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On Galerkin spectral element method for solving Riesz fractional diffusion equation based on Legendre polynomials

Mouhssine Zakaria*, **Abdelaziz Moujahid**

LaR2A, FS, Abdelmalek Essaadi University, Tetouan, Morocco
Email(s): mouhssine.zakaria@etu.uae.ac.ma, moujahid.abdelaziz@uae.ac.ma

Abstract. This paper presents a Galerkin spectral element method for solving a fractional diffusion equation, considering initial and boundary conditions. We construct a discrete scheme for time, employing the Crank-Nicolson method to approximate the Caputo fractional derivative on a uniform mesh. Then we introduce a Galerkin variational formulation to establish the unconditional stability of the scheme. Moreover, we apply the spectral element method based on Legendre polynomials in the space direction and obtain the fully discrete scheme. The error analysis of the fully discrete scheme is treated in L_2 sense. we present a computational analysis to deal with the Galerkin spectral element method, to compute the corresponding bilinear form, on the implementation process. Finally, we prove the effectiveness of the method through numerical experiments and some simulations using MATLAB software.

Keywords: Fractional diffusion equation (FDE), Riesz derivative, Caputo derivative;, Galerkin spectral element method, Legendre polynomials, stability, error estimates.
AMS Subject Classification 2010: 26A33, 35R11,65M06, 65M12.

1 Introduction

Fractional differential equations(FDEs) represent an exciting domain within applied mathematics, providing crucial tools to describe natural phenomena with memory and hereditary properties. These equations have wide applications, for further exploration, we recommend this relevant literature. In physics, [16], and [17] where a fractional Bloch model was utilized, to control the fundamental processes of nuclear magnetic resonance. In chemistry and biology, the FDEs were able to describe the collective behavior of molecules in transport [27], and the transfer of heat and mass [12]. In finance, a fractional-time Black-Scholes model has been proposed in [18], to model the variation in option pricing for a fractal transmission system. Economics [1], and engineering [19].

*Corresponding author
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Solving and analyzing the FDEs has received increasing attention from researchers, where mathematicians have developed tools and techniques over the years to simulate its solutions as finding an analytical solution remains demanding and complicated. This has led to a wealth of studies that contribute significantly to the legacy of this field, inspiring us to explore several of them. As in [9], the authors addressed a mobile-immobile advection-dispersion equation, which incorporates the Caputo fractional derivative to model solute transport in porous and fractured media. They employed meshless methods using radial basis functions, along with the finite difference method in time direction. The second-order convolution quadrature and the weighted and shifted Grnwald formula was used to approximate the RiemannLiouville fractional integral and the distributed-order time-fractional derivative, for a multi-dimensional fractional integrodifferential models in [10]. Additionally, the authors employed the energy method to conduct the error analysis for the derived scheme. The existence and uniqueness was treated via the BanachAlaoglu theorem in a Sobolev space, for a nonlinear Kortewegde Vries-Rosenau-regularized long-wave problem proposed to model the dynamics of dispersive shallow water waves along lake shores and beaches in [13], and was simulated using the finite element method (FEM) combining with a second-order Crank-Nicolson scheme. The authors designed a finite element analysis to simulate a nonlinear advection-diffusion equation in [14], where a backward Eulers and Crank-Nicolson schemes was used to derive a full discretization, and the error analysis is performed in Bchner norms. In [20] the authors establish a non-standard algorithm via an uniform scale-3 Haar wavelets of space and time variables for a two-dimensional fractional advection dispersion model, which arises in complex network, fluid dynamics, biology, chemistry, system control.

The spectral element method (SEM) was created to solve fluid dynamics models, the combination of the FEM and spectral method led to the development of SEM, see [21]. In the literature, SEM is an efficient method to approximate solutions for complex models in applied sciences, for time-fractional PDEs. It is also used to approximate the spatial operators for the integer order, see [4, 25]. In [8], an FDEs are used to describe mobile/immobile fractional transport in complex dynamical systems, via a fractal mobile-immobile transport problem based on the CaputoFabrizio derivative in both the linear and the quasi-linear source term. The authors used a non-standard finite difference and a SEM to approach the model. A hybrid SEM is constructed for both Caputo and Riemann-Liouville fractional derivatives to solve fractional two-point boundary value equations in [28]. Further, in [15] the authors proposed a numerical method via the SEM and hierarchical matrix approximation to solve an equation with two-sided second-order Riemann-Liouville operators. In [32], the authors applied the FEM for the one-sided/two-dimensional fractional BlochTorrey equation, using the $L_2 - 1_\sigma$ formula to approximate the temporal Caputo derivative, and the FEM to deal with the approximations in the spatial direction leading to a fully discrete scheme solved using linear piecewise polynomials. Also, in [3], the same model was treated with the Caputo fractional derivative with multi-order, which is approximated using the standard L_1 formula and the FEM to approximate the Riesz fractional derivative. An analytical study is employed for the fully discrete scheme, by using the energy method, and by numerical experiments on irregular domains validated the theoretical results.

This work deals with a numerical method for a two-dimensional/two-sided fractional diffusion equation:

$$\begin{cases} {}_0^C\mathfrak{D}_t^\alpha \Theta(x,y,t) = \Psi_{1,x}\frac{\partial^{2\gamma}\Theta}{\partial|x|^{2\gamma}} + \Upsilon_{1,y}\frac{\partial^{2\beta}\Theta}{\partial|y|^{2\beta}} + \Psi_{2,x}\frac{\partial^{2\zeta}\Theta}{\partial|x|^{2\zeta}} + \Upsilon_{2,y}\frac{\partial^{2\varsigma}\Theta}{\partial|y|^{2\varsigma}} + \mathfrak{F}(x,y,t), \\ \Theta(x,y,t) = 0, \quad (x,y,t) \in \partial\Xi \times (0,T], \quad \text{and} \quad \Theta(x,y,0) = \mathfrak{C}_{\mathfrak{I}}(x,y), \quad (x,y) \in \Xi, \end{cases} \tag{1}$$

where $0<\alpha,2\gamma,2\beta<1, 1<2\zeta,2\varsigma<2$, in which $\Xi=(0,H)\times(0,L)\subset\mathbb{R}^2$, $\mathfrak{C}_\mathfrak{J}$ and $\mathfrak{F}$ are known smooth functions and show source terms. Here, $\Psi_\gamma,\Psi_\beta,\Psi_\zeta,\Psi_\varsigma$ are the non-negative weight coefficients.

The Caputo fractional derivative is defined in [22], as follows

$$ {}_0^C\mathfrak{D}_t^\zeta\Theta(t)=\frac{1}{\Gamma(n-\zeta)}\int_0^t\frac{\Theta^{(n)}(\tau)d\tau}{(t-\tau)^{\zeta+1-n}},\quad n-1<\zeta<n,\quad n\in\mathbb{N}, $$

where Γ denoting the Gamma function. The Riesz fractional derivative on a finite domain [22, 26], is given by:

$$ \frac{\partial^{2\eta}\Theta}{\partial|x|^{2\eta}}=-\frac{1}{2\cos(\eta\pi)}\left({}_0\mathfrak{D}_x^{2\eta}\Theta+{}_x\mathfrak{D}_H^{2\eta}\Theta\right), $$

for $n-1<2\eta<n, n\in\mathbb{N}$, and the operators ${}_0\mathfrak{D}_z^{2\eta}\Theta$, and ${}_z\mathfrak{D}_D^{2\eta}a\Theta$ are defined as

$$ {}_0\mathfrak{D}_z^{2\eta}\Theta={}_z\mathfrak{D}_L^{2\eta}\Theta=\frac{1}{\Gamma(n-2\eta)}\frac{\partial^n}{\partial z^n}\int_0^z(z-\chi)^{n-2\eta-1}\Theta(\chi)d\chi, $$
$$ {}_z\mathfrak{D}_D^{2\eta}\Theta={}_z\mathfrak{D}_R^{2\eta}\Theta=\frac{(-1)^n}{\Gamma(n-2\eta)}\frac{\partial^n}{\partial z^n}\int_z^D(\chi-z)^{n-2\eta-1}\Theta(\chi)d\chi. $$

The main contributions of this paper differ from the general timespace fractional diffusion equations (TSFDEs) by focusing on a two-dimensional model that incorporates a two-sided for the spatial direction. This adds greater generality for modeling complex diffusion processes. Moreover, the study presents a non-standard procedures to error analysis through new techniques and results. This provides novel methods for the analytical study of TSFDEs. The paper also features a subsection on computational analysis, explaining the application of the spectral elements method, with Legendre polynomials as the spatial basis. This provides insights into the process of implementing the method and solving our problem numerically by means of the MATLAB software.

The rest of the study is organized as follows. Section 2 gives preliminary tools and background on fractional derivative spaces. In Section 3, the Crank-Nicolson scheme is applied for time to approximate the Caputo derivative. Further, a Galerkin variational formulation is derived to establish unconditional stability for the semi-discrete scheme. In Section 4, the spectral element method based on Legendre polynomials is employed in spatial direction, we obtain the fully Galerkin spectral element scheme. We introduce a computational study concerning the implementation process. Next, the error analysis of the fully discrete scheme is showcased. In Section 5, two test problems are illustrated to validate the effectiveness of our numerical method, with simulated graphs. Finally, the results obtained are summarized in Section 6.

2 Preliminaries and backgrounds

To begin with, we present notions and lemmas for the theoretical studies.

2.1 Fractional derivative spaces and properties

We present definitions and lemmas concerned norms and semi-norms related to the fractional spaces.

Definition 1 ([7, 24]). *For $\xi > 0$, we define the semi-norm and norm respectively as*

$$|\Theta|_{\mathfrak{J}_L^\xi(\Xi)} := \left(\|_x\mathfrak{D}_L^\xi\Theta\|_{L_2(\Xi)}^2 + \|_y\mathfrak{D}_L^\xi\Theta\|_{L_2(\Xi)}^2\right)^{1/2}, \quad \|\Theta\|_{\mathfrak{J}_L^\xi(\Xi)} := \left(\|\Theta\|_{L_2(\Xi)}^2 + |\Theta|_{\mathfrak{J}_L^\xi(\Xi)}^2\right)^{1/2},$$

and denote $\mathfrak{J}_L^\xi(\Xi)(\mathfrak{J}_{L,0}^\xi(\Xi))$ as the closure of $C^\infty(\Xi)(C_0^\infty(\Xi))$ with respect to $\|\cdot\|_{\mathfrak{J}_L^\xi(\Xi)}$.

