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Exponential decay for a general class of nonautonomous abstract semilinear evolution equations with time-varying delay feedback

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Abstract. In this paper, we consider a general class of nonautonomous abstract delayed evolution equations with a nonlinear source term. Under appropriate assumptions on the time-independent operator and the initial data, we establish global existence using the method of steps and employing classical results from the theory of inhomogeneous evolution problems. Then, by assuming that the operator associated with the non-delayed part of the system generates an exponentially stable semigroup, we obtain an exponential decay estimate. This is achieved through a direct proof based on Duhamel's formula combined with Gronwall's inequality, under Lipschitz continuity conditions on the nonlinear source term. Finally, we conclude the paper by providing illustrative examples that validate the generalized setting of our system.

Keywords: Duhamel's formula, energy function, evolutionary family, Lipschitz continuous.
AMS Subject Classification 2010: 35B35, 34K20, 37L10, 93C20.

1 Introduction

This paper addresses a class of nonautonomous evolution equations with time-varying delays and a nonlinear source term

$$\begin{cases} U_t(t) = A(t)U(t) + \alpha(t)BU(t-\tau(t)) + F(U(t)), & t \in (0,+\infty), \\ BU(t) = f(t) & t \in [-\tau(0),0], \\ U(0) = U_0, \end{cases} \tag{1}$$

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where $A(t)$ be the infinitesimal generator of C_0 semigroup $\{S_t(s)\}_{s\geq 0}$ on $\mathscr{H}$ and $B : \mathscr{H} \longrightarrow \mathscr{H}$ is a continuous linear operator. $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a function from the space L^∞, for all $t \geq 0$. The function of time delay $\tau : \mathbb{R}_+ \to (0,+\infty)$ belonging to $W^{2,\infty}([0,T])$, for all $T > 0$, moreover, we suppose there exist two positive constants τ_1, τ_2 and $d < 1$ such that

$$0 < \tau_1 \leq \tau(t) \leq \tau_2, \tag{2}$$

and

$$\tau'(t) \leq d < 1, \tag{3}$$

for all $t > 0$. The initial datum (U_0, f) belongs to $\mathscr{H}$ and $C([-\tau(0),0];\mathscr{H})$, respectively. The nonlinear function F defined from $\mathscr{H}$ into itself satisfies some Lipschitz continuity assumption.

The study of evolution equations with delays has attracted significant attention in recent years due to its applications in various fields such as control theory, population dynamics, and neural networks. The inclusion of delays in the evolution of the system introduces significant challenges in terms of stability, well-posedness, and the long-term behavior of solutions, see for example, [3,10,22]. In the autonomous case, where the operator A is not time-dependent, many papers have focused on global existence and asymptotic behavior of solutions in the presence of memory terms. Various estimates of decay have been established depends on the growth of memory kernel at infinity. For similar results, we refer to [5,6,27,30] and the references therein.

On the other hand, works such as [19] studied the exponential stability of solutions under a suitable condition between the coefficient of the delay and undelayed terms, for the following problem, for all $x \in \Omega$ and $t > 0$,

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\, ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0,$$

where $\Omega \subset \mathbb{R}^n$ is a regular domain, Δ denotes the Laplacian operator, g be a given function representing the kernel of memory term defined on $\mathbb{R}_+$ into itself, τ represents the time delay, μ_1 is a positive constant and μ_2 is a real number. A similar result is obtained in [15] by considering an abstract operators and setting $\mu_1 = 0$. The exponential stability in the presence of delay is proven by showing that the memory term is strong enough to stabilize the system, which is the unique dissipation. Related works, such as [7,8,11,14,19,20,23–25], investigate the stability of delayed evolution equations, both in abstract and non-abstract settings, for cases involving delays, whether constant, time-varying or distributed delay. In these studies, the authors demonstrated that, under appropriate conditions on the relation between the coefficients of the delayed and undelayed dissipative terms, stability can be guaranteed.

In the nonautonomous case of systems where the operators A is time-dependent, this type of equations has a practical significance in various fields: biological, mechanical, engineering and economic systems. The earlier papers have interested by evolution equations, evolution families and their applications. An evolutionary family is defined by a two-parameter family $\{\mathscr{U}(t,s)\}_{0\leq s\leq t}$ of bounded linear operators on a Banach space $\mathscr{H}$. It is called an evolution family (or strongly continuous evolution), if it satisfies the following conditions

1. $\mathscr{U}(t,t) = 0$, for all $t \geq 0$.
2. $\mathscr{U}(t,s)\mathscr{U}(s,r) = \mathscr{U}(t,r)$, for all $0 \leq r \leq s \leq t$.

