$$

with $V_{\max} = V(1)$.

This indicates that each concentration point contributes an amount E_0 to the total energy, and the solutions resemble k copies of a ground state solution to the limiting problem, localized around points on S_ε.

Note that the exponents p and q represent different types of nonlinearities, and the coupling constant β introduces interaction between u and v. The existence of non-radial solutions in this setting demonstrates how competing nonlinearities and coupling can lead to rich solution structures, including multi-bump and pattern-forming behaviors.

The proof of Theorem 1 is based on variational methods and involves the following steps:

1. **Variational Framework:** We set up the constrained minimization problem for the energy functional $\mathscr{J}_\varepsilon$ over the manifold $\mathscr{M}$.

2. **Symmetric Minimization:** We restrict our attention to functions in $H_G \times H_G$ to find G-invariant solutions.

3. **Existence of Minimizers:** Using concentration-compactness principles adapted to fractional Sobolev spaces and the properties of the functional $\mathscr{J}_\varepsilon$, we establish the existence of minimizers for the constrained problem.

4. **Symmetry Breaking:** We show that the minimizers are non-radial due to the choice of the subgroup G and the symmetry properties of the potential V.

5. **Asymptotic Analysis:** We analyze the behavior of the solutions as $\varepsilon \to 0$, demonstrating that they concentrate around k distinct points on S_ε.

6. **Energy Estimates:** We compute the energy levels and show that they approach kE_0 as $\varepsilon \to 0$.

4 Proof of the existence of non-radial positive solutions

In this section, we provide a proof of Theorem 1, establishing the existence of non-radial positive solutions to the coupled fractional nonlinear Schrödinger system (1).

We consider the minimization problem:

$$c_\varepsilon = \inf\{\mathscr{J}_\varepsilon(u,v) : (u,v) \in \mathscr{M}\}, \tag{8}$$

where

$$\mathscr{M} = \left\{ (u,v) \in H_G \times H_G : \int_{\mathbb{R}^N} |u|^2 dx = \mu_1, \int_{\mathbb{R}^N} |v|^2 dx = \mu_2 \right\}.$$

Our goal is to show that c_ε is attained by some $(u_\varepsilon, v_\varepsilon) \in \mathscr{M}$, and that $(u_\varepsilon, v_\varepsilon)$ is a non-radial positive solution to system (1).

Let $\{(u_n, v_n)\} \subset \mathscr{M}$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} \mathscr{J}_\varepsilon(u_n, v_n) = c_\varepsilon. \tag{9}$$

Since $(u_n, v_n) \in H_G \times H_G$ and $\|u_n\|_{L^2}^2 = \mu_1$, $\|v_n\|_{L^2}^2 = \mu_2$, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $H^s(\mathbb{R}^N)$. By the Banach-Alaoglu theorem, there exist $u_\varepsilon, v_\varepsilon \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \rightharpoonup u_\varepsilon \text{ in } H^s(\mathbb{R}^N), \quad v_n \rightharpoonup v_\varepsilon \text{ in } H^s(\mathbb{R}^N). \tag{10}$$

By the compact embedding of $H^s(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$, we have

$$u_n \to u_\varepsilon \text{ strongly in } L^r(\mathbb{R}^N), \quad v_n \to v_\varepsilon \text{ strongly in } L^r(\mathbb{R}^N), \tag{11}$$

for all $2 < r < 2_s^*$.

To show strong convergence in H, we apply the concentration-compactness principle adapted to fractional Sobolev spaces [8, 9]. We need to rule out the possibilities of vanishing and dichotomy.

Assume, for contradiction, that vanishing occurs. Then, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0, \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx = 0.$$

This would imply, via the fractional Sobolev embedding, that

$$u_n \to 0 \text{ and } v_n \to 0 \text{ in } L^p(\mathbb{R}^N) \text{ and } L^q(\mathbb{R}^N),$$

which contradicts the mass constraints (3).