Definition 2 ([7, 24]). *For $\xi > 0$, we define the semi-norm and norm respectively as*

$$|\Theta|_{\mathfrak{J}_R^\xi(\Xi)} := \left(\|_x\mathfrak{D}_R^\xi\Theta\|_{L_2(\Xi)}^2 + \|_y\mathfrak{D}_R^\xi\Theta\|_{L_2(\Xi)}^2\right)^{1/2}, \quad \|\Theta\|_{\mathfrak{J}_R^\xi(\Xi)} := \left(\|\Theta\|_{L_2(\Xi)}^2 + |\Theta|_{\mathfrak{J}_R^\xi(\Xi)}^2\right)^{1/2},$$

and denote $\mathfrak{J}_R^\xi(\Xi)(\mathfrak{J}_{R,0}^\xi(\Xi))$ as the closure of $C^\infty(\Xi)(C_0^\infty(\Xi))$ with respect to $\|\cdot\|_{\mathfrak{J}_R^\xi(\Xi)}$.

Definition 3 ([7, 24]). *For $\xi > 0, \xi \neq n - \frac{1}{2}, n \in \mathbb{N}$, we define the semi-norm and norm respectively as*

$$|\Theta|_{\mathfrak{J}_S^\xi(\Xi)} = \left(\left|\left({}_x\mathfrak{D}_L^\xi\Theta, {}_x\mathfrak{D}_R^\xi\Theta\right)\right| + \left|\left({}_y\mathfrak{D}_L^\xi\Theta, y\mathfrak{D}_R^\xi\Theta\right)\right|\right)^{1/2}, \quad \|\Theta\|_{\mathfrak{J}_S^\xi(\Xi)} := \left(\|\Theta\|_{L_2(\Xi)}^2 + |\Theta|_{\mathfrak{J}_S^\xi(\Xi)}^2\right)^{1/2},$$

and denote $\mathfrak{J}_S^\xi(\Xi)(\mathfrak{J}_{S,0}^\xi(\Xi))$ as the closure of $C^\infty(\Xi)(C_0^\infty(\Xi))$ with respect to $\|\cdot\|_S^\xi(\Xi)$.

Definition 4 ([7, 24]). *For $\eta > 0$, we define the semi-norm and norm respectively as*

$$|\Theta|_{\mathfrak{H}^\eta(\Xi)} := \left\||\xi|^\eta \mathscr{F}(\hat{\Theta})(\xi)\right\|_{L_2(\Xi)}, \quad \|\Theta\|_{\mathfrak{H}^\eta(\Xi)} := \left(\|\Theta\|_{L_2(\Xi)}^2 + |\Theta|_{\mathfrak{H}^\eta(\Xi)}^2\right)^{1/2},$$

where $\mathscr{F}(\hat{\Theta})(\xi)$ is the Fourier transform of $\hat{\Theta}$ and $\hat{\Theta}$ is the zero extension of Θ outside Ξ, and denote $\mathfrak{H}^\eta(\Xi)\left(\mathfrak{H}_0^\eta(\Xi)\right)$ as the closure of $C^\infty(\Xi)\,(C_0^\infty(\Xi))$ with respect to $\|\cdot\|_{\mathfrak{H}^\eta(\Xi)}$.

Definition 5 ([7, 24]). *We define the spaces $\mathfrak{J}_{L,0}^\kappa(\Xi), \mathfrak{J}_{R,0}^\kappa(\Xi), \mathfrak{J}_{S,0}^\kappa(\Xi)$ and $\mathfrak{H}_0^\kappa(\Xi)$ with respect to related norms, as the closer of $C_0^\infty(\Xi)$.*

Lemma 1 ([24]). *Let $\xi > 0, \Theta^n \in \mathfrak{J}_{L,0}^\xi \cap \mathfrak{J}_{R,0}^\xi$, therefore*

$$\left({}_x\mathfrak{D}_L^\xi\Theta, {}_x\mathfrak{D}_R^\xi\Theta\right)_{L^2(\Xi)} = \left({}_x\mathfrak{D}_L^\xi\hat{\Theta}, {}_x\mathfrak{D}_R^\xi\hat{\Theta}\right)_{L^2(\mathbb{R}^2)} = \cos(\xi\pi)\left\|{}_x\mathfrak{D}_L^\xi\hat{\Theta}\right\|_{L^2(\mathbb{R}^2)}$$
$$\left({}_y\mathfrak{D}_L^\xi\Theta, {}_y\mathfrak{D}_R^\xi\Theta\right)_{L^2(\Xi)} = \left({}_y\mathfrak{D}_L^\xi\hat{\Theta}, {}_y\mathfrak{D}_R^\xi\hat{\Theta}\right)_{L^2(\mathbb{R}^2)} = \cos(\xi\pi)\left\|{}_y\mathfrak{D}_L^\xi\hat{\Theta}\right\|_{L^2(\mathbb{R}^2)},$$

where $\hat{\Theta}$ is the zero extension of Θ outside Ξ.

Lemma 2 ([6]). *For all $\kappa > 0, \kappa \neq n - \frac{1}{2}, n \in \mathbb{N}$, let $\Theta \in \mathfrak{J}_{L,0}^\kappa(\Xi) \cap \mathfrak{J}_{R,0}^\kappa(\Xi) \cap \mathfrak{H}_0^\kappa(\Xi)$. Then we have*

$$\mathfrak{v}_1|\Theta|_{\mathfrak{H}^\kappa(\Xi)} \leq \max\left\{|\Theta|_{\mathfrak{J}_L^\kappa(\Xi)}, |\Theta|_{\mathfrak{J}_R^\kappa(\Xi)}\right\} \leq \mathfrak{v}_2|\Theta|_{\mathfrak{H}^\kappa(\Xi)},$$

where $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are positive constants which are independent of Θ.

Lemma 3 ([7, 24]). *For all $\kappa > 0$, the spaces $\mathfrak{J}_L^\kappa\left(\mathbb{R}^2\right), \mathfrak{J}_R^\kappa\left(\mathbb{R}^2\right), \mathfrak{J}_S^\kappa\left(\mathbb{R}^2\right)$ and $\mathfrak{H}^\kappa\left(\mathbb{R}^2\right)$ are equal with respected semi-norms and norms. For $\kappa > 0, \kappa \neq n - \frac{1}{2}, n \in \mathbb{N}$, spaces $\mathfrak{J}_{L,0}^\kappa(\Xi), \mathfrak{J}_{R,0}^\kappa(\Xi), \mathfrak{J}_{S,0}^\kappa(\Xi)$ and $\mathfrak{H}_0^\kappa(\Xi)$ are equal with respected semi-norms and norms.*

Lemma 4 ([7,24]). *For $\Theta \in \mathfrak{J}^{\kappa}_{L,0}(\Xi), 0 < p < \kappa$ then, there exist reels $\mathfrak{v}_q, \mathfrak{v} > 0$ and $\mathfrak{v}_i > 0, i = 1,2,3,4$, such that*

$$\|\Theta\|_{L^2(\Xi)} \le \mathfrak{v}|\Theta|_{\mathfrak{J}^{\kappa}_{L}(\Xi)}, \quad |\Theta|_{\mathfrak{J}^{\nu}_{L}(\Xi)} \le \mathfrak{v}|\Theta|_{\mathfrak{J}^{\kappa}_{L}(\Xi)}, \quad \|\Theta\|_{\mathfrak{J}^{\kappa}_{L}(\Xi)} \le \mathfrak{v}_q|\Theta|_{\mathfrak{J}^{\kappa}_{L}(\Xi)},$$
$$\|\Theta\|_{L^2(\Xi)} \le \mathfrak{v}_1 \left\|{}_x\mathfrak{D}^{p}_{L}\Theta\right\|_{L^2(\Xi)} \le \mathfrak{v}_2 \left\|{}_x\mathfrak{D}^{\kappa}_{L}\Theta\right\|_{L^2(\Xi)}, \quad \|\Theta\|_{L^2(\Xi)} \le \mathfrak{v}_3 \left\|{}_y\mathfrak{D}^{p}_{L}\Theta\right\|_{L^2(\Xi)} \le \mathfrak{v}_4 \left\|{}_y\mathfrak{D}^{\kappa}_{L}\Theta\right\|_{L^2(\Xi)},$$

In the same way, the results mentioned above apply to the spaces. $\mathfrak{J}^{\kappa}_{R,0}(\Xi), \mathfrak{J}^{\kappa}_{S,0}(\Xi)$ and $\mathfrak{H}^{\kappa}_{0}(\Xi)$ $\left(\nu \ne n - \frac{1}{2}\right)$.

Lemma 5 ([7,24]). *If $\xi \in (0,1), e, f \in \mathfrak{J}^{2\xi}_{L}(\Xi)$ and $e|_{\partial\Xi} = 0$, and $f|_{\partial\Xi} = 0$, then*

$$\left({}_x\mathfrak{D}^{2\xi}_{R} e, f\right) = \left({}_x\mathfrak{D}^{\xi}_{R} e, {}_x\mathfrak{D}^{\xi}_{L} f\right), \quad \left({}_y\mathfrak{D}^{2\xi}_{L} e, f\right) = \left({}_y\mathfrak{D}^{\xi}_{L} e, {}_y\mathfrak{D}^{\xi}_{R} f\right),$$
$$\left({}_y\mathfrak{D}^{2\xi}_{R} e, f\right) = \left({}_y\mathfrak{D}^{\xi}_{R} e, {}_y\mathfrak{D}^{\xi}_{L} f\right), \quad \left({}_y\mathfrak{D}^{2\xi}_{L} e, f\right) = \left({}_y\mathfrak{D}^{\xi}_{L} e, {}_y\mathfrak{D}^{\xi}_{R} f\right).$$

Lemma 6 ([23]). *(Discrete Gronwall Inequality). Let E_m denote a non-negative sequence, and let the sequence w_m satisfy*

$$\begin{cases} w_0 \le x_0, \\ w_m \le x_0 + \sum_{i=0}^{m-1} b_i + \sum_{i=0}^{m-1} E_i w_i, \quad m \ge 1. \end{cases}$$

If $x_0 \ge 0$ and $b_0 \ge 0$, we have $w_m \le \left(x_0 + \sum_{i=0}^{m-1} b_i\right) \exp\left(\sum_{i=0}^{m-1} E_i\right), \quad m \ge 1.$

