Journal of Mathematical Modeling
Vol. 13, No. 2, 2025, pp. 327-339. Research Article

JMM

Balancing efficiency and shortage costs: optimal production and inventory control in a distribution network

Mohammad Saffari[†*], **Mohammad Ali Mehrpouya**[‡]

[†]*Department of Industrial Engineering, Tafresh University, Tafresh 39518-79611, Iran*
[‡]*Department of Mathematics, Tafresh University, Tafresh 39518-79611, Iran*
Email(s): saffari@tafreshu.ac.ir, mehrpouya@tafreshu.ac.ir

Abstract. This article explores the integration of control theory techniques into production and inventory management systems within distribution networks. We examine a distribution network model where each node encompasses production and inventory segments, aiming to enhance overall benefit by adjusting key decision variables like production rate and total release. The proposed model formulates an optimal control problem which emphasizes the nonlinear nature of shortage costs and their impact on decision-making, especially concerning high-cost escalations due to shortages. The study proposes numerical solutions using the Legendre pseudospectral method, demonstrating its effectiveness in solving complex optimal control problems with multiple variables.

Keywords: Inventory, Network production control, Nonlinear demand, Pseudospectral method, Direct optimization method
AMS Subject Classification 2010: 90B05, 34H05, 65L60, 65K05.

1 Introduction

Traditionally control theory has been developed and applied in fields such as aerospace, robotics, and process control. These areas often involve complex systems with well-defined dynamics and control objectives. While the control theory has traditionally found its application in other fields, the importance of efficient production and inventory control systems has sparked interest in exploring its potential in this domain. Although, there are challenges to overcome, ongoing research and development efforts aim to enhance the applicability of the control theory techniques to production and inventory control, leading to improved operational performance and competitive advantages for industries. By bridging the gap between the control theory and the production-inventory control, industries stand to gain significant advantages. In the meantime, the optimal control strategy, which is one of the branches of the control theory, can help minimize costs, improve production efficiency, optimize inventory levels, and enhance

*Corresponding author
Received: 24 July 2024 / Revised: 25 November 2024 / Accepted: 22 December 2024
DOI: 10.22124/jmm.2024.28014.2464

customer service. With this motivation, many researchers started their research in this field. For instance, we can refer to the some of them.

Simon [20] made the initial effort to employ the control theory in addressing production-inventory challenges involved regulating the production rate of a single item using principles from servomechanism theory. According to Axsater [2], there was significant enthusiasm for utilizing the control theory in the production-inventory scenarios during the 1960s, but this interest waned by the 1980s. The 1994 publication by Wikner [22], titled "dynamic modelling and analysis of information flows in production-inventory and supply chain systems", thoroughly explores the utilization of dynamic modeling and analytical techniques to optimize the flow of information in manufacturing, inventory, and supply chain setups. The author underscores the inclusion of crucial components such as prediction methodologies, lead times, and inventory renewal tactics when constructing models for industrial systems. Through comprehensive dialogues, the text deeply examines diverse predictive approaches like moving averages and primary exponential smoothing, showcasing their conversion using Laplace and Z transforms. To explore research conducted prior to 2004, refer to the work by Ortega and Lin [18]. In a more recent investigation, Chazal et al. [5] examined the optimization of production planning and inventories using a backward approach, focusing on scenarios with convex storage costs. Another recent study by Gayon et al. [11] delved into the optimal control of a production- inventory system featuring two disposal choices and product return considerations. The dynamics of inventory in the context of sustained sales were explored by Maccini et al. [16]. Addressing the complexities of base-stock inventory systems with state-dependent demand rates, Olsson [17] introduced various modeling techniques. Dolgui et al. [6] with emphasis on exploring the deterministic maximum principle, presented a comprehensive overview of how the optimal control is used in scheduling for production, supply chain, and industry 4.0 systems. It is noted that, the industry 4.0, a widely recognized concept, refers to the ongoing transformation of manufacturing and production through the integration of advanced digital technologies. It involves automation, data exchange, artificial intelligence, the internet of things and smart systems to create highly efficient, interconnected, and intelligent industrial processes. This shift aims to enhance productivity, flexibility, and real-time decision-making in modern industries. In the paper published recently, Ignaciuk [13] investigated how the utilization of the linear-quadratic optimal control can be applied to manage the goods distribution process in logistic systems that involve multiple transportation alternatives.

In this paper, a distribution network is examined where each node comprises both production and inventory segments. It is noted that, in a supply chain, a distribution network consists of a connected system of manufacturers, storage facilities, and transportation services that handle incoming inventories and deliver them to customers. It serves as a crucial link between the manufacturer and the final consumer, either through direct delivery or via a retail network. For the first time, we address the challenge of simultaneously optimizing production and release policies in a distribution network that involves multiple production and inventory centers, continuously deteriorating items over time, and back ordering. The model is designed in such a way that can be applicable to any distribution network which experiences continuous demand with predictable seasonal patterns, characterized by low forecasting error. It can be effectively utilized in scenarios where demand fluctuations follow regular, seasonal trends. This makes it suitable for businesses that rely on reliable demand forecasting to optimize inventory levels, transportation, and overall supply chain operations. Utilizing optimal control theory, we aim to enhance the overall benefit function by adjusting key decision variables, specifically the production rate and total release. The subsequent section outlines the assumptions, notations, and mathematical framework used in our analysis. Section 3 delves into some details about optimal control theory. Sections 4 and 5 are dedicated to presenting numerical investigations, while our findings and a concise summary are summarized in

Section 6.