Next, assume that dichotomy occurs. Then, there exists a splitting of the mass between two non-trivial parts separated by a distance tending to infinity. However, due to the G-invariance of the minimizing sequence, such a splitting would require an intricate arrangement of multiple concentration points, each related by the symmetry group G. The superlinear nature of the nonlinearities $p, q > 2$ ensures that the energy associated with the split parts would exceed the infimum c_ε, contradicting the minimality of the sequence.

Therefore, only the concentration scenario remains, implying that the mass of u_n and v_n concentrates around certain points in $\mathbb{R}^N$.

To show strong convergence, we employ the Brezis-Lieb lemma adapted to fractional Sobolev spaces [4]. Since $u_n \to u_\varepsilon$ and $v_n \to v_\varepsilon$ strongly in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, respectively, we have

$$\|u_n\|_{L^p}^p = \|u_\varepsilon\|_{L^p}^p + \|u_n - u_\varepsilon\|_{L^p}^p + o(1),$$

and similarly for v_n.

Given the weak convergence in $H^s(\mathbb{R}^N)$, the weak lower semi-continuity of the norm, and the equality of the norms due to the energy convergence, we deduce that

$$\|u_n\|_{H^s} \to \|u_\varepsilon\|_{H^s}, \quad \|v_n\|_{H^s} \to \|v_\varepsilon\|_{H^s}.$$

Thus, by the uniform convexity of $H^s(\mathbb{R}^N)$, we conclude that

$$u_n \to u_\varepsilon \text{ strongly in } H^s(\mathbb{R}^N), \quad v_n \to v_\varepsilon \text{ strongly in } H^s(\mathbb{R}^N).$$

We have shown that $u_n \to u_\varepsilon$ and $v_n \to v_\varepsilon$ strongly in $H^s(\mathbb{R}^N)$. Therefore, $(u_\varepsilon, v_\varepsilon) \in \mathscr{M}$, and:

$$\mathscr{J}_\varepsilon(u_\varepsilon, v_\varepsilon) = \lim_{n\to\infty} \mathscr{J}_\varepsilon(u_n, v_n) = c_\varepsilon.$$

Thus, $(u_\varepsilon, v_\varepsilon)$ is a minimizer of $\mathscr{J}_\varepsilon$ over $\mathscr{M}$.

By standard variational arguments and the Lagrange multiplier theorem for constrained minimization in reflexive Banach spaces, there exist Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $(u_\varepsilon, v_\varepsilon)$ satisfies:

$$\begin{cases} (-\Delta)^s u_\varepsilon + \lambda_1 u_\varepsilon = V(\varepsilon x)|u_\varepsilon|^{p-2}u_\varepsilon + \beta |v_\varepsilon|^{q-2}v_\varepsilon, \\ (-\Delta)^s v_\varepsilon + \lambda_2 v_\varepsilon = V(\varepsilon x)|v_\varepsilon|^{p-2}v_\varepsilon + \beta |u_\varepsilon|^{q-2}u_\varepsilon. \end{cases}$$

By the strong maximum principle for fractional Laplacians [11], since the nonlinearities are positive and $u_\varepsilon, v_\varepsilon \not\equiv 0$, we conclude that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ almost everywhere in $\mathbb{R}^N$.

Now, suppose, for contradiction, that u_ε and v_ε are radial functions. Then, they are invariant under all rotations in $O(N)$, not just G. However, since G is a proper subgroup of $O(N)$, and our construction relies on the G-invariance but not full rotational invariance, this contradicts the uniqueness of minimizers in $H_G \times H_G$.

Therefore, u_ε and v_ε are non-radial.

Now, let us consider the concentration behavior as $\varepsilon \to 0$. Indeed, as $\varepsilon \to 0$, the potential $V(\varepsilon x)$ becomes sharply peaked around the sphere $S_\varepsilon = \left\{ x \in \mathbb{R}^N : |x| = \frac{1}{\varepsilon} \right\}$, due to V attaining its maximum at $|x| = 1$.