3. The map $(t,s) \to \mathscr{U}(t,s)x$ is continuous for every $x \in \mathscr{H}$.

The solution of the linear part of system (1) without delay ($\alpha = 0$ and $F = 0$) is given by

$$U(t) = \mathscr{U}(t,s)v, \quad \text{for all} \quad 0 \le s \le t \le T \quad \text{with} \quad U(s) = v.$$

For further details, the reader is referred to [2, 9, 17, 29]. Since then, many papers have been appeared, see [4, 12, 16, 21, 26]. In [1], the authors abstained the global existence and the exponential stability for the following nonautonomous second-order hyperbolic equations with infinite memory

$$u_{tt}(t) + A(t)u(t) - \int_0^\infty g(s)B(t)u(t-s)\,ds = 0, \quad \forall t > 0, \tag{4}$$

where $A(t) : D(A(t)) \subset H \to H$ and $B(t) : D(B(t)) \subset H \to H$, be linear operators with H an Hilbert space. For non abstract setting, the same results have been obtained in [13] for constant delayed viscoelastic wave equations under some appropriate conditions between the coefficients of dissipative terms.

In this paper, we extend these results to a more general setting by considering a class of nonautonomous evolution equations with both time-varying delays and nonlinear source terms. We assume that the operator associated with the non-delayed part of the system generates an exponentially stable semigroup, and we use classical results from the theory of inhomogeneous evolution problems to prove the well-posedness of system (1). Furthermore, we establish an exponential decay estimate for the solution by applying Duhamel's formula combined with Gronwall's inequality, under the assumption of Lipschitz continuity for the nonlinear source term. This provides a comprehensive framework that generalizes the previous works [13, 15, 19, 23–25, 28, 30] for studying the long-term behavior of solutions to delayed evolution equations.

The structure of our paper is outlined in the coming manner. In Section 2, we introduce the assumptions on the given data. We then demonstrate the global existence of the solution using the theory of semi-group, and the stability result is established through a direct proof using Duhamel's formula; these results are presented in Section 3. Finally, Section 4 is dedicated to presenting some practical applications.

2 Assumptions

In order to derive the main results and ensure their validity, we begin by stating the key assumptions underlying our framework. These assumptions are stated below

(H_1) The domain of A is independent on time.

(H_2) For all $t \in [0,T]$, $A(t)$ generates a strongly continuous semigroup on $\mathscr{H}$, and the family $A = \{A(t) : t \in [0,T]\}$ is stable with stability constants c and m independent of t.

(H_3) $\partial_t A$ belongs to $L^\infty_*([0,T], \mathscr{B}(Y,\mathscr{H}))$ the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $\mathscr{B}(Y,\mathscr{H})$ of bounded operators from Y into $\mathscr{H}$.

(H_4) The source term F is globally Lipschitz continuous, meaning that there exists a positive constant η, such that

$$\|F(U) - F(V)\| \le \eta \|U - V\|, \quad \forall U, V \in \mathscr{H}. \tag{5}$$

In addition, we suppose that $F(0) = 0$.

In the coming theorem, we state a classical result concerning the nonautomonous case of systems.

Theorem 1. *[2, 29] Let $A(t)$, $0 \leq t \leq T$, be the infinitesimal generator of C_0 semigroup $\{S_t(s)\}_{s\geq 0}$ on $\mathscr{H}$. If $A(t)$ satisfies the conditions (H_1), (H_2) and (H_3), for all $0 \leq t \leq T$, then there exists a unique evolution system $\{\mathscr{U}(t,s)\}$, $0 \leq s \leq t \leq T$.*

Remark 1. *1. We can drop the assumptions (H_1)-(H_3) and instead assume that there exists a unique evolution system$\{\mathscr{U}(t,s)\}$, $0 \leq s \leq t \leq T$ generated by $A(t)$. Note that those assumptions are very classical, see [29], this as enriching our abstract and general setting.*
2. Under (H_1)-(H_3), The triplet $\{A, \mathscr{H}, Y\}$, where $A = \{A(t),\ t \in [0,T]\}$ for some T fixed and $Y = D(A(0))$, forms a constant domain system, which leads to some existence and uniqueness results, see [17, 18].