2 Model Description

We are investigating a network composed of nodes, each consisting of production and inventory divisions. All of the production units within this network manufacture the same product. In addition to the costs associated with holding inventory, the inventory itself deteriorates as time passes. Demand function during the planning horizon is known and deterministic. While shortage is permitted, the cost of shortages is exceptionally high and escalates. This phenomenon is primarily attributed to the nature of the product. For instance, let's consider the production of petrol nationwide. The decision-maker faces the challenge of determining the optimal production rate for each facility, as well as the overall total release. These decisions are critical for ensuring efficient operations and minimizing costs. The decision-maker must carefully consider several factors, including the production capacity of each facility, the current inventory levels, the rate of deterioration, and the increasing cost associated with shortages. This paper shows, how to achieve the desired goal in such networks with the help of utilizing the optimal control strategy.

It is noted that, in our model, there are some variables which are called the state variables and are shown by $I_i(t)$, $i = 1,\dots,M$, and there are some variables which are called the control variables and are shown by $R_i(t)$ and $p_i(t)$, $i = 1,\dots,M$, and have the following concepts. The state variables $I_i(t)$, $i = 1,\dots,M$, are inventory level in the store i at time t and the control variables $R_i(t)$ and $p_i(t)$, $i = 1,\dots,M$, are total release and production rates in the facility i at time t, respectively. In addition, all parameters and functions appearing in the model are summarized in Table 1. So, the explained networked production planning model can be formulated as

Table 1: Parameters and functions used in the proposed model.

Notation	Meaning
T	Planning time horizon
h_i	Holding cost per unit time in the store i
α_i	Rate of disposal in the store i
r	Unit sale price
s	Per unit shortage cost
β	Degree of nonlinearity of shortage cost function
v_i	Initial value for $I_i(t)$
$p_i^{\max}$	The maximum rate of production in the facility i
φ_i	Functions of production rates in the facility i at time t
$d(t)$	Demand rate at time t
$S(t)$	Sale rate at time t
$H(t)$	Shortage rate at time t

Maximize:

$$J = \int_0^T \Big\{ rS(t) - sH(t)^\beta - \sum_i \big(\varphi_i(p_i(t)) + h_i I_i(t)\big) \Big\} dt, \tag{1}$$

such that:

$$\frac{dI_i(t)}{dt} = p_i(t) - R_i(t) - \alpha_i I_i(t), \tag{2}$$

$$I_i(0) = v_i, \tag{3}$$

$$I_i(t) \geq 0, \tag{4}$$

$$R_i(t) \geq 0, \tag{5}$$

$$\sum_i R_i(t) \leq d(t), \tag{6}$$

$$0 \leq p_i(t) \leq p_i^{\max}, \tag{7}$$

where, $S(t) = \sum_i R_i(t)$, $H(t) = d(t) - \sum_i R_i(t)$, the parameters h_i, r and s are non-negative, $0 \leq \alpha_i \leq 1$, $\beta \geq 1$, $v_i \in \mathbb{R}$, $p_i^{\max} > 0$ and $i = 1, \ldots, M$. It is noted that the objective functional of the problem presents benefit function which accumulates total sale and release benefits minus production, holding and shortages costs. Furthermore, the cost of shortages is non-linear, exhibiting exceptionally high escalation as shortage increases. The dynamical equations can be expressed as follows: as production adds to on-hand inventory, release and depreciation reduce it.

3 Optimal control: some preliminary concepts

The goal of the optimal control is to find controls that induce a system to encounter certain physical constraints while also maximizing (minimizing) a performance measure. Optimal control has been the focus of much interest for several decades. Solving an optimal control problem requires meeting some initial and boundary conditions imposed by the differential equations. Analytical solution of optimal control problems is not possible except in a few cases where the objective functional is simple enough and the dynamical equations are linear. Here, the optimal control problems are most often solved numerically. Consequently, significant advancements have been achieved in advancing both direct and indirect numerical techniques [3, 19].

The direct methods, known for their reliance on discretization and nonlinear programming, have been widely utilized in various optimal control problems due to their effectiveness and ease of use. Their advantage over indirect approaches, which is based on solving the necessary conditions of optimality derived from the Pontryagin's minimum principle, lies in their broader range of convergence towards an optimal solution [9]. Additionally, direct methods can be used to solve a range of optimal control problems without the need to derive the necessary conditions of optimality. The next section aims to propose a direct-based method for efficient solution of the explained networked production planning model (1)-(7).