2.2 Legendre polynomials and properties

The Legendre polynomials are an important special case of the Jacobi polynomials. We now introduce a set of fundamental formulas for Legendre polynomials [29, 30]. The Legendre polynomials $\mathfrak{L}_p(x), p = 0,1,\ldots$, are the eigenfunctions of the singular Sturm-Liouville problem:

$$\left(\left(1-x^2\right)\mathfrak{L}'_p(x)\right)' + \lambda_p \mathfrak{L}_p(x) = 0, \quad \lambda_p = p(p+1),$$

or equivalently

$$\left(1-x^2\right)\mathfrak{L}''_p(x) - 2x\mathfrak{L}'_p(x) + p(p+1)\mathfrak{L}_p(x) = 0.$$

If $\mathfrak{L}_p(x)$ is normalized so that $\mathfrak{L}_p(1) = 1$, then the Legendre polynomials has the following expansion:

$$\mathfrak{L}_p(x) = \frac{1}{2^p} \sum_{i=0}^{[p/2]} (-1)^i \binom{p}{i} \binom{2p-2i}{p} x^{p-2i}, \qquad p \ge 0,$$

where $[p/2]$ denotes the integral part of $p/2$. The Legendre polynomials satisfy the three-term recurrence relation and Rodrigues' formula, respectively:

$$\mathfrak{L}_{p+1}(x) = \frac{2p+1}{p+1} x \mathfrak{L}_p(x) - \frac{p}{p+1} \mathfrak{L}_{p-1}(x), \quad \mathfrak{L}_p(x) = \frac{1}{2^p p!} \frac{d^p}{dx^p}\left[\left(x^2-1\right)^p\right], \quad p \ge 0,$$

where the first few Legendre polynomials are:

$$\mathfrak{L}_0(x) = 1, \quad \mathfrak{L}_1(x) = x, \quad \mathfrak{L}_2(x) = \frac{1}{2}\left(3x^2-1\right), \quad \mathfrak{L}_3(x) = \frac{1}{2}\left(5x^3-3x\right).$$

It has symmetric property:

$$\mathfrak{L}_p(-x) = (-1)^p \mathfrak{L}_p(x), \quad \mathfrak{L}_p(\pm 1) = (\pm 1)^p, \qquad p \geq 0.$$

Hence, $\mathfrak{L}_p(x)$ is an odd (resp. even) function, if n is odd (resp. even). Moreover, it is uniformly bound:

$$|\mathfrak{L}_p(x)| \leq 1, \quad \forall x \in [-1,1], \ \ p \geq 0.$$

3 Formulation of the time discrete scheme

In this section, we will use the finite difference method in the temporal direction. Let $\Delta t = \frac{T}{\mathcal{N}}$ be the time step and $t_n = n\Delta t$ such that $n = 0,1,\ldots,\mathcal{N}, \mathcal{N} \in \mathbb{N}^+$. For $\Theta \in C(\Xi \times [0,T])$, we consider $\Theta^n(\cdot) = \Theta(\cdot,t_n)$.

To approximate the fractional Caputo derivative, we bring to the standard L_1 formula on a uniform mesh, then ${}_0^C\mathfrak{D}_t^\alpha \Theta(t)$ at $t = t_n$, for $n = 1,2,\ldots,\mathcal{N}$ can be approximated by

$$\mathfrak{Y}_t^\alpha \Theta^n = \frac{1}{\Gamma(2-\alpha)} \left[d_1^n \Theta^n + \sum_{i=1}^{n-1} \left(d_{i+1}^n - d_i^n \right) \Theta^{n-i} - d_n^n \Theta^0 \right] = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n} q_i^n \Theta^i, \tag{2}$$

where $\Delta t^{-\alpha} q_{n-i}^n = d_{i+1}^n - d_i^n, \quad 1 \leq i \leq n-1$, and

$$\Delta t^{-\alpha} q_0^n = -d_n^n, \quad q_n^n = 1, \quad \mathit{and} \quad d_i^n = \frac{(t_n - t_{n-i})^{1-\alpha} - (t_n - t_{n-i+1})^{1-\alpha}}{\Delta t}, \quad i \geq 1.$$

We can get via the mean value theorem that

$$\Delta t^{-\alpha} = d_1^n \leq d_i^n \leq d_{i+1}^n, \quad i \geq 2. \tag{3}$$

Lemma 7 ([31]). *Let $\Theta \in C^2(0,T] \bigcap C(0,T]$, suppose $\left|\frac{d^p\Theta}{dt}\right| \leq C_1 \left(1 + t^{\eta - l}\right)$ for $p = 0,1,2$ and $0 < \eta < 1$. Then there exists C_2, where the truncation error of the L_1 formula (2) satisfies*

$$|\mathfrak{R}_1^n| = \left|{}_0^c\mathfrak{D}_t^\eta \Theta(t_n) - \mathfrak{Y}_t^\eta \Theta(t_n)\right| \leq C_2 \Delta t^{2-\eta}.$$

We conclude the discrete scheme in the time direction at the point $t = t_n$ as

$$\mathfrak{Y}_t^\alpha \Theta^n = \Psi_{1,x} \frac{\partial^{2\gamma}\Theta^n}{\partial |x|^{2\gamma}} + \Upsilon_{1,y} \frac{\partial^{2\beta}\Theta^n}{\partial |y|^{2\beta}} + \Psi_{2,x} \frac{\partial^{2\zeta}\Theta^n}{\partial |x|^{2\zeta}} + \Upsilon_{2,y} \frac{\partial^{2\varsigma}\Theta^n}{\partial |y|^{2\varsigma}} + \mathfrak{F}^n + \mathfrak{R}_1^n, \tag{4}$$

Eliminating the small term $\mathfrak{R}_1^n$, we have

$$\mathfrak{Y}_t^\alpha \theta^n = \Psi_{1,x} \frac{\partial^{2\gamma}\theta^n}{\partial |x|^{2\gamma}} + \Upsilon_{1,y} \frac{\partial^{2\beta}\theta^n}{\partial |y|^{2\beta}} + \Psi_{2,x} \frac{\partial^{2\zeta}\theta^n}{\partial |x|^{2\zeta}} + \Upsilon_{2,y} \frac{\partial^{2\varsigma}\theta^n}{\partial |y|^{2\varsigma}} + \mathfrak{F}^n, \tag{5}$$

The Galerkin variational formulation of Eq. (5) is as follows:

Find $\theta^n \in \mathfrak{H}_0^\zeta \cap \mathfrak{H}_0^\varsigma \subset \mathfrak{H}_0^\gamma \cap \mathfrak{H}_0^\beta$ verified

$$(\mathfrak{Y}_t^\alpha \theta^n, e) + \mathfrak{A}(\theta^n, e) = (\mathfrak{F}^n, e), \quad \forall e \in \mathfrak{H}_0^\zeta \cap \mathfrak{H}_0^\varsigma, \tag{6}$$

where

$$\mathfrak{A}(e,f)=\frac{\Upsilon_{1,x}}{2\cos(\gamma\pi)}\left[\left({}_x\mathfrak{D}_L^{2\gamma}e,f\right)+\left({}_x\mathfrak{D}_R^{2\gamma}e,f\right)\right]+\frac{\Psi_{1,y}}{2\cos(\beta\pi)}\left[\left({}_y\mathfrak{D}_L^{2\beta}e,f\right)+\left({}_y\mathfrak{D}_R^{2\beta}e,f\right)\right]$$
$$+\frac{\Upsilon_{2,x}}{2\cos(\zeta\pi)}\left[\left({}_x\mathfrak{D}_L^{2\zeta}e,f\right)+\left({}_x\mathfrak{D}_R^{2\zeta}e,f\right)\right]+\frac{\Psi_{2,y}}{2\cos(\varsigma\pi)}\left[\left({}_y\mathfrak{D}_L^{2\varsigma}e,f\right)+\left({}_y\mathfrak{D}_R^{2\varsigma}e,f\right)\right],$$

and $\mathfrak{A}$ can be expressed as

$$\mathfrak{A}(e,f)=\frac{\Upsilon_{1,x}}{2\cos(\gamma\pi)}\left[\left({}_x\mathfrak{D}_L^{\gamma}e,{}_x\mathfrak{D}_R^{\gamma}f\right)+\left({}_x\mathfrak{D}_R^{\gamma}e,{}_x\mathfrak{D}_L^{\gamma}f\right)\right]+\frac{\Psi_{1,y}}{2\cos(\beta\pi)}\left[\left({}_y\mathfrak{D}_L^{\beta}e,{}_y\mathfrak{D}_R^{\beta}f\right)+\left({}_y\mathfrak{D}_R^{\beta}e,{}_y\mathfrak{D}_L^{\beta}f\right)\right]$$
$$+\frac{\Upsilon_{2,x}}{2\cos(\zeta\pi)}\left[\left({}_x\mathfrak{D}_L^{\zeta}e,{}_x\mathfrak{D}_R^{\zeta}f\right)+\left({}_x\mathfrak{D}_R^{\zeta}e,{}_x\mathfrak{D}_L^{\zeta}f\right)\right]+\frac{\Psi_{2,y}}{2\cos(\varsigma\pi)}\left[\left({}_y\mathfrak{D}_L^{\varsigma}e,{}_y\mathfrak{D}_R^{\varsigma}f\right)+\left({}_y\mathfrak{D}_R^{\varsigma}e,{}_y\mathfrak{D}_L^{\varsigma}f\right)\right].$$

Theorem 1. *For $\theta^n\in\mathfrak{H}_0^{\zeta}\cap\mathfrak{H}_0^{\varsigma}$ solution of the time discrete scheme (6) there exists $\mathscr{H}_1$, $\mathscr{H}_2>0$, where we have the following estimation*

$$\|\theta^n\|_{L^2(\Xi)}\leq\mathscr{H}_1\left\|\theta^0\right\|_{L^2(\Xi)}+\mathscr{H}_2\max_{0\leq j\leq n}\left\|\mathfrak{F}^j\right\|_{L^2(\Xi)}.$$

Proof. Replacing $e=\theta^n$ in Eq. (6), we get

$$\left(\mathfrak{Y}_t^{\alpha}\theta^n,\theta^n\right)+\mathfrak{A}\left(\theta^n,\theta^n\right)=\left(\mathfrak{F}^n,\theta^n\right). \tag{7}$$