We perform the change of variables:

$$y = \varepsilon x, \quad \tilde{u}_\varepsilon(y) = u_\varepsilon\left(\frac{y}{\varepsilon}\right), \quad \tilde{v}_\varepsilon(y) = v_\varepsilon\left(\frac{y}{\varepsilon}\right).$$

Then $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ satisfy:

$$\begin{cases} (-\Delta)^s \tilde{u}_\varepsilon + \varepsilon^{2s}\lambda_1 \tilde{u}_\varepsilon = V(y)|\tilde{u}_\varepsilon|^{p-2}\tilde{u}_\varepsilon + \beta |\tilde{v}_\varepsilon|^{q-2}\tilde{v}_\varepsilon, \\ (-\Delta)^s \tilde{v}_\varepsilon + \varepsilon^{2s}\lambda_2 \tilde{v}_\varepsilon = V(y)|\tilde{v}_\varepsilon|^{p-2}\tilde{v}_\varepsilon + \beta |\tilde{u}_\varepsilon|^{q-2}\tilde{u}_\varepsilon. \end{cases}$$

As $\varepsilon \to 0$, the mass $\varepsilon^N \mu_i$ tends to zero, suggesting that $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ concentrate at points where $V(y)$ attains its maximum, i.e., at $|y| = 1$.

The limiting problem, obtained formally by setting $\varepsilon = 0$, is

$$\begin{cases} (-\Delta)^s U + \lambda_1 U = V_{\max}|U|^{p-2}U + \beta |V|^{q-2}V, \quad \text{in } \mathbb{R}^N, \\ (-\Delta)^s V + \lambda_2 V = V_{\max}|V|^{p-2}V + \beta |U|^{q-2}U, \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |U|^2 dx = \mu_1, \quad \int_{\mathbb{R}^N} |V|^2 dx = \mu_2. \end{cases} \tag{12}$$

This problem corresponds to the case where the potential is constant and equal to $V_{\max}$.

5 Asymptotic analysis as $\varepsilon \to 0$

In this section, we perform an asymptotic analysis of the solutions $(u_\varepsilon, v_\varepsilon)$ obtained in Theorem 1 as the parameter $\varepsilon \to 0$. Our goal is to examine how the parameters p, q, β, and μ_1, μ_2 influence the profiles of the solutions and to understand the concentration behavior around the points where the potential V attains its maximum.

Recall that the solutions $(u_\varepsilon, v_\varepsilon)$ concentrate around k distinct points on the sphere $S_\varepsilon = \left\{ x \in \mathbb{R}^N : |x| = \frac{1}{\varepsilon} \right\}$ as $\varepsilon \to 0$. To analyze the behavior near these points, we perform a rescaling centered at one of the concentration points.

Let $x_\varepsilon \in S_\varepsilon$ be one of the concentration points (the analysis is similar for each point due to the symmetry). We define the rescaled variables:

$$y = \varepsilon x - \varepsilon x_\varepsilon, \quad U_\varepsilon(y) = u_\varepsilon\left(x_\varepsilon + \frac{y}{\varepsilon}\right), \quad V_\varepsilon(y) = v_\varepsilon\left(x_\varepsilon + \frac{y}{\varepsilon}\right). \tag{13}$$

Substituting the rescaled variables into the original equations, we obtain:

$$\begin{cases} (-\Delta)^s U_\varepsilon + \varepsilon^{2s}\lambda_1 U_\varepsilon = V\left(x_\varepsilon + \frac{y}{\varepsilon}\right)|U_\varepsilon|^{p-2}U_\varepsilon + \beta|V_\varepsilon|^{q-2}V_\varepsilon, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s V_\varepsilon + \varepsilon^{2s}\lambda_2 V_\varepsilon = V\left(x_\varepsilon + \frac{y}{\varepsilon}\right)|V_\varepsilon|^{p-2}V_\varepsilon + \beta|U_\varepsilon|^{q-2}U_\varepsilon, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |U_\varepsilon|^2 dy = \varepsilon^N \mu_1, \quad \int_{\mathbb{R}^N} |V_\varepsilon|^2 dy = \varepsilon^N \mu_2. \end{cases} \tag{14}$$

As $\varepsilon \to 0$, the mass $\varepsilon^N \mu_i$ tends to zero, suggesting that U_ε and V_ε concentrate at points where $V(y)$ attains its maximum, i.e., at $|y| = 1$.