4 Numerical solution of the explained networked production planning model

Let's go back to the target problem (1)-(7) which is actually an optimal control problem with the state variables $I_i(t)$ and control variables $R_i(t)$ and $p_i(t)$ for $i = 1, \ldots, M$. So, the explained networked production planning model has M state variables and $2M$ control variables. In this section, a Legendre pseudospectral method [1, 7, 8, 12, 14, 15, 21] is presented for solving this problem. It should be noted that, the method presented in this paper includes two steps. First, the global polynomial approximations for the state and control variables in terms of their values at the well-known Legendre-Gauss-Lobatto (LGL) points are considered. Then in the second step, with replacement these approximations in the target optimal control problem and utilizing the collocation process at the LGL points, the problem at hand is transformed into a discrete nonlinear programming problem (NLP) that can be tackled with the established algorithms.

To better explain the method, let us consider some symbols. Since in the model, there are M state variables and $2M$ control variables, therefore, the following symbols are used to show the state and control variables.

$$\begin{aligned}&\mathbf{I}(t) = [I_1(t), \ldots, I_M(t)]^T \in \mathbb{R}^M,\\ &\mathbf{u}(t) = [R_1(t), \ldots, R_M(t), p_1(t), \ldots, p_M(t)]^T \in \mathbb{R}^{2M}.\end{aligned}$$

Consequently, the dynamical system of the model governed by some differential equations with fixed initial values, can be represented as follows:

$$\dot{\mathbf{I}}(t) = \mathbf{f}(\mathbf{I}(t), \mathbf{u}(t), t), \;\; \mathbf{I}(0) = \mathbf{v}_0, \tag{8}$$

where, $\mathbf{f} : \mathbb{R}^M \times \mathbb{R}^{2M} \times \mathbb{R} \to \mathbb{R}^M$ is assumed to be a function of the variables $(\mathbf{I}, \mathbf{u}, t)$. Finally, the objective functional of the model can be expressed by

$$\begin{aligned}\text{Minimize }\; \Gamma &= -\int_0^T \Big\{ rS(t) - sH(t)^\beta - \sum_i \big(\varphi_i(p_i(t)) + h_i I_i(t)\big) \Big\} dt\\ &= \int_0^T g(\mathbf{I}(t), \mathbf{u}(t), t) dt,\end{aligned} \tag{9}$$

where, $g : \mathbb{R}^M \times \mathbb{R}^{2M} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, M$ and the state and control constraints are given as following constraints

$$\mathbf{I}(t) \geq \mathbf{0}, \tag{10}$$

$$u_1(t) \geq 0, \tag{11}$$

$$\vdots$$

$$u_M(t) \geq 0, \tag{12}$$

$$\sum_{i=1}^{M} u_i(t) \leq d(t), \tag{13}$$

$$0 \leq u_{M+1}(t) \leq p_{M+1}^{\max}, \tag{14}$$

$$\vdots$$

$$0 \leq u_{2M}(t) \leq p_{2M}^{\max}. \tag{15}$$

Here we point out that in the standard case, the objective functional in an optimal control problem is considered in the form of minimization. Obviously, according to this convention, the objective functional (1) should be multiplied by a negative sign, so that, it can be assumed as a minimization form (9).

Now, let

$$\hat{\theta}_j = \frac{1}{2} T(\theta_j + 1), \quad j = 1, \ldots, N+1,$$

be the associated LGL points to the interval $[0,T]$, for $(N+1)$ LGL points $\{\theta_j\}_{j=1}^{N+1}$, where $\theta_1 = -1$ and $\theta_{N+1} = +1$. It is important to mention that, the LGL points $\{\theta_j\}_{j=1}^{N+1}$ are the $N+1$ roots of $(1-t^2)(d/dt)P_N(t)$, where the famous Legendre polynomials P_N can be found using the given recurrence relations [4]

$$
\begin{aligned}
&P_0(t) = 1, \;\; P_1(t) = t, \\
&P_{N+1}(t) = \frac{2N+1}{N+1} t P_N(t) - \frac{N}{N+1} P_{N-1}(t), \;\; N = 1,2,\ldots
\end{aligned}
$$