We denote the growth bound

$$\omega(\mathscr{U}) = \inf\{\omega : \ \exists M = M(\omega) \geq 1 \text{ such that } \|\mathscr{U}(t,s)\|_{\mathscr{L}(\mathscr{H})} \leq Me^{-\omega(t-s)} \text{ for } \ t \geq s\}.$$

Definition 1. *The evolution family $\{\mathscr{U}(t,s)\}_{0\leq s\leq t}$ is (uniformly) exponentially stable if the growth bound $\omega(\mathscr{U})$ is positive.*

For more details, we refer to [2, 9, 29]. The following assumption is crucial in our setting, we assume that the linear and non-delayed part of the system (1) described by the evolution operator $A(t)$ in equation (1) is exponentially stable.

(H_5) The evolutionary family $\{\mathscr{U}(t,s)\}_{0\leq s\leq t}$ generated by $A(t)$ is (uniformly) exponentially stable, i.e., there exist positive constants M and ω such that

$$\|\mathscr{U}(t,s)\|_{\mathscr{L}(\mathscr{H})} \leq Me^{-\omega(t-s)}, \quad 0 \leq s \leq t. \tag{6}$$

3 Main results

This section is devoted to stating and proving the global existence and exponential decay results for the solution of the system (1), by applying classical results from the theory of evolution equations based on semigroup theory.

3.1 Well-posedness

The existence of the solution of the system (1) is given by the following theorem.

Theorem 2. *Given $U_0 \in \mathscr{H}$ and a continuous function $f : [-\tau(0), 0] \to \mathscr{H}$, the problem (1) has a unique (weak) solution $U(t) \in C([0,\infty); \mathscr{H})$, which is expressed by Duhamel's formula, for all $t \geq 0$*

$$U(t) = \mathscr{U}(t,0)U_0 + \int_0^t \mathscr{U}(t,s)\alpha(s)[BU(s-\tau(s)) + F(U(s))]ds. \tag{7}$$

Proof. We aim to establish the existence and uniqueness of a continuous solution U for the system (1) over the entire time domain $[0,+\infty)$. To do so, we divide the time domain into intervals of length τ_1 and analyze each interval iteratively.

Firstly, let us partition the time domain $[0,+\infty)$ into intervals of length τ_1, i.e.,

$$[0,+\infty) = \bigcup_{k=0}^{\infty} [k\tau_1,(k+1)\tau_1].$$

We will show that a solution exists on each subinterval $[k\tau_1,(k+1)\tau_1]$, and use the solution on one interval to construct the solution on the next interval.

Let justify the choice of the length τ_1, we note that for $t \geq 0$

$$t-\tau(t) \leq t-\tau_1,$$

which ensures that for any $t \in [k\tau_1,(k+1)\tau_1]$, we have

$$t-\tau(t) \leq k\tau_1.$$

Furthermore, by differentiating $t-\tau(t)$ with respect to t, we get

$$(t-\tau(t))' = 1-\tau'(t) > 0, \quad \text{a.e.}, \quad \forall t \geq 0,$$

which implies that

$$t-\tau(t) \geq -\tau(0), \quad \forall t \geq 0.$$

On other hand, we have

$$t-\tau(t) \leq t-\tau_1, \quad \forall t \geq 0.$$

Therefore, if $t \in [k\tau_1,(k+1)\tau_1]$, we deduce

$$t-\tau(t) \leq k\tau_1.$$

Let's first consider the interval $[0,\tau_1]$. For $t \in [0,\tau_1]$, we define the function

$$G_1(t) = \alpha(t)BU(t-\tau(t)),$$

where $U(t-\tau(t))$ is evaluated at times in the past. For $k=0$, we have

$$-\tau(0) \leq t-\tau(t) \leq 0,$$

and so, the term $G_1(t) = \alpha(t)f(t)$, where $f(t)$ is given in the problem statement.