According to Lemma 1, $\mathfrak{A}\left(\theta^n,\theta^n\right)\geq 0$, then

$$d_1^n\|\theta^n\|^2{}_{L^2(\Xi)}\leq d_n^n(\theta^0,\theta^n)-\sum_{i=1}^{n-1}\left(d_{i+1}^n-d_i^n\right)\left(\theta^{n-i},\theta^n\right)+\Gamma(2-\alpha)\left(\mathfrak{F}^n,\theta^n\right).$$

From (3), and by employing the Cauchy-Schwarz, Young inequality, we obtain

$$\|\theta^n\|_{L^2(\Xi)}\leq\frac{d_n^n}{d_1^n}\left\|\theta^0\right\|_{L^2(\Xi)}+\frac{1}{d_1^n}\sum_{i=1}^{n-1}\left(d_i^n-d_{i+1}^n\right)\left\|\theta^{n-i}\right\|_{L^2(\Xi)}+\frac{\Gamma(2-\alpha)}{d_1^n}\|\mathfrak{F}^n\|_{L^2(\Xi)},$$

which is equivalent to

$$\|\theta^n\|_{L^2(\Xi)}\leq\frac{d_n^n}{d_1^n}\left\|\theta^0\right\|_{L^2(\Xi)}+\frac{\Gamma(2-\alpha)}{d_1^n}\|\mathfrak{F}^n\|_{L^2(\Xi)}+\frac{1}{d_1^n}\sum_{i=1}^{n-1}\left(d_{n-i}^n-d_{n-i+1}^n\right)\left\|\theta^{i}\right\|_{L^2(\Xi)}.$$

Using (3) and Lemma 6, we conclude

$$\|\theta^n\|_{L^2(\Xi)}\leq\left(\frac{d_n^n}{d_1^n}\left\|\theta^0\right\|_{L^2(\Xi)}+\frac{\Gamma(2-\alpha)}{d_1^n}\|\mathfrak{F}^n\|_{L^2(\Xi)}\right)\exp\left(\frac{1}{d_1^n}\sum_{i=1}^{n-1}\left(d_{n-i}^n-d_{n-i+1}^n\right)\right).$$

We obtain

$$\|\theta^n\|_{L^2(\Xi)}\leq\frac{d_n^n}{d_1^n}\exp\left(\frac{1}{d_1^n}\left(d_1^n-d_n^n\right)\right)\left\|\theta^0\right\|_{L^2(\Xi)}+\frac{\Gamma(2-\alpha)}{d_1^n}\exp\left(\frac{1}{d_1^n}\left(d_1^n-d_n^n\right)\right)\max_{0\leq j\leq n}\left\|\mathfrak{F}^j\right\|_{L^2(\Xi)}.$$

Using the mean value theorem, we get

$$\begin{aligned}
&d_1^n - d_n^n = \Delta t^{-\alpha}(1 - n^{1-\alpha} + (n-1)^{1-\alpha}) \leq \Delta t^{-\alpha},\\
&\frac{d_n^n}{d_1^n} = (n^{1-\alpha} - (n-1)^{1-\alpha}) \leq 1-\alpha.
\end{aligned}$$

Finally, we get

$$\|\theta^n\|_{L^2(\Xi)} \leq \exp(1)(1-\alpha)\left\|\theta^0\right\|_{L^2(\Xi)} + \exp(1)T^{\alpha}\Gamma(2-\alpha)\max_{0\leq j\leq n}\left\|\mathfrak{F}^j\right\|_{L^2(\Xi)}.$$

The proof is completed. □

4 The fully discrete method

This section focuses on the computational study and error analysis.

4.1 Computational analysis

The problem in the implementation process of the spectral element method is to compute the bilinear form $\mathfrak{A}(\theta_n, \xi)$. This leads us to evaluate $({}_x\mathfrak{D}_R^{2\eta}U_n, \xi)$ and $({}_x\mathfrak{D}_R^{2\eta}U_n, \xi)$. Let $m = 1, 2$, and $m-1 < 2\eta < m$. We will divide the domain into non-overlapping K elements. First, we introduced some backgrounds.

Lemma 8 ([29]). *Let us denote $g_p = \frac{1}{\sqrt{4p+6}}$, $\psi_p(x) = g_p\left(\mathfrak{L}_p(x) - \mathfrak{L}_{p+2}(x)\right)$ for $1 \leq x \leq -1$, and $\mathbf{b}_{ip} = \int_{-1}^{1}\left(\psi_p(x), \psi_i(x)\right)dx$. Then*

$$\mathbf{b}_{ip} = \mathbf{b}_{pi} = \begin{cases} g_p g_i\left(\frac{2}{2i+1} + \frac{2}{2i+5}\right), & p = i,\\ -g_p g_i \frac{2}{2p+1}, & p = i+2,\\ 0, & \textit{Otherwise.}\end{cases}$$

Lemma 9 ([11]). *For all $\eta > 0$, we have*

$$\begin{aligned}
{}_{-1}\mathfrak{D}_w^{\eta}\mathfrak{L}_j(w) &= \frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}(1+w)^{-\eta}\mathscr{P}_j^{\eta,-\eta}(w), \quad -1 \leq w \leq 1,\\
{}_w\mathfrak{D}_1^{\eta}\mathfrak{L}_j(w) &= \frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}(1-w)^{-\delta}\mathscr{P}_j^{\eta,-\eta}(w), \quad -1 \leq w \leq 1,
\end{aligned}$$

where $\mathscr{P}_j^{\eta_1,\eta_2}(w), (\eta_1, \eta_2 > -1)$ are Jacobi polynomials.

For the 1D case, we give the approximation of the solution θ^n as:

$$\theta(x, t_n) = \sum_{j=0}^{N}\theta(x_j, t_n)\psi_j(x), \quad n = 1, 2, \ldots, \mathscr{N}.$$

Therefore we compute $\left({}_x\mathfrak{D}_L^{2\eta}\theta^n,\xi\right)$ and $\left({}_x\mathfrak{D}_R^{2\eta}\theta^n,\xi\right)$ as follows

$$\begin{aligned}\left({}_x\mathfrak{D}_L^{2\eta}\theta^n,\xi\right) &= \sum_{e=1}^{K}\left({}_x\mathfrak{D}_L^{2\eta}\theta^n,\xi\right)_{\Xi_e}\\ &= \sum_{e=1}^{K}\left(\sum_{q=1}^{e-1}\int_{\Xi e}\frac{1}{\Gamma(m-2\eta)}\frac{d^m}{dx^m}\int_{x_{q-1}}^{x_q}(x-s)^{m-2\eta-1}\theta^n(s)ds\xi d\Xi_e+\left({x_{e-1}}\mathfrak{D}_x^{2\eta}\theta^n,\xi\right)_{\Xi_e}\right)\\ &= \sum_{e=1}^{K}\left(\sum_{q=1}^{e-1}\int_{\Xi e}\frac{1}{\Gamma(m-2\eta)}\frac{d^m}{dx^m}\int_{x_{q-1}}^{x_q}(x-s)^{m-2\eta-1}\theta^n(s)ds\xi d\Xi_e+\left({x_{e-1}}\mathfrak{D}_x^{\eta}\theta^n,{}_x\mathfrak{D}_{x_e}^{\eta}\xi\right)_{\Xi_e}\right),\end{aligned}$$

and

$$\begin{aligned}\left({}_x\mathfrak{D}_R^{2\eta}\theta^n,\xi\right) &= \sum_{e=1}^{K}\left({}_x\mathfrak{D}_R^{2\eta}\theta^n,\xi\right)_{\Xi_e}\\ &= \sum_{e=1}^{K}\left(\left({}_x\mathfrak{D}_{x_e}^{2\eta}\theta^n,\xi\right)_{\Xi_e}+\sum_{u=e+1}^{K}\int_{\Xi e}\frac{(-1)^m}{\Gamma(m-2\eta)}\frac{d^m}{dx^m}\int_{x_{u-1}}^{x_u}(s-x)^{m-2\eta-1}\theta^n(s)ds\xi d\Xi_e\right)\\ &= \sum_{e=1}^{K}\left(\left({}_x\mathfrak{D}_{x_e}^{\eta}\theta^n,{x_{e-1}}\mathfrak{D}_x^{\eta}\xi\right)_{\Xi_e}+\sum_{u=e+1}^{K}\int_{\Xi e}\frac{(-1)^m}{\Gamma(m-2\eta)}\frac{d^m}{dx^m}\times\int_{x_{u-1}}^{x_u}(s-x)^{m-2\eta-1}\theta^n(s)ds\xi d\Xi_e\right).\end{aligned}$$

Therefore we need to calculate $\left({x_{e-1}}\mathfrak{D}_x^{\eta}\mathfrak{L}_j(w),{}_x\mathfrak{D}_{x_e}^{\eta}\mathfrak{L}_k(w)\right)_{\Xi_e}$.

As in [5], for $h_e = x_e - x_{e-1}$ be the length of the element e, the map function from the e into $[-1,1]$, and its inverse can be given by

$$x(z)=\frac{1}{2}\left[(x_e-x_{e-1})z+(x_e+x_{e-1})\right],\quad -1\le z\le 1,\quad z(x)=\frac{2}{h_e}(x-x_{e-1})-1.$$

Therefore, we have

$$\mathbf{b}_{ip}^e=\int_{x_{e-1}}^{x_e}\psi_i(x)\psi_p(x)dx=\frac{h_e}{2}\int_{-1}^{1}\psi_i(z)\psi_p(z)dz.$$