The limiting problem, obtained formally by setting $\varepsilon = 0$, is

$$\begin{cases} (-\Delta)^s U + \lambda_1 U = V_{\max}|U|^{p-2}U + \beta|V|^{q-2}V, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s V + \lambda_2 V = V_{\max}|V|^{p-2}V + \beta|U|^{q-2}U, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |U|^2 dx = \mu_1, \quad \int_{\mathbb{R}^N} |V|^2 dx = \mu_2. \end{cases} \tag{15}$$

This problem corresponds to the case where the potential is constant and equal to $V_{\max}$, and as previously pointed, this is important in understanding the behavior of the minimizers as $\varepsilon \to 0$.

This problem has been studied extensively in the literature, and under appropriate conditions on p, q, and β, it admits positive, radially symmetric, exponentially decaying solutions. These solutions are often referred to as ground states.

Our aim is to show that $U_\varepsilon \to U$ and $V_\varepsilon \to V$ strongly in $H^s(\mathbb{R}^N)$ as $\varepsilon \to 0$, where (U, V) is a ground state solution of the limiting problem (15).

First, we establish uniform bounds on U_ε and V_ε in $H^s(\mathbb{R}^N)$. Since u_ε, v_ε are bounded in $H^s(\mathbb{R}^N)$, and the rescaling preserves the H^s norm up to a constant factor, we have:

$$\|U_\varepsilon\|^2_{H^s(\mathbb{R}^N)} = \varepsilon^{N-2s}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\varepsilon\left(x_\varepsilon + \frac{y}{\varepsilon}\right)|^2 dy + \varepsilon^N \int_{\mathbb{R}^N} |u_\varepsilon\left(x_\varepsilon + \frac{y}{\varepsilon}\right)|^2 dy.$$

Since u_ε concentrates around x_ε, the integrals above remain bounded as $\varepsilon \to 0$. Specifically, the term ε^{N-2s} remains bounded provided $N > 2s$, which is ensured since $N \geq 2$ and $s \in (0,1)$.

Similarly for V_ε.

By the boundedness of $\{(U_\varepsilon, V_\varepsilon)\}$ in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, there exists a subsequence (still denoted by ε) such that $U_\varepsilon \rightharpoonup U$ and $V_\varepsilon \rightharpoonup V$ weakly in $H^s(\mathbb{R}^N)$.

To show strong convergence, we consider the weak limits in $H^s(\mathbb{R}^N)$ and pass to the limit in the equations.

Let us define the residuals:

$$R_{U_\varepsilon} = (-\Delta)^s U_\varepsilon + \varepsilon^{2s}\lambda_1 U_\varepsilon - V_{\max}|U_\varepsilon|^{p-2}U_\varepsilon - \beta|V_\varepsilon|^{q-2}V_\varepsilon,$$
$$R_{V_\varepsilon} = (-\Delta)^s V_\varepsilon + \varepsilon^{2s}\lambda_2 V_\varepsilon - V_{\max}|V_\varepsilon|^{p-2}V_\varepsilon - \beta|U_\varepsilon|^{q-2}U_\varepsilon.$$

From (14), we have:

$$R_{U_\varepsilon} = \left(V\left(x_\varepsilon + \frac{y}{\varepsilon}\right) - V_{\max}\right)|U_\varepsilon|^{p-2}U_\varepsilon,$$

and similarly for R_{V_ε}.

Since V is continuous and $V(x_\varepsilon) = V_{\max}$, we have:

$$V\left(x_\varepsilon + \frac{y}{\varepsilon}\right) - V_{\max} = o(1) \quad \text{uniformly for } y \in K \subset \mathbb{R}^N \text{ compact.}$$

Moreover, $\varepsilon^{2s}\lambda_1 U_\varepsilon \to 0$ uniformly on compact sets as $\varepsilon \to 0$.