Next, we rewrite the system (1) as a standard non-homogeneous problem, which is given by the following system

$$\begin{cases} U'(t) = A(t)U(t)+G_1(t)+F(U(t)), & \forall t \in (0,\tau_1), \\ U(0) = U_0. \end{cases} \tag{8}$$

According to Theorem 1, there exists a unique evolution system $\mathscr{U}(t,s)$, $0 \leq s \leq t \leq T$ under the assumptions are verified for all $T>0$. Note that $G_1 \in L^1((0,\tau);\mathscr{H})$, since $k \in L^\infty([0,\infty);\mathbb{R})$ and $f \in C([-\tau(0),0];\mathscr{H})$, and under the hypothesis (5), the system (8) admit a unique solution $U \in C([0,\tau_1];\mathscr{H})$ on the interval $[0,\tau_1]$, based on the results of Theorems 5.3,1.2 , 2.1 in [29]. This solution satisfies the Duhamel's formula

$$U(t) = \mathscr{U}(t,0)U_0 + \int_0^t \mathscr{U}(t,s)[G_1(t)+F(U(s))]ds, \quad \forall t \in [0,\tau_1],$$

and then, it gives

$$U(t) = \mathscr{U}(t,0)U_0 + \int_0^t \mathscr{U}(t,s)\alpha(s)[BU(s-\tau(s)) + F(U(s))]ds, \quad \forall t \in [0,\tau_1]. \tag{9}$$

Now, for the next interval $[\tau_1, 2\tau_1]$, we define

$$G_2(t) = \alpha(t)BU(t-\tau(t)),$$

where $U(t-\tau(t))$ is given by the solution on the previous interval $[0,\tau_1]$. Therefore, the system on $[\tau_1, 2\tau_1]$ is

$$\begin{cases} U'(t) = A(t)U(t) + G_2(t) + F(U(t)) & \text{in} \quad (\tau_1, 2\tau_1), \\ U(\tau_1) = U_{\tau_1}, \end{cases} \tag{10}$$

The value of U_{τ_1} is determined from the solution on the previous interval $[0,\tau_1]$, and it satisfies

$$U_{\tau_1} = \mathscr{U}(\tau_1,0)U_0 + \int_0^{\tau_1} \mathscr{U}(\tau_1,s)\alpha(s)\left[BU(s-\tau(s)) + F(U(s))\right] ds.$$

By the standard theory for Cauchy problems and applying the same arguments, the system on $[\tau_1, 2\tau_1]$ admits a unique solution $U \in C([\tau_1, 2\tau_1]; \mathscr{H})$, and we can apply Duhamel's formula again

$$U(t) = \mathscr{U}(t,\tau_1)U_{\tau_1} + \int_{\tau_1}^t \mathscr{U}(t,s)\left[G_2(s) + F(U(s))\right] ds, \quad t \in [\tau_1, 2\tau_1].$$

Thus, we obtain

$$U(t) = \mathscr{U}(t,\tau_1)U_{\tau_1} + \int_{\tau_1}^t \mathscr{U}(t,s)\alpha(s)\left[BU(s-\tau(s)) + F(U(s))\right] ds, \quad t \in [\tau_1, 2\tau_1].$$

By iterating this process over subsequent intervals, we construct a continuous solution $U(t)$ on the entire time domain $[0,\infty)$. Specifically, for any $t \in [0,(k+1)\tau_1]$, we can express $U(t)$ in the form

$$U(t) = \mathscr{U}(t,0)U_0 + \int_0^t \mathscr{U}(t,s)\alpha(s)\left[BU(s-\tau(s)) + F(U(s))\right] ds.$$

By induction, this gives a unique solution $U(t) \in C([0,\infty); \mathscr{H})$ to the problem expressed by (7), completing the proof. □

3.2 Exponential stability

The next theorem proves that the solution to the problem (1) is exponentially stable under a suitable condition on the system's parameters. This stability result is established based on the exponentially stable semigroup of the non-delayed part of the system.