Then we get

$$\begin{aligned}{x_{e-1}}\mathfrak{D}_x^{\eta}\mathfrak{L}_i(w) &= \frac{1}{\Gamma(1-\eta)}\frac{d}{dx}\int_{x_{e-1}}^{x}(x-s)^{-\eta}\mathfrak{L}_i(\hat{s})ds=\left(\frac{h_e}{2}\right)^{-\eta}{}_{-1}\mathfrak{D}_{\mu}^{\eta}\mathfrak{L}_i(w),\ \ 0<\eta<1,\\ {}_x\mathfrak{D}_{x_e}^{\eta}\mathfrak{L}_i(w) &= \frac{-1}{\Gamma(1-\eta)}\frac{d}{dx}\int_{x}^{x_e}(s-x)^{-\eta}\mathfrak{L}_i(\hat{s})ds=\left(\frac{h_e}{2}\right)^{-\eta}{}_w\mathfrak{D}_1^{\eta}\mathfrak{L}_i(w),\ \ 0<\eta<1,\end{aligned}$$

where $-1\le\hat{s}\le 1$. This gives

$$\begin{aligned}&\left({x_{e-1}}\mathfrak{D}_x^{\eta}\mathfrak{L}_j(w),{}_x\mathfrak{D}_{x_e}^{\eta}\mathfrak{L}_k(w)\right)_{\Xi_e}\\ &= \int_{x_{e-1}}^{x_e}{}_{x_{e-1}}\mathfrak{D}_x^{\eta}\mathfrak{L}_j(w){}_x\mathfrak{D}_{x_e}^{\eta}\mathfrak{L}_k(w)dx=\left(\frac{h_e}{2}\right)^{1-2\eta}\int_{-1}^{1}{}_{-1}\mathfrak{D}_w^{\eta}\mathfrak{L}_j(w){}_w\mathfrak{D}_1^{\eta}\mathfrak{L}_k(w)dw\\ &= \left(\frac{h_e}{2}\right)^{1-2\eta}\frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}\frac{\Gamma(k+1)}{\Gamma(k-\eta+1)}\int_{-1}^{1}(1+w)^{-\eta}(1-w)^{\eta}\mathscr{P}_j^{\eta,-\eta}\mathscr{P}_k^{-\eta,\eta}dw\\ &= \left(\frac{h_e}{2}\right)^{1-2\eta}\frac{\Gamma(j+1)}{\Gamma(j-\eta+1)}\frac{\Gamma(k+1)}{\Gamma(k-\eta+1)}\sum_{l=0}^{N}\mathscr{W}_l\mathscr{P}_j^{\eta,-\eta}(w_l)\,\mathscr{P}_k^{-\eta,\eta}(w_l).\end{aligned}$$

where $\{\mathscr{W}_l\}_{l=0}^{N}$ are the weights functions and $\{w_l\}_{l=0}^{N}$ are the Jacobi-Gauss-Lobatto quadrature [30].

In the 2D case, we give the approximation of the solution θ^n as:

$$\theta^n(x,y,t_n) = \sum_{i=0}^{N}\sum_{j=0}^{N} \theta^n(x_i,y_j,t_n)\,\psi_i(x)\psi_j(y), \quad 1 \le n \le \mathscr{N}.$$

The polynomials of the order no more than N are the selected basis, then the element matrices are given by

$$\mathbf{b}^{Qe} = \mathbf{b}_x \otimes \mathbf{b}_y, \qquad \mathbf{D}_{lx}^{Qe} = \mathbf{D}_{lx}^{e} \otimes \mathbf{b}_y, \quad \mathbf{D}_{rx}^{Qe} = \mathbf{D}_{rx}^{e} \otimes \mathbf{b}_y,$$
$$\mathbf{G}_x^{Qel} = \mathbf{G}_x^{el} \otimes \mathbf{b}_y, \quad \mathbf{L}_x^{Qer} = \mathbf{L}_x^{er} \otimes \mathbf{b}_y,$$

where

$$(\mathbf{D}_{lx}^{e})_{ij} = \int_{x_{e-1}}^{x_e} {}_{x_{e-1}}D_x^{\eta}\psi_j^{(e)}(x)\,{}_xD_{x_e}^{\eta}\psi_i^{(e)}(x)dx, \quad (\mathbf{D}_{rx}^{e})_{ij} = \int_{x_{e-1}}^{x_e} {}_xD_{x_e}^{\eta}\psi_j^{(e)}(x)\,{}_{x_{e-1}}D_x^{\eta}\psi_i^{(e)}(x)dx,$$
$$\left(\mathbf{G}_x^{el}\right)_{ij} = \int_{x_{e-1}}^{x_e} \frac{1}{\Gamma(m-2\eta)}\frac{d^m}{dx^m}\int_{x_{l-1}}^{x_l}(x-s)^{m-2\eta-1}\psi_j^{(l)}(s)ds\,\psi_i^{(e)}(x)dx,$$
$$(\mathbf{L}_x^{er})_{ij} = \int_{x_{e-1}}^{x_e} \frac{(-1)^m}{\Gamma(m-2\eta)}\frac{d^m}{dx^m}\int_{x_{r-1}}^{x_r}(s-x)^{m-2\eta-1}\psi_j^{(r)}(s)ds\,\psi_i^{(e)}(x)dx.$$

Using integration by parts recursively, we can directly compute $\mathbf{G}^{el}$ and $\mathbf{L}^{er}$. The same process is used to calculate $({}_y\mathfrak{D}_R^{2\eta}\theta_n,\xi)$ and $({}_y\mathfrak{D}_R^{2\eta}\theta_n,\xi)$, also to conclude the element matrices.

4.2 Error estimation

This subsection showcases the error analysis of the fully Galerkin spectral element method. For that, we declare for some notions and lemmas. We denote $\|\cdot\|_{L_2(\Xi)} = \|\cdot\|$. We define the space $\mathscr{V}_h^0$ as

$$\mathscr{V}_h^0 = \left\{ w \in \mathfrak{H}_0^{\zeta} \cap \mathfrak{H}_0^{\varsigma} : w|_{\Xi_e} \in \mathbf{P}_N \right\},$$

where $\mathbf{P}_N$ is the space of polynomials of degree no more than $N \in \mathbb{N}$.

Let $\mathscr{P}_h^{\zeta,\varsigma,0}$ be the orthogonal projection operator, map from $\mathfrak{H}_0^{\zeta}(\Xi)\cap\mathfrak{H}_0^{\varsigma}(\Xi) \to \mathscr{V}_h^0$ is defined as

$$\mathfrak{A}\left(\Theta - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta, \xi\right) = 0, \quad \forall \xi \in \mathscr{V}_h^0. \tag{8}$$

Additionally, we introduce the distributed seminorm $|\cdot|_{\zeta,\varsigma}$ and the distributed norm $\|\cdot\|_{\zeta,\varsigma}$ as follows:

$$|\Theta|_{\zeta,\varsigma} = \left(\left\| {}_0\mathfrak{D}_x^{\zeta}\Theta \right\|_{L^2(\Xi)}^2 + \left\| {}_0\mathfrak{D}_y^{\varsigma}\Theta \right\|_{L^2(\Xi)}^2 \right)^{1/2}, \quad \|\Theta\|_{\zeta,\varsigma} = \left(\|\Theta\|_{L^2(\Xi)}^2 + |\Theta|_{\zeta,\varsigma}^2 \right)^{1/2}.$$

Lemma 10. *For $\Theta \in \mathfrak{H}_0^{\kappa_1} \cap \mathfrak{H}_0^{\kappa_2}, 0 < \kappa_1, \kappa_2 \le 1$, there exists a positive constant k, where we have the following estimation*

$$\|\Theta\|_{\kappa_1,\kappa_2} \le k|\Theta|_{\kappa_1,\kappa_2}.$$

Proof. According to Lemma 4, we obtain

$$\begin{aligned}
\|\Theta\|^2_{\kappa_1,\kappa_2} &= \|\Theta\|^2 + \|{}_0\mathfrak{D}_x^{\zeta}\Theta\|^2 + \|{}_0\mathfrak{D}_y^{\kappa_2}\Theta\|^2 \\
&\leq (1+k_1)\left\|{}_0\mathfrak{D}_x^{\kappa_1}\Theta\right\|^2 + (1+k_2)\left\|{}_0\mathfrak{D}_y^{\kappa_2}\Theta\right\|^2 \\
&\leq k\left(\left\|{}_0\mathfrak{D}_x^{\kappa_1}\Theta\right\|^2 + \left\|{}_0\mathfrak{D}_y^{\kappa_2}\Theta\right\|^2\right) \\
&= k|\Theta|^2_{\kappa_1,\kappa_2}.
\end{aligned}$$

□

Lemma 11. *For $\Theta \in \mathfrak{H}_0^{\kappa_1} \cap \mathfrak{H}_0^{\kappa_2}, 0 < \kappa_1, \kappa_2 \leq 1$, there exists a positive constant k, where we have the following estimation*

$$\|\Theta\|_{\kappa_1,\kappa_2} \leq k\|\Theta\|_{\mathfrak{H}(\Xi)^{\max(\kappa_1,\kappa_2)}}.$$

Proof. According to Lemma 4, we can get

$$\begin{aligned}
|\Theta|^2_{\kappa_1,\kappa_2} &= \left\|{}_0\mathfrak{D}_x^{\kappa_1}\Theta\right\|^2 + \left\|{}_0\mathfrak{D}_y^{\kappa_2}\Theta\right\|^2 \\
&\leq \bar{k}\left(\|{}_0\mathfrak{D}_x^{\max(\kappa_1,\kappa_2)}\Theta\|^2 + \|{}_0\mathfrak{D}_y^{\max(\kappa_1,\kappa_2)}\Theta\|^2\right) \\
&= \bar{k}|\Theta|^2_{\mathfrak{I}_L(\Xi)^{\max(\kappa_1,\kappa_2)}}.
\end{aligned}$$

Using Lemma 2, we conclude

$$\|\Theta\|^2_{\kappa_1,\kappa_2} \leq |\Theta|^2_{\kappa_1,\kappa_2} \leq k|\Theta|^2_{\mathfrak{H}(\Xi)^{\max(\kappa_1,\kappa_2)}},$$

and the proof is completed. □

Theorem 2. *The bilinear form $\mathfrak{A}(\cdot,\cdot)$ is coercive and continuous.*

Proof. Similar to Theorem 3,1 in [3], we obtain the continuity and the coercivity, by combining Lemmas 1 and 4, and 10, respectively. □

Lemma 12. *Let $\xi^n \in \mathscr{V}_h^0$, then the bilinear form $\mathfrak{A}(\cdot,\cdot)$ satisfies the following relations*

$$2\mathfrak{A}\left(\xi^n, \sum_{i=0}^{n} q_i^n \xi^i\right) = \mathfrak{A}(\xi^n,\xi^n) + \sum_{i=0}^{n-1} q_i^n \mathfrak{A}\left(\xi^i,\xi^i\right) - \sum_{i=0}^{n-1} q_i^n \mathfrak{A}\left(\xi^i - \xi^n, \xi^i - \xi^n\right), \tag{9}$$

and

$$2\mathfrak{A}\left(\xi^n, \sum_{i=0}^{n} q_i^n \xi^i\right) \geq \mathfrak{A}(\xi^n,\xi^n) + \sum_{i=0}^{n-1} q_i^n \mathfrak{A}\left(\xi^i,\xi^i\right). \tag{10}$$