In addition, the LGL quadrature method relies on the LGL points $\{\theta_j\}_{j=1}^{N+1}$ can be applied to estimate the integral of a function over the range $[0,T]$ in the following manner:

$$
\int_0^T F(t)dt \approx \frac{T}{2}\sum_{j=1}^{N+1} \omega_j F(T(\theta_j+1)/2) = \frac{T}{2}\sum_{j=1}^{N+1} \omega_j F(\hat{\theta}_j), \tag{16}
$$

where,

$$
\omega_j = \frac{2}{N(N+1)} \frac{1}{(P_N(\theta_j))^2}, \quad j = 1,\ldots,N+1,
$$

are LG quadrature weights [10]. To apply the Legendre pseudospectral method, suppose that, $\hat{L}_j(t), j = 1,2,\ldots,N+1$ be the Lagrange polynomials associated with $\{\hat{\theta}_j\}_{j=1}^{N+1}$, in which

$$
\hat{L}_j(t) = L_j((2/T)t - 1),
$$

and $\{L_j(t)\}_{j=1}^{N+1}$ are the Nth-order Lagrange interpolating polynomials, defined by

$$
L_j(t) = \prod_{l=1, l\neq j}^{N+1} \frac{t-\theta_l}{\theta_j - \theta_l}, \;\; j = 1,\ldots,N+1,
$$

with the Kronecker property

$$
L_j(\theta_l) = \delta_{jl} = \begin{cases} 0, & \text{if } j \neq l, \\ 1, & \text{if } j = l. \end{cases} \tag{17}
$$

Currently, the state and control functions $\mathbf{I}(t)$ and $\mathbf{u}(t)$ are estimated by a polynomial up to degree N utilizing the Lagrange polynomials as

$$
\mathbf{I}(t) \approx \sum_{j=1}^{N+1} \mathbf{I}(\hat{\theta}_j)\hat{L}_j(t), \tag{18}
$$

$$
\mathbf{u}(t) \approx \sum_{j=1}^{N+1} \mathbf{u}(\hat{\theta}_j)\hat{L}_j(t), \tag{19}
$$

in which, $\mathbf{I}(\hat{\theta}_j)$ and $\mathbf{u}(\hat{\theta}_j)$ are unknown parameters. Differentiating the expression in Eq. (18), we have

$$
\dot{\mathbf{I}}(t) \approx \sum_{j=1}^{N+1} \mathbf{I}(\hat{\theta}_j)\dot{\hat{L}}_j(t). \tag{20}
$$

Consequently, by substituting the Eqs. (18), (19) and (20) into the dynamical system of the model, i.e. Eq. (8), next collocating in $\{\hat{\theta}_j\}_{j=1}^{N+1}$, we obtain

$$\sum_{j=1}^{N+1} \mathbf{I}(\hat{\theta}_j)\dot{\hat{L}}_j(\hat{\theta}_j) \approx \mathbf{f}(\sum_{j=1}^{N+1} \mathbf{I}(\hat{\theta}_j)\hat{L}_j(\hat{\theta}_j), \sum_{j=1}^{N+1} \mathbf{u}(\hat{\theta}_j)\hat{L}_j(\hat{\theta}_j), \hat{\theta}_j).$$

Now, by using the kronecker property (17), we have

$$\begin{aligned}
&\frac{2}{T}\sum_{j=1}^{N+1} \mathbf{I}(\hat{\theta}_j)d_{jl} \approx \mathbf{f}(\mathbf{I}(\hat{\theta}_l), \mathbf{u}(\hat{\theta}_l), \hat{\theta}_l), \quad l = 1,\dots,N+1, \\
&\mathbf{I}(\hat{\theta}_1) - \mathbf{v}_0 \approx \mathbf{0},
\end{aligned}$$

where

$$d_{jl} = \dot{L}_j(\theta_l),$$

represents the element at position (j,l) in the square matrix $\mathbf{D}$, which is known as the Legendre pseudospectral differentiation matrix [10]. Subsequently, the constraints related to the path in Eqs. (10)-(15) are enforced at the $(N+1)$ collocation points for $l = 1,\dots,N+1$ as follows:

$$\begin{aligned}
&\mathbf{I}(\hat{\theta}_l) \geq \mathbf{0}, \\
&u_1(\hat{\theta}_l) \geq 0, \\
&\quad\vdots \\
&u_M(\hat{\theta}_l) \geq 0, \\
&\sum_{i=1}^{M} u_i(\hat{\theta}_l) \leq d(\hat{\theta}_l), \\
&0 \leq u_{M+1}(\hat{\theta}_l) \leq p_{M+1}^{\max}, \\
&\qquad\quad\vdots \\
&0 \leq u_{2M}(\hat{\theta}_l) \leq p_{2M}^{\max},
\end{aligned}$$

and $u_1(\hat{\theta}_j) \geq 0,\dots,u_M(\hat{\theta}_j) \geq 0$, $0 \leq u_{M+1}(\hat{\theta}_j) \leq p_{M+1}^{\max},\dots,$ $0 \leq u_{2M}(\hat{\theta}_j) \leq p_{2M}^{\max}$. Finally, using the LGL quadrature rule (16), the objective functional (9) is approximated by

$$\Gamma \approx \frac{T}{2}\sum_{j=1}^{N+1} \omega_j \mathbf{g}(\mathbf{I}(\hat{\theta}_j), \mathbf{u}(\hat{\theta}_j), \hat{\theta}_j).$$