Theorem 3. *Assume that, for all $t \geq 0$,*

$$M\left(\|\alpha\|_\infty \|B\| \frac{e^{\omega\tau_2}}{1-d} + \eta\right) < \omega. \tag{11}$$

Hence, there exist two positive constants M' and ω' such that the solution of (1) is exponentially stable, namely

$$\|U(t)\| \leq M' e^{-\omega' t}, \quad \forall \quad t \geq 0. \tag{12}$$

Proof. The proof starts by expressing the solution $U(t)$ using Duhamel's formula and then analyzing the bounds of each term. From the formula for the solution, the estimate (7), it follows that

$$\|U(t)\| \leq \|\mathscr{U}(t,0)U_0\| + \left\|\int_0^t \mathscr{U}(t,s)\alpha(s)BU(s-\tau(s))\,ds\right\| + \left\|\int_0^t \mathscr{U}(t,s)F(U(s))ds\right\|.$$

By using (6), it results in

$$\begin{aligned}\|U(t)\| &\leq Me^{-\omega t}\|U_0\| + \int_0^t Me^{-\omega(t-s)}\|\alpha(s)BU(s-\tau(s))\|\,ds + \int_0^t Me^{-\omega(t-s)}\|F(U(s))\|\,ds,\\ &\leq Me^{-\omega t}\|U_0\| + Me^{-\omega t}\int_0^t e^{\omega s}\|\alpha(s)BU(s-\tau(s))\|\,ds + Me^{-\omega t}\int_0^t e^{\omega s}\|F(U(s))\|\,ds,\end{aligned}$$

So, by using the fact that F is a globally Lipschitz continuous function with $F(0)=0$ and $\alpha \in L^\infty([0,\infty);\mathbb{R})$, we deduce

$$e^{\omega t}\|U(t)\| \leq M\|U_0\| + M\|\alpha\|_\infty \int_0^t e^{\omega s}\|BU(s-\tau(s))\|\,ds + M\eta\int_0^t e^{\omega s}\|U(s)\|\,ds.$$

Then,

$$e^{\omega t}\|U(t)\| \leq M\|U_0\| + M\|\alpha\|_\infty \int_0^t e^{\omega(s+\tau(s)-\tau(s))}\|BU(s-\tau(s))\|\,ds + M\eta\int_0^t e^{\omega s}\|U(s)\|\,ds,$$

and by using (2), we get

$$e^{\omega t}\|U(t)\| M\|U_0\| + M\|\alpha\|_\infty e^{\omega\tau_2}\int_0^t e^{\omega(s-\tau(s))}\|BU(s-\tau(s))\|\,ds + M\eta\int_0^t e^{\omega s}\|U(s)\|\,ds. \tag{13}$$

Now, using the change of variable $\sigma = s-\tau(s)$, and the fact that $\tau'(t)\leq d$, we obtain

$$\begin{aligned}\int_0^t e^{\omega(s-\tau(s))}\|BU(s-\tau(s))\|\,ds &= \int_{-\tau(0)}^{t-\tau(t)} e^{\omega\sigma}\|BU(\sigma)\|\frac{d\sigma}{1-\tau'(s)}\\ &\leq \frac{1}{1-d}\int_{-\tau(0)}^{t-\tau(t)} e^{\omega\sigma}\|BU(\sigma)\|\,d\sigma.\end{aligned} \tag{14}$$

By inserting (14) in (13), we arrive at

$$\begin{aligned}e^{\omega t}\|U(t)\| &\leq M\|U_0\| + M\|\alpha\|_\infty\frac{e^{\omega\tau_2}}{1-d}\int_{-\tau(0)}^{t-\tau(t)} e^{\omega s}\|BU(s)\|\,ds\\ &\quad + M\eta\int_0^t e^{\omega s}\|U(s)\|\,ds,\\ &\leq M\|U_0\| + M\|\alpha\|_\infty\frac{e^{\omega\tau_2}}{1-d}\int_{-\tau(0)}^{0} e^{\omega s}\|BU(s)\|\,ds\\ &\quad + M\|\alpha\|_\infty\frac{e^{\omega\tau_2}}{1-d}\int_0^{t-\tau(t)} e^{\omega s}\|BU(s)\|\,ds + M\eta\int_0^t e^{\omega s}\|U(s)\|\,ds,\\ &\leq M\|U_0\| + M\|\alpha\|_\infty\frac{e^{\omega\tau_2}}{1-d}\int_{-\tau(0)}^{0} e^{\omega s}\|f(s)\|\,ds\\ &\quad + M\|\alpha\|_\infty\frac{e^{\omega\tau_2}}{1-d}\int_0^{t} e^{\omega s}\|BU(s)\|\,ds + M\eta\int_0^t e^{\omega s}\|U(s)\|\,ds.\end{aligned}$$