Proof. Note that

$$\begin{aligned}
\mathfrak{A}\left(\xi^n, \sum_{i=0}^{n} q_i^n \xi^i\right) &= \mathfrak{A}(\xi^n,\xi^n) + \mathfrak{A}\left(\xi^n, \sum_{i=0}^{n-1} q_i^n \xi^i\right) \\
&= \mathfrak{A}(\xi^n,\xi^n) + q_{n-1}^n \mathfrak{A}\left(\xi^{n-1},\xi^{n-1}\right) - q_{n-1}^n B\left(\xi^{n-1},\xi^{n-1}\right) + \mathfrak{A}\left(\xi^n, \sum_{i=0}^{n-1} q_i^n \xi^i\right) \\
&= \mathfrak{A}(\xi^n,\xi^n) + q_{n-1}^n \mathfrak{A}\left(\xi^{n-1},\xi^{n-1}\right) - q_{n-1}^n \mathfrak{A}\left(\xi^{n-1} - \xi^n,\xi^{n-1}\right) + \mathfrak{A}\left(\xi^n, \sum_{i=0}^{n-2} q_i^n \xi^j\right).
\end{aligned} \tag{11}$$

Hence

$$\mathfrak{A}\left(\xi,\sum_{i=0}^{n}b_i^n\xi^i\right)-q_{n-1}^n\mathfrak{A}\left(\xi^n,\xi^n-\xi^{n-1}\right)=\mathfrak{A}\left(\xi^n,\xi^n\right)+q_{n-1}^n\mathfrak{A}\left(\xi^{n-1},\xi^{n-1}\right)$$
$$-q_{n-1}^n\mathfrak{A}\left(\xi^{n-1}-\xi^n,\xi^{n-1}-\xi^n\right)+\mathfrak{A}\left(\xi^n,\sum_{i=0}^{n-2}q_i^n\xi^i\right), \tag{12}$$

Repeating the processes (11)-(12), we get

$$\mathfrak{A}\left(\xi^n,\sum_{i=0}^{n}q_i^n\xi^i\right)-\sum_{i=0}^{n-1}q_i^n\mathfrak{A}\left(\xi^n,\xi^n-\xi^i\right)=\mathfrak{A}\left(\xi^n,\xi^n\right)+\sum_{i=0}^{n-1}q_i^n\mathfrak{A}\left(\xi^i,\xi^i\right)$$
$$-\sum_{i=0}^{n-1}q_i^n\mathfrak{A}\left(\xi^i-\xi^n,\xi^i-\xi^n\right). \tag{13}$$

As

$$-\sum_{i=0}^{n-1}q_i^n=q_n^n,$$

from (13) we conclude

$$\mathfrak{A}\left(\xi^n,\sum_{i=0}^{n}q_i^n\xi^i\right)=\frac{1}{2}\left(\mathfrak{A}\left(\xi^n,\xi^n\right)+\sum_{i=0}^{n-1}q_i^n\mathfrak{A}\left(\xi^i,\xi^i\right)-\sum_{i=0}^{n-1}q_i^n\mathfrak{A}\left(\xi^i-\xi^n,\xi^i-\xi^n\right)\right).$$

Using the facts $-q_i^n>0$, and $\mathfrak{A}\left(\xi^i-\xi^n,\xi^i-\xi^n\right)\geq 0$, $i=0,\dots,n-1$ leads to (10). □

Lemma 13 ([2])**.** *Suppose e and d are real numbers with $0\leq e\leq d$. Then, there exists a positive constant* $\mathbf{Q}$ *independent of d, such that for any function Θ belonging to both $\mathfrak{H}_0^e(\Xi)$ and $\mathfrak{H}^d(\Xi)$, the estimate inequality holds:*

$$\left\|\Theta-\mathscr{P}_h^{1,0}\Theta\right\|_{\mathfrak{H}^e(\Xi)}\leq \mathbf{Q}h_l^{\min(d,M)-e}N^{e-d}\|\Theta\|_{\mathfrak{H}^d(\Xi)}.$$

The diameters h_l of the element l satisfies $h\leq h_l\leq qh$ for all l, where h and q are positive constants.

Lemma 14. *Consider real numbers ζ, ς, and $\mathfrak{L}$ satisfying $0<\zeta,\varsigma<1<\mathfrak{L}$, then there exists a positive constant* $\mathbf{Q}$ *independent of N such that, for any function $\Theta\in\mathfrak{H}_0^{\zeta}(\Xi)\cap\mathfrak{H}_0^{\varsigma}(\Xi)\cap\mathfrak{H}^{\mathfrak{L}}(\Xi)$, the following estimate holds:*

$$\|\Theta-\mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma}\leq\mathbf{Q}\left(h_l^{\min(d,M)-\max(\zeta,\varsigma)}N^{\max(\zeta,\varsigma)-\mathfrak{L}}\right)\|\Theta\|_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}.$$

Proof. Based on Theorem 2, we use the V-elliptic of $\mathscr{A}$, then

$$\|\Theta-\mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma}^2\leq\mathbf{Q}\mathfrak{A}\left(\Theta-\mathscr{P}_h^{\zeta,\varsigma,0}\Theta,\Theta-\mathscr{P}_h^{\zeta,\varsigma,0}\Theta\right).$$

Eq. (8) gives

$$\|\Theta-\mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma}^2\mathbf{Q}\mathfrak{A}\left(\Theta-\mathscr{P}_h^{\zeta,\varsigma,0}\Theta,\Theta-\Theta_h\right),\quad \Theta_h\in\mathscr{V}_h^0.$$

Replacing $\Theta_h = \mathscr{P}_h^{1,0}\Theta$, using the continuity of $\mathscr{A}$, and from the Theorem 2, we have

$$\|\Theta - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma} \leq \mathbf{Q}\|\Theta - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma}\|\Theta - \mathscr{P}_h^{1,0}\Theta\|_{\zeta,\varsigma}.$$

According to Lemma 10, we obtain

$$\|\Theta - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma} \leq \mathbf{Q}\|\Theta - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma}\|\Theta - \mathscr{P}_h^{1,0}\Theta\|_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}.$$

Applying Lemma 13, we conclude

$$\|\Theta - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta\|_{\zeta,\varsigma} \leq \mathbf{Q}\left(h_l^{\min(d,M)-\max(\zeta,\varsigma)}N^{\max(\zeta,\varsigma)-\mathfrak{L}}\right)\|\Theta\|_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}.$$

□

Theorem 3. *Let Θ be the solution of problem (1), where $\Theta \in \mathfrak{H}_0^{\zeta}(\Xi)\cap\mathfrak{H}_0^{\varsigma}(\Xi)\cap\mathfrak{H}^{\mathfrak{L}}(\Xi)$ and θ^n be the solution of the fully discrete scheme. Then, there exists $\mathbf{Q} > 0$, independent of n, Δt, and N, such that*

$$\|\mathbf{e}^n\|^2 \leq \mathbf{Q}\left(\Delta t^{4-2\alpha} + N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\right). \tag{14}$$

Proof. Let $\mathscr{Z}^n = \Theta^n - \mathscr{P}_h^{\zeta,\varsigma,0}\Theta^n$, $\mathscr{E}^n = \mathscr{P}_h^{\zeta,\varsigma,0}\Theta^n - \theta^n$, and $\mathbf{e}^n = \mathscr{E}^n + \mathscr{Z}^n$. Assume that $\mathbf{e}^0$=0. Using Eqs. (4) and (6), we get

$$(\mathfrak{Y}_t^{\alpha}\Theta^n,\xi) + \mathfrak{A}(\Theta^n,\xi) = (\mathfrak{F}^n,\xi) + (\mathfrak{R}_1^n,\xi), \quad \forall\xi \in \mathscr{V}_h^0, \tag{15}$$
$$(\mathfrak{Y}_t^{\alpha}\theta^n,\xi) + \mathfrak{A}(\theta^n,\xi) = (\mathfrak{F}^n,\xi), \quad \forall\xi \in \mathscr{V}_h^0. \tag{16}$$

Then, we have

$$(\mathfrak{Y}_t^{\alpha}(\Theta^n - \theta^n),\xi) + \mathfrak{A}(\Theta^n - \theta^n,\xi) = (\mathfrak{R}_1^n,\xi), \quad \forall\xi \in \mathscr{V}_h^0. \tag{17}$$

Using the relation (8), we obtain

$$(\mathfrak{Y}_t^{\alpha}\mathscr{E}^n,\xi) + \mathfrak{A}(\mathscr{E}^n,\xi) = (\mathfrak{R}_1^n,\xi) - (\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n,\xi), \quad \forall\xi \in \mathscr{V}_h^0. \tag{18}$$

Replacing $\xi = \mathfrak{Y}_t^{\alpha}\mathscr{E}^n$, we get

$$\|\mathfrak{Y}_t^{\alpha}\mathscr{E}^n\|^2 + \mathfrak{A}(\mathscr{E}^n,\mathfrak{Y}_t^{\alpha}\mathscr{E}^n) = (\mathfrak{R}_1^n,\mathfrak{Y}_t^{\alpha}\mathscr{E}^n) - (\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n,\mathfrak{Y}_t^{\alpha}\mathscr{E}^n). \tag{19}$$

Using

$$\begin{aligned}
|(\mathfrak{R}_1^n,\mathfrak{Y}_t^{\alpha}\mathscr{E}^n)| &\leq \frac{1}{2}\|\mathfrak{R}_1^n\|^2 + \frac{1}{2}\|\mathfrak{Y}_t^{\alpha}\mathscr{E}^n\|^2, \\
|(\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n,\mathfrak{Y}_t^{\alpha}\mathscr{E}^n)| &\leq \frac{1}{2}\|\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n\|^2 + \frac{1}{2}\|\mathfrak{Y}_t^{\alpha}\mathscr{E}^n\|^2,
\end{aligned}$$

we obtain

$$\mathfrak{A}\left(\mathscr{E}^n,\sum_{i=0}^{n}q_i^n\mathscr{E}^i\right) \leq \frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{2}\|\mathfrak{R}_t^n\|^2 + \frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{2}\|\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n\|^2.$$

According to Lemma 12, we have

$$\mathfrak{A}\left(\mathscr{E}^n,\mathscr{E}^n\right) \geq -\sum_{i=0}^{n-1} q_i^n \mathfrak{A}\left(\mathscr{E}^i,\mathscr{E}^i\right) + \Delta t^{\alpha}\Gamma(2-\alpha)\left\|\mathfrak{R}_t^n\right\|^2 + \Delta t^{\alpha}\Gamma(2-\alpha)\left\|\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n\right\|^2. \tag{20}$$