So, the target optimal control problem is transformed into the following NLP

Minimize:

$$\Gamma = \frac{T}{2}\sum_{j=1}^{N+1} \omega_j \mathbf{g}(\mathbf{I}(\hat{\theta}_j), \mathbf{u}(\hat{\theta}_j), \hat{\theta}_j), \tag{21}$$

such that:

$$\frac{2}{T}\sum_{j=1}^{N+1}\mathbf{I}(\hat{\theta}_j)d_{jl} = \mathbf{f}(\mathbf{I}(\hat{\theta}_l), \mathbf{u}(\hat{\theta}_l), \hat{\theta}_l), \tag{22}$$

$$\mathbf{I}(\hat{\theta}_1) - \mathbf{v}_0 = \mathbf{0}, \tag{23}$$

$$\mathbf{I}(\hat{\theta}_l) \geq \mathbf{0}, \tag{24}$$

$$u_1(\hat{\theta}_l) \geq 0, \tag{25}$$

$$\vdots$$

$$u_M(\hat{\theta}_l) \geq 0, \tag{26}$$

$$\sum_{i=1}^{M} u_i(\hat{\theta}_l) \leq d(\hat{\theta}_l), \tag{27}$$

$$0 \leq u_{M+1}(\hat{\theta}_l) \leq p_{M+1}^{\max}, \tag{28}$$

$$\vdots$$

$$0 \leq u_{2M}(\hat{\theta}_l) \leq p_{2M}^{\max}, \tag{29}$$

where, $l = 1, \ldots, N+1$ and $\mathbf{I}(\hat{\theta}_l)$ and $\mathbf{u}(\hat{\theta}_l)$ are decision parameters.

5 Numerical illustrations

The optimal states and controls will be determined by the Legendre pseudospectral method in this section. We present optimal solutions for one set of fixed parameters and functions appearing in the model. The NLP in Eqs. (21)-(29) is solved using the MATLAB function `fmincon` with the `sqp` numerical algorithm specified as the solver. In this solver, we can set termination tolerances for the objective function value, `TolFun`, constraint violation, `TolCon`, and decision variables, `TolX`, to be less than 10^{-12}. In addition, all calculations are carried out on a 2.53 GHz Core i5 laptop with 4 GB of RAM using MATLAB R2016a.

5.1 Test example

In our numerical example, we specifically analyze a national network of seven facilities, i.e., $i = 1, \ldots, 7$, each with its own inventory stock and production constraints. The model is parameterized based on realistic assumptions for production capacities, inventory holding and deterioration rates, and shortage penalties to closely simulate real-world conditions. Key fixed parameters and functions that define the system dynamics, demand profiles, and cost structure are outlined in Table 2, ensuring clarity and reproducibility of the model setup. Now, the problem is solved by using the presented method. The approximated solutions for the states and controls for $N = 25$ discretization points are shown in Figs. 1-3. Additionally, to demonstrate the convergence and accuracy of the presented method, the values of the objective functional, as well as the initial values of $R_1(0)$, $R_7(0)$, $p_1(0)$, $p_7(0)$ and final values of $I_1(1)$ and $I_7(1)$ for different numbers of discretization points are reported in Table 3.

As can be seen from the results, we find that the presented method has a good performance. Moreover, the graphs of optimal control policy illustrate that a limited shortage is permitted when demand is at its peak to avoid excessive production expenses surpassing the usual production capacity or storage

Table 2: All values of parameters and functions used in the proposed model in Test example.

Notation	Formula/value
T	1
r	4
s	2
β	2
$[h_1, h_2, h_3, h_4, h_5, h_6, h_7]$	[8,6,6.6,7,8,7.5,9]
$[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7]$	[0.1,0.12,0.14,0.11,0.11,0.1,0.05]
$[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$	[0.6,0.5,0.5,0.4,0.2,0.2,0]
$[p_8^{\max}, p_9^{\max}, p_{10}^{\max}, p_{11}^{\max}, p_{12}^{\max}, p_{13}^{\max}, p_{14}^{\max}]$	[2.5,2.5,2,3,2.5,2.5,1]
$\varphi_1(p_1(t))$	$0.7(1.2p_1^3(t) - 3p_1^2(t) + 5p_1(t))$
$\varphi_2(p_2(t))$	$0.8(1.2p_2^3(t) - 3p_2^2(t) + 5p_2(t))$
$\varphi_3(p_3(t))$	$0.7(1.2p_3^3(t) - 3p_3^2(t) + 5p_3(t))$
$\varphi_4(p_4(t))$	$0.8(1.2p_4^3(t) - 3p_4^2(t) + 5p_4(t))$
$\varphi_5(p_5(t))$	$0.7(1.2p_5^3(t) - 3p_5^2(t) + 5p_5(t))$
$\varphi_6(p_6(t))$	$0.8(1.2p_6^3(t) - 3p_6^2(t) + 5p_6(t))$
$\varphi_7(p_7(t))$	$0.9(1.2p_7^3(t) - 3p_7^2(t) + 5p_7(t))$
$d(t)$	$15 + 7t - 6t^2$

costs. It is important to highlight that, the cost of shortages follows a quadratic function, and even though higher shortages incur significant costs, the expense associated with smaller shortages is tolerable. It is both logical and significant that a higher proportion of releases are allocated to the first and seventh facilities at the start of the period. This strategy aims to expedite the reduction of their inventory levels, which is particularly important given the relatively higher holding costs associated with these facilities.