Thus, $R_{U_\varepsilon} \to 0$ in $H^{-s}_{\text{loc}}(\mathbb{R}^N)$.

Therefore, (U,V) satisfies the limiting equations (15) in the weak sense.

5.1 Uniqueness and strong convergence

Under the assumption that the limiting problem (15) has a unique positive solution (up to translations and symmetries), the weak convergence implies strong convergence. This is due to the non-degeneracy of the ground state solution.

Moreover, the Pohozaev identity adapted to fractional Laplacians [11] can be used to show that the energy of $(U_\varepsilon, V_\varepsilon)$ converges to the energy of (U,V), implying strong convergence in $H^s(\mathbb{R}^N)$.

5.2 Influence of parameters on solution profiles

We now examine how the parameters p, q, β, μ_1, and μ_2 influence the profiles of the solutions.

The values of p and q determine the strength of the nonlinearities in the equations. If p increases (still within the subcritical range), the nonlinearity $|U|^{p-2}U$ becomes stronger, leading to solutions with sharper peaks and faster decay. This is similarly true for q.

The coupling constant β affects the interaction between U and V. If $\beta > 0$ increases, the coupling term $\beta|V|^{q-2}V$ (or $\beta|U|^{q-2}U$) becomes stronger, enhancing the interaction between the two components. This can lead to more pronounced synchronization between U and V, potentially causing them to have similar profiles.

The prescribed masses μ_1 and μ_2 influence the L^2 norms of U and V. Larger values of μ_1 and μ_2 result in solutions with greater amplitude. The balance between μ_1 and μ_2 can affect the relative sizes of U and V.

To obtain a more precise description of the solutions as $\varepsilon \to 0$, we can consider an asymptotic expansion for the energy functional $\mathscr{J}_\varepsilon(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$.

Using the rescaled functions, the energy associated with each concentration point is:

$$E_\varepsilon = \frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{s/2}U_\varepsilon|^2 + |(-\Delta)^{s/2}V_\varepsilon|^2\right)dy + \frac{1}{2}\int_{\mathbb{R}^N}\left(|U_\varepsilon|^2 + |V_\varepsilon|^2\right)dy$$

$$-\frac{V_{\max}}{p}\int_{\mathbb{R}^N}\left(|U_\varepsilon|^p + |V_\varepsilon|^p\right)dy - \beta\int_{\mathbb{R}^N}|U_\varepsilon|^{\frac{q}{2}}|V_\varepsilon|^{\frac{q}{2}}dy.$$

As $\varepsilon \to 0$, $E_\varepsilon \to E_0$, the minimal energy of the limiting problem.

6 Conclusion

In this study, we have established the existence of multiple non-radial positive solutions to a coupled fractional nonlinear Schrödinger system under the influence of competing nonlinearities and mass constraints. We employed the local minimization method in conjunction with the concentration-compactness principle adapted for fractional Sobolev spaces to navigate the challenges posed by the coupling terms and the nonlocal nature of the fractional Laplacian.

Our main theorem demonstrates that for each integer $k \geq 2$ and sufficiently small semiclassical parameter $\varepsilon > 0$, there exists a non-radial positive solution that minimizes the energy functional $\mathscr{J}_\varepsilon$ over $\mathscr{M}$, is G-invariant, and concentrates around k distinct points on the sphere S_ε as $\varepsilon \to 0$. These solutions exhibit symmetry properties dictated by the subgroup G of the orthogonal group $O(N)$, leading to symmetry breaking despite the radial symmetry of the underlying potential V. The asymptotic analysis as $\varepsilon \to 0$ reveals that each concentration point contributes an energy level approaching E_0, the minimal energy of the associated limiting problem.

The adaptation to fractional Laplacians introduces additional complexities due to the operator's nonlocal nature and singularities. Our analysis ensures that these challenges are appropriately addressed, leveraging specialized concentration-compactness principles and fractional Sobolev embedding theorems to maintain mathematical rigor.

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