Lemmas 7 and 13 lead to the following error bounds,

$$\begin{aligned}
&\|R_t^n\|_0^2 \leq C_1 \Delta t^{4-2\alpha} \max_{0\leq t\leq T}\|\Theta_{tt}\|^2,\\
&\|\mathfrak{Y}_t^{\alpha}\mathscr{Z}^n\|^2 \leq \left\|{}_0^C\mathfrak{D}_t^{\alpha}\mathscr{Z}^n\right\|^2 + C\Delta t^{4-2\alpha}\max_{0\leq t\leq T}\|\mathscr{Z}_{tt}\|^2\\
&\qquad\qquad \leq C_2\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta\left(t_n\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)} + \Delta t^{4-2\alpha}\right),\\
&\left\|{}_0^C\mathfrak{D}_t^{\alpha}\mathscr{Z}^n - \mathfrak{Y}_t^{\alpha}\mathscr{Z}^n\right\| \leq C\Delta t^{2-\alpha}\max_{0\leq t\leq T}\|\mathscr{Z}_{tt}\|,\\
&\mathfrak{A}\left(\mathscr{E}^0,\mathscr{E}^0\right) \leq C_a\left\|\mathscr{E}^0\right\|^2_{\zeta,\varsigma} \leq C_3\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta\left(t_0\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).
\end{aligned}$$

Then the estimate (20) leads to

$$\begin{aligned}
\mathfrak{A}\left(\mathscr{E}^n,\mathscr{E}^n\right) &\leq -C_3 q_0^n\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta\left(t_0\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right) - \sum_{i=1}^{n-1} q_i^n\mathfrak{A}\left(\mathscr{E}^i,\mathscr{E}^i\right)\\
&\quad + \Gamma(2-\alpha)\Delta t^{\alpha} C_1\Delta t^{4-2\alpha} + \Gamma(2-\alpha)\Delta t^{\alpha} C_2\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta\left(t_n\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)} + \Delta t^{4-2\alpha}\right),\\
&\leq -C_3 q_0^n\Gamma(2-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta\left(t_0\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\
&\quad + C_*\Gamma(2-\alpha)\Delta t^{\alpha}\left(\Delta t^{4-2\alpha} + N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta\left(t_n\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right) - \sum_{i=1}^{n-1} q_i^n\mathfrak{A}\left(\mathscr{E}^i,\mathscr{E}^i\right).
\end{aligned} \tag{21}$$

For $n=1$, we have

$$\begin{aligned}
\mathfrak{A}\left(\mathscr{E}^1,\mathscr{E}^1\right) &\leq C_3\Gamma(1-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta\left(t_0\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\
&\quad + C_*\Gamma(1-\alpha)\Delta t^{\alpha}\left(\Delta t^{4-2\alpha} + N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta\left(t\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).
\end{aligned}$$

Assume that $n \leq l, (l \geq 1)$, the inequality follows

$$\begin{aligned}
\mathfrak{A}\left(\mathscr{E}^n,\mathscr{E}^n\right) &\leq C_3\Gamma(1-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta\left(t_0\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\
&\quad + C_*\Gamma(1-\alpha)n^{\alpha}\Delta t^{\alpha}\left(\Delta t^{4-2\alpha} + N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta\left(t\right)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).
\end{aligned} \tag{22}$$

Then it follows from Eq. (21) that

$$\begin{aligned}\mathfrak{A}\left(\mathscr{E}^{l+1},\mathscr{E}^{l+1}\right) &\leq C_3(-q_0^{l+1})\Gamma(2-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(2-\alpha)\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t_n)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)-\sum_{i=1}^{l}q_i^{l+1}\mathfrak{A}\left(\mathscr{E}^i,\mathscr{E}^i\right)\\ &\leq C_3(-q_0^{l+1})\Gamma(2-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(2-\alpha)\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t_n)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad-C_3\Gamma(1-\alpha)\sum_{i=1}^{l}q_i^{l+1}\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad-C_*\Gamma(1-\alpha)\sum_{i=1}^{l}q_i^{l+1}i^{\alpha}\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).\end{aligned}\tag{23}$$

Since

$$-\sum_{i=1}^{l}q_i^{l+1}=1+q_0^{l+1},\quad q_0^{l+1}=-\left((l+1)^{1-\alpha}-l^{1-\alpha}\right)\leq-(1-\alpha)(l+1)^{-\alpha},$$

we can write

$$\begin{aligned}\mathfrak{A}\left(\mathscr{E}^{l+1},\mathscr{E}^{l+1}\right) &\leq C_3\Gamma(2-\alpha)(-q_0^{l+1})\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(2-\alpha)\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t_n)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_3\Gamma(1-\alpha)(1+q_0^{l+1})\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(1-\alpha)(1+q_0^{l+1})l^{\alpha}\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).\end{aligned}\tag{24}$$

Furthermore

$$\begin{aligned}\mathfrak{A}\left(\mathscr{E}^{l+1},\mathscr{E}^{l+1}\right) &\leq C_3\Gamma(1-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(2-\alpha)\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t_n)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(1-\alpha)(1+q_0^{l+1})(l+1)^{\alpha}\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).\end{aligned}\tag{25}$$

Using $q_0^{l+1}\leq-(1-\alpha)(l+1)^{-\alpha}$, we obtain directly

$$\begin{aligned}\mathfrak{A}\left(\mathscr{E}^{l+1},\mathscr{E}^{l+1}\right) &\leq C_3\Gamma(1-\alpha)\left(N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\\ &\quad+C_*\Gamma(1-\alpha)(l+1)^{\alpha}\Delta t^{\alpha}\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right).\end{aligned}\tag{26}$$

By $\mathfrak{A}(\mathscr{E}^n,\mathscr{E}^n)\geq C_a\|\mathscr{E}^n\|^2_{\zeta,\varsigma}$, the mathematical induction leads to

$$\|\mathscr{E}^n\|^2_{\zeta,\varsigma}\leq\mathbf{C}_4\left(\Delta t^{4-2\alpha}+N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\left(\max_{0\leq t\leq T}\left\|{}_0^C\mathfrak{D}_t^{\alpha}\Theta(t)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}+\left\|\Theta(t_0)\right\|^2_{\mathfrak{H}^{\mathfrak{L}}(\Xi)}\right)\right).\tag{27}$$

Table 1: L_∞ errors, C.O and CPU time for K values with $T = 1$, for Example 1.

K	$\|\theta^n - \Theta_n\|_\infty$	C.O	CPU time (s)
2	7.9870e − 05	-	22.0155
3	2.0899e − 05	3.3066	45.1623
4	5.0195e − 06	4.9582	67.2784
5	1.8988e − 06	4.3564	88.9169
6	5.3896e − 07	6.9072	121.701

According to Lemma 14, we get

$$\|\mathbf{e}^n\|^2 \leq \|\mathbf{e}^n\|^2_{\zeta,\varsigma} \leq c_q(\|\mathscr{E}^n\|^2_{\zeta,\varsigma} + \|\mathscr{Z}^n\|^2_{\zeta,\varsigma}) \leq \mathbf{Q}_1\left(\Delta t^{4-2\alpha} + N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\right) + \mathbf{Q}_2 N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}. \tag{28}$$

We conclude

$$\|\mathbf{e}^n\|^2 \leq \mathbf{Q}\left(\Delta t^{4-2\alpha} + N^{2\max(\zeta,\varsigma)-2\mathfrak{L}}\right). \tag{29}$$

□

5 Numerical results

This section aims to validate the theoretical findings through numerical demonstrations implemented using MATLAB software. Our numerical investigation encompasses the numerical solution, the error measured in $\|\cdot\|_\infty$ sense, and the convergence orders (C.O), as follows

$$C.O = \frac{\log\left(\|\text{ error }_1\|/\|\text{ error }_2\|\right)}{\log\left(\kappa_1/\kappa_2\right)}, \tag{30}$$

where error $= \|\Theta^n - \theta^n\|$ is the error equation, and $\kappa_1 \neq \kappa_2$.

Example 1. Considering the problem (1) in 1D case, as follow: $\Theta(x,0) = \mathfrak{C}_\mathfrak{J}(x) = (x(x-1))^4$, $x \in \Xi$, with $H = 1$, $\Psi_{1,x} = \Psi_{2,x} = 0.5$, $\Upsilon_{1,y} = \Upsilon_{2,y} = 0$, and $\mathfrak{F}$ is defined as

$$\begin{aligned}\mathfrak{F}(x,t) = (x(x-1))^4 &\left(\frac{\Gamma(5)t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{3\Gamma(3)t^{2-\alpha}}{4\Gamma(3-\alpha)}\right)\\ &+0.5\left(t^4 + \frac{3t^2}{4}\right)\left[\sec(\pi\gamma)\mathfrak{S}_1(x,2\gamma) + \sec(\pi\zeta)\mathfrak{S}_1(x,2\zeta)\right],\end{aligned}$$

where

$$\begin{aligned}\mathfrak{S}_1(x,\xi) = (&\frac{12(x^{4-\xi} + (1-x)^{4-\xi})}{\Gamma(5-\xi)} - \frac{240(x^{5-2\xi} + (1-x)^{5-\xi})}{\Gamma(6-\xi)}\\ &+ \frac{2160(x^{6-\xi} + (1-x)^{6-\xi})}{\Gamma(7-\xi)} - \frac{10080(x^{7-\xi} + (1-x)^{7-\xi})}{\Gamma(8-\xi)} + \frac{20160(x^{8-\xi} + (1-x)^{8-\xi})}{\Gamma(9-\xi)}).\end{aligned}$$

We solve the problem (1) with $T = 1$, our focus is on the error estimations in the case of $\alpha = \gamma = 0.5$, $2\zeta = 1.2$, by considering the exact solution $\Theta(x,t) = \left(t^4 + \frac{3t^2}{4}\right)(x(1-x))^4$. Figure 1 presents the

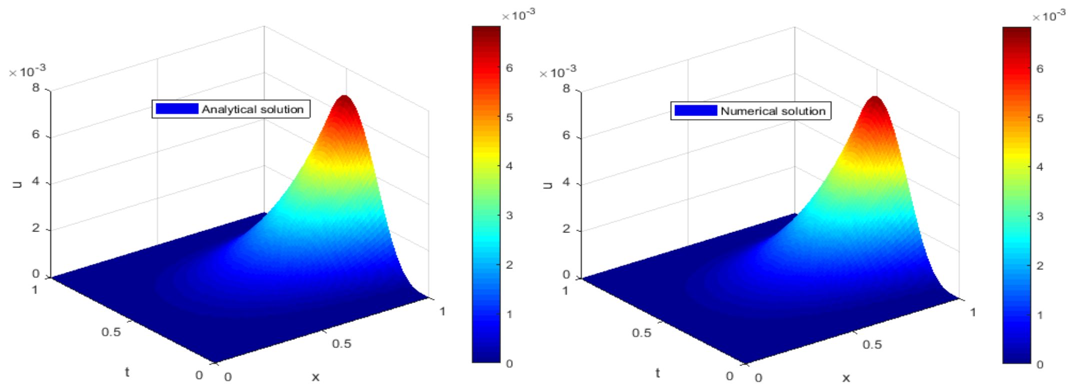

Figure 1: Analytical and numerical solution, with $N = 6$, $K = 8$ and $\Delta T = 2e-3$, for T=1.