6 Conclusion and Discussion

This research delves into the integration of control theory into the optimization of production and inventory management in distribution networks. By examining historical and contemporary studies, the paper outlines a model emphasizing the adjustment of critical variables to maximize overall benefit. The proposed numerical solution method, utilizing the Legendre pseudospectral method, showcases its efficacy in resolving intricate optimal control problems involving various factors. Ultimately, this work offers a promising avenue for enhancing operational efficiency and reducing total costs within industrial systems through advanced control theory applications.

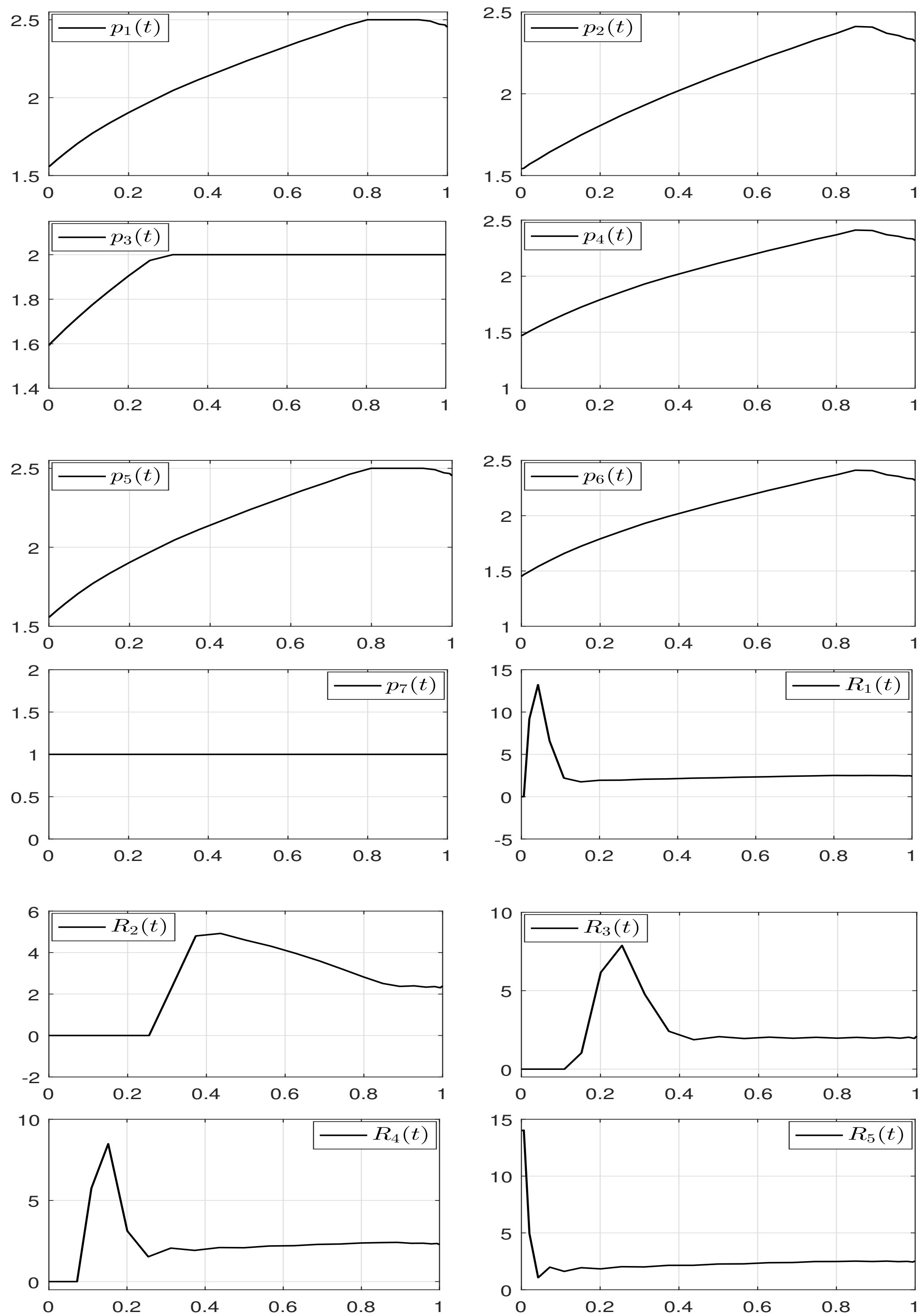

Figure 1: Solution of Test example, for $N = 25$.