Table 2: L_∞ errors, C.O and CPU time for Δt values with $T = 1$ for Example 1.

Δt	$\|\theta^n - \Theta_n\|_\infty$	C.O	CPU time (s)
$1/100$	$3.6670e-04$	-	11.4103
$1/144$	$2.1852e-04$	1.4197	18.3197
$1/256$	$9.9293e-05$	1.3710	26.1170
$1/400$	$4.9316e-05$	1.5681	43.2582
$1/576$	$2.8834e-05$	1.4718	64.9263

Table 3: L_∞ errors, C.O and CPU time for N values with $T = 1$ for Example 1.

N	$\|\theta^n - \Theta_n\|_\infty$	C.O	CPU time (s)
2	$3.2905e-04$	-	21.1972
3	$8.2995e-05$	1.9872	34.2031
4	$1.7347e-05$	3.8607	51.8751
5	$3.0098e-06$	6.0885	74.0185
6	$7.2143e-07$	6.4012	114.152

numerical solution with $N = 6$, $K = 8$, and $\Delta T = 2e-3$, in the right panel. Additionally, the analytical solution was displayed in the left panel. Table 1 shows the L_∞ errors, the C.O, and CPU time, for different values of K, where $N = 6$ and Δt= 2e-3. For $N = K = 8$, the Table 2 presents the L_∞ errors, the C.O, and CPU time, for different values of Δt. At the end, we take K=8 and Δt= 2e-3, for different values of N, the Table 3 gives the L_∞ errors, the C.O, and CPU time.

Example 2. Consider problem (1) as a two-dimensional/two-sided fractional diffusion equation: $\Theta(x,y,0) = \mathfrak{C}_\mathfrak{I}(x,y) = (xy(x-1)(y-1))^{10}$, $(x,y) \in \Xi$, with $H = L = 1$, $\Psi_{1,x} = \Upsilon_{1,y} = 0.5$, $\Psi_{2,x} = \Upsilon_{2,y} = 0.75$, and $\mathfrak{F}$ is defined as

$$\begin{aligned}\mathfrak{F}(x,y,t) = (xy(x-1)(y-1))^{10} \frac{\Gamma(\alpha+\gamma+\beta+\zeta+\varsigma+1) t^{\gamma+\beta+\zeta+\varsigma}}{\Gamma(\gamma+\beta+\zeta+\varsigma+1)} \\ +0.5t^{\alpha+\gamma+\beta+\zeta+\varsigma} \left[\sec(\pi\gamma)\mathfrak{S}_2(x,y,2\gamma) + \sec(\pi\beta)\mathfrak{S}_2(y,x,2\beta)\right] \\ +0.75t^{\alpha+\gamma+\beta+\zeta+\varsigma} \left[\sec(\pi\zeta)\mathfrak{S}_2(x,y,2\zeta) + \sec(\pi\varsigma)\mathfrak{S}_2(y,x,2\varsigma)\right],\end{aligned}$$

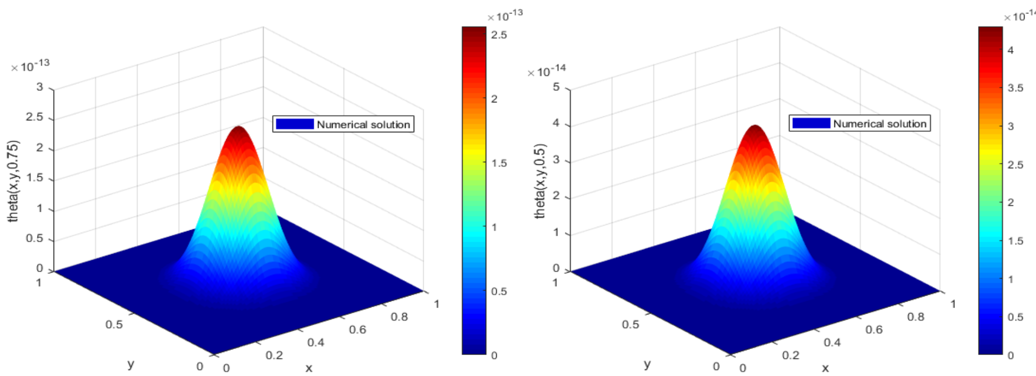

Figure 2: Numerical solution at time t =0.5, 0.75, with $T = 1$.

where

$$\begin{aligned}\mathfrak{S}_2(x,y,\xi) = \Bigg\{ &\frac{\Gamma(11)}{\Gamma(11-\xi)}\left[x^{10-\xi}+(1-x)^{10-\xi}\right] - \frac{10\Gamma(12)}{\Gamma(12-\xi)}\left[x^{11-\xi}+(1-x)^{11-\xi}\right] \\ &+ \frac{45\Gamma(13)}{\Gamma(13-\xi)}\left[x^{12-\xi}+(1-x)^{12-\xi}\right] - \frac{120\Gamma(14)}{\Gamma(14-\xi)}\left[x^{13-\xi}+(1-x)^{13-\xi}\right] \\ &+ \frac{210\Gamma(15)}{\Gamma(15-\xi)}\left[x^{14-\xi}+(1-x)^{14-\xi}\right] - \frac{252\Gamma(16)}{\Gamma(16-\xi)}\left[x^{15-\xi}+(1-x)^{15-\xi}\right] \\ &+ \frac{210\Gamma(17)}{\Gamma(17-\xi)}\left[x^{16-\xi}+(1-x)^{16-\xi}\right] - \frac{120\Gamma(18)}{\Gamma(18-\xi)}\left[x^{17-\xi}+(1-x)^{17-\xi}\right] \\ &+ \frac{45\Gamma(19)}{\Gamma(19-\xi)}\left[x^{18-\xi}+(1-x)^{18-\xi}\right] - \frac{10\Gamma(20)}{\Gamma(20-\xi)}\left[x^{19-\xi}+(1-x)^{19-\xi}\right] \\ &+ \frac{\Gamma(21)}{\Gamma(21-\xi)}\left[x^{20-\xi}+(1-x)^{20-\xi}\right] \Bigg\} y^{10}(y-1)^{10}.\end{aligned}$$

We solve the problem (1) with $T = 1$. Figure 2 shows the graphs of numerical solution for $N = 6$, Δt= 2e-4, and $K = 16$ at $t = 0.5, 0.75$. Further, by considering the analytical solution given by $\Theta(x,y,t) = t^{\alpha+\gamma+\beta+\zeta+\varsigma}(xy(1-x)(1-y))^{10}$, Figure 3 presents the numerical error with different values of N (left graph with $K = 8$ and $\Delta T = 2e-3$) and different values of K (right graph with $N = 6$ and $\Delta T = 2e-3$), in three different cases.

6 Conclusions

This study explores a numerical spectral method for a two-sided fractional diffusion equation. The Crank-Nicolson method of order $\mathscr{O}\left(\Delta t^{2-\alpha}\right)$ was applied to approximate the Caputo fractional derivative operator ${}_0^C\mathfrak{D}_t^{\alpha}$, the unconditional stability was shown of the time-discrete scheme. Next, the spectral element method was employed for the Riesz fractional derivative operators to derive a fully discrete scheme. Further, the error estimate was analyzed, and the numerical method was proven to be convergent, where $\mathscr{O}\left(\Delta t^{2-\alpha} + N^{\max(\zeta,\varsigma)-\mathfrak{L}}\right)$ is the order of accuracy. We solved two numerical problems and simulated them using MATLAB software, to demonstrate the efficiency of the method.

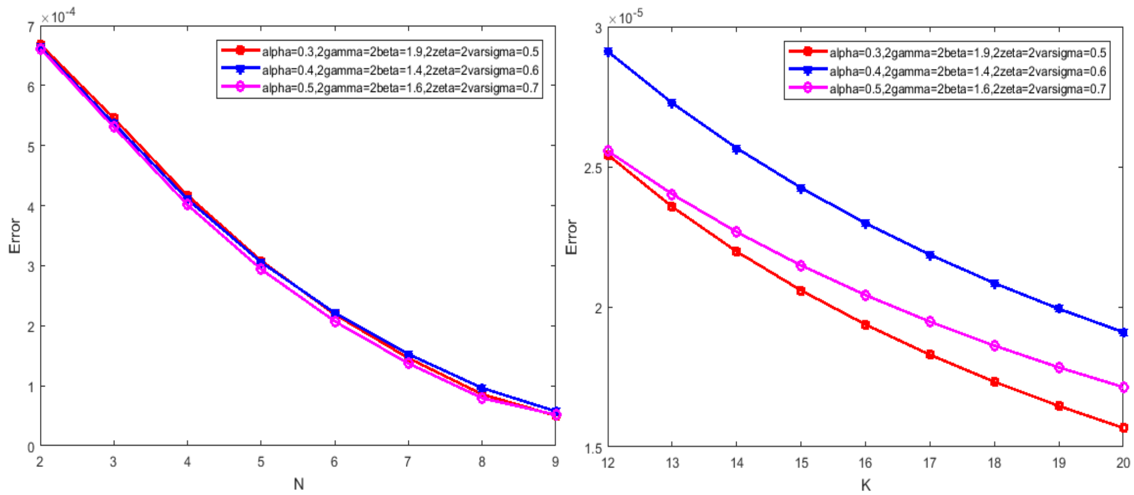

Figure 3: L_∞ Errors for different values of N, and K, with $T = 1$.

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