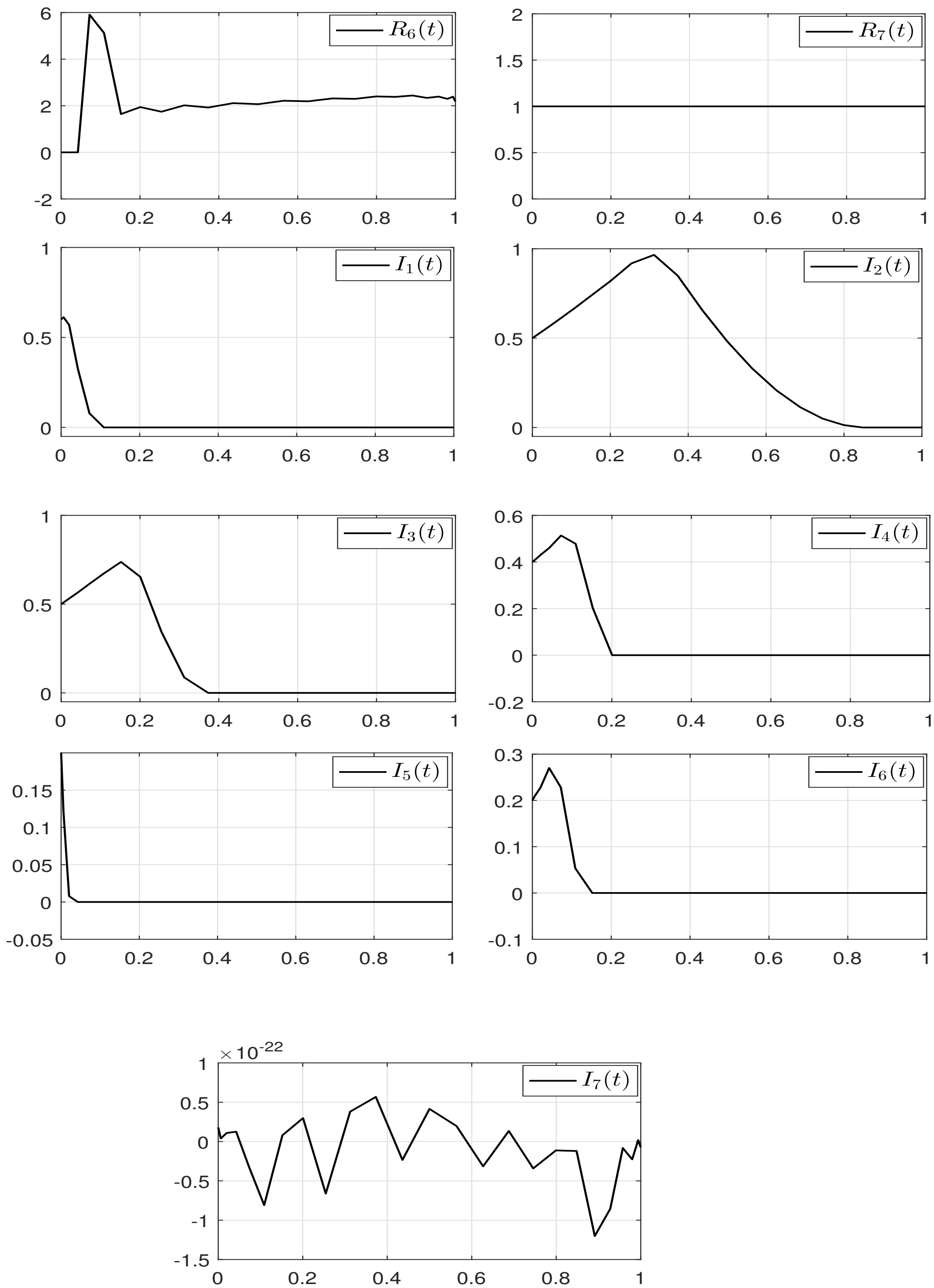

Figure 2: Solution of Test example, for $N = 25$.

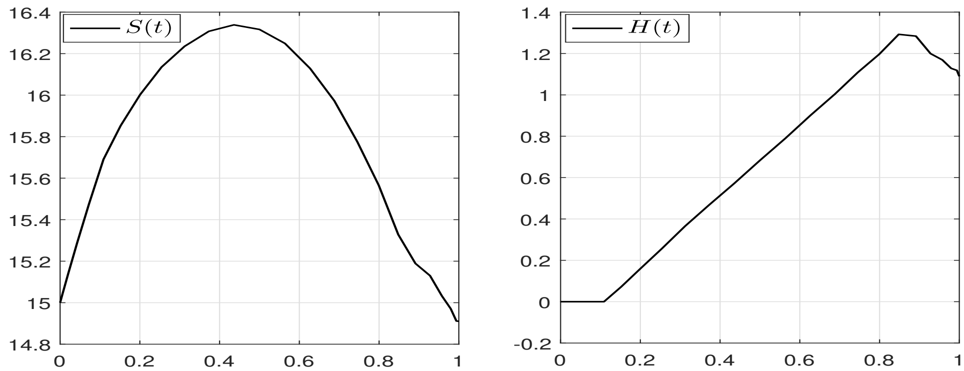

Figure 3: Solution of Test example, for $N = 25$.

Table 3: The values of objective functional and some state and control variables for different numbers of discretization points obtained by the presented method in Test example.

N	$\Gamma = -J$	$R_1(0)$	$R_7(0)$	$p_1(0)$	$p_7(0)$	$I_1(1)$	$I_7(1)$
15	-15.9272896816	$1.1668e-22$	1	1.55037837327	1	$5.1218e-24$	$2.6928e-23$
20	-15.9297624099	$1.6818e-22$	1	1.55505960788	1	$1.8688e-23$	$1.7668e-23$
25	-15.9322380564	$8.25570e-26$	1	1.55644443585	1	$3.7113e-23$	$7.3934e-24$

Acknowledgements

The authors are very grateful to the anonymous reviewers for their valuable comments and suggestions which have improved the paper.

References

[1] Y.E. Aghdam, H. Mesgarani, Z. Asadi, *Estimate of the fractional advection-diffusion equation with a time-fractional term based on the shifted Legendre polynomials*, J. Math. Model. **11** (2023) 731-744.

[2] S. Axsater, *Control theory concepts in production and inventory control*, Int. J. Syst. Sci. **16** (1985) 161–169.

[3] J. Betts, *Survey of numerical methods for trajectory optimization*, J. Guid. Control Dyn. **21** (1998) 193–207.

[4] C. Canuto, M. Hussaini, A. Quarteroni, T. Zang, *Spectral Methods in Fluid Dynamics*, Springer Series in Computational Physics, Springer, Berlin, 1991.

[5] M. Chazal, E. Jouini, R. Tahraoui, *Production planning and inventories optimization: A backward approach in the convex storage cost case*, J. Math. Econ. **44**(2008) 997–1023.

[6] A. Dolgui, D. Ivanov, S.P. Sethi, B. Sokolov, *Scheduling in production, supply chain and industry 4.0 systems by optimal control: fundamentals, state-of-the-art and applications*, Int. J. Prod. Res. **57** (2019) 411-432.

[7] G. Elnagar, M.A. Kazemi, M. Razzaghi, *The pseudospectral Legendre method for discretizing optimal control problems*, IEEE Trans. Autom. Control **40** (1995) 1793–1796.

[8] F. Fahroo, I.M. Ross, *Costate estimation by a Legendre pseudospectral method*, J. Guid. Control Dyn. **24** (2001) 270-277.

[9] F. Fahroo, I. M. Ross, *Direct trajectory optimization by a Chebyshev pseudospectral method*, J. Guid. Control Dyn. **25** (2002) 160-166.

[10] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, 1998.

[11] J.-P. Gayon, S. Vercraene, S.D.P. Flapper, *Optimal control of a production-inventory system with product returns and two disposal options*, Eur. J. Oper. Res. **262** (2017) 499–508.

[12] S.M. Hoseini, *Approximate solution of nonlinear optimal control problems with scale delay function via a composite pseudospectral approach*, Int. J. Syst. Sci. **54** (2023) 2407–2422.

[13] P. Ignaciuk, *Linear-quadratic optimal control of multi-modal distribution systems with imperfect channels*, Int. J. Prod. Res. **60** (2022) 5523–5538.

[14] C.-G. Jung, C.-H. Lee, M.-J. Tahk, *Legendre pseudo-spectral method for missile trajectory optimization with free final time*, Lect. Notes Electr. Eng. **913** (2023) 569–581.

[15] Z. Li, F. Zhang, J. Zhou, *Partial Newton-correction method for multiple fixed points of semilinear differential operators by Legendre-Gauss-Lobatto pseudospectral method*, J. Sci. Comput. **97**(2023) 32.

[16] L.J. Maccini, B. Moore, H. Schaller, *Inventory behavior with permanent sales shocks*, J. Econ. Dyn. Control **53** (2015) 290-313.

[17] F. Olsson, *Simple modeling techniques for base-stock inventory systems with state dependent demand rates*, Math. Methods Oper. Res. **90** (2019) 61–76.

[18] M. Ortega, L. Lin, *Control theory applications to the production–inventory problem: a review*, Int. J. Prod. Res. **42** (2004) 2303–2322.

[19] A.V. Rao, *A survey of numerical methods for optimal control*, Adv. Astronaut. Sci. **135** (2009) 497-528.

[20] H.A. Simon, *On the application of servomechanism theory in the study of production control*, Econometrica (1952) 247–268.

[21] L. Tu, Y. Wang, C. Yan, Y. Yang, *Optimal transfer orbit design of spacecraft with finite thrust based on Legendre pseudospectral method*, Proc. Inst. Mech. Eng. G: J. Aerosp. Eng. **237** (2023) 1791–1797.

[22] J. Wikner, *Dynamic Modelling and Analysis of Information Flows in Production-Inventory and Supply Chain Systems*, Profil (Linköping). Linköping, 1994.