By using the fact that B is continuous, we conclude

$$\begin{aligned} e^{\omega t}\|U(t)\| \leq\ & M\|U_0\| + M\|\alpha\|_\infty \frac{e^{\omega\tau_2}}{1-d}\int_{-\tau(0)}^{0} e^{\omega s}\|f(s)\|\,ds \\ & + M\left(\|\alpha\|_\infty \|B\| \frac{e^{\omega\tau_2}}{1-d} + \eta\right)\int_0^t e^{\omega s}\|U(s)\|\,ds. \end{aligned} \tag{15}$$

At this position, for all $t \geq 0$, define $u(t) := e^{\omega t}\|U(t)\|$. Then, we have the inequality

$$u(t) \leq \alpha + \int_0^t \beta(s)u(s)ds, \quad \forall t \geq 0$$

where

$$\alpha := M\left(\|U_0\| + \|\alpha\|_\infty \frac{e^{\omega\tau_2}}{1-d}\int_{-\tau(0)}^{0} e^{\omega s}\|f(s)\|\,ds\right)$$

and

$$\beta(s) = M\left(\|\alpha\|_\infty \|B\| \frac{e^{\omega\tau_2}}{1-d} + \eta\right).$$

As a result, the inequality (15) takes the form

$$u(t) \leq \alpha + \int_0^t \beta(s)u(s)ds, \quad \forall t \geq 0.$$

By applying the Gronwall inequality, we come to

$$u(t) \leq \alpha e^{\beta t},$$

for all $t \geq 0$. Thus,

$$\|U(t)\| \leq \alpha e^{(\beta-\omega)t} \quad \forall t \geq 0,$$

Finally, by considering the condition (11), we conclude that $\omega' = \omega - \beta$ is positive, ensuring the exponential stability of the solution, where

$$M' = M\left(\|U_0\| + \|\alpha\|_\infty \frac{e^{\omega\tau_2}}{1-d}\int_{-\tau(0)}^{0} e^{\omega s}\|f(s)\|\,ds\right),$$

and

$$\omega' = \omega - M\left(\|\alpha\|_\infty \|B\| \frac{e^{\omega\tau_2}}{1-d} + \eta\right).$$

This completes the proof. □

4 Applications

In this section, we apply the theoretical results presented in the previous section to practical problems: the study of a nonautonomous delayed wave equation with a variable delay in time and a nonlinear source term and other classes.

4.1 Nonautonomous delayed wave equation

In this subsection, we consider a regular domain Ω of $\mathbb{R}^n$ and the Hilbert space $H = L^2(\Omega)$ where $\langle .,. \rangle$ and $\|.\|$ are the inner product and norm associated with it, respectively. Moreover, we define the space V by $V = H^2(\Omega) \cap H_0^1(\Omega)$.

Let us consider the following delayed wave equation

$$\begin{cases} u_{tt}(x,t) - b(t)\Delta u(x,t) + CC^* u_t(x,t) = \alpha(t) D u_t(x,t-\tau(t)) + \psi(u(x,t)), \quad x \in \Omega \quad t > 0. \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \\ Du_t(t) = f_0(t) \quad t \in [-\tau(0), 0], \end{cases} \tag{16}$$

where

- b is a given function belongs to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}_+^*)$ and Δ design the Laplacian operator;
- C and D are linear bounded operators from H into itself;
- The time-varying delay function $\tau : \mathbb{R}_+ \to \mathbb{R}_+^*$ belonging to $W^{2,\infty}([0,T])$, for all $T > 0$ and satisfies the inequalities (2) and (3);
- The nonlinear function $\psi : H \to H$ is globally Lipschitz continuous with $\psi(0) = 0$;
- The initial datum (u_0, u_1, f_0) belongs to $V \times H \times C([-\tau(0), 0]; H)$.

Remark 2. *1. For $U = (u, u_t)^T$, the system (16) can be reformulated in the more abstract form of equation (1), where the Hilbert space $\mathscr{H} = V \times H$ and the operator $A(t)$ is defined by*

$$A(t) = \begin{pmatrix} 0 & 1 \\ b(t)\Delta & -CC^* \end{pmatrix}, \tag{17}$$

with domain

$$D(A(t)) = \{(\varphi_1, \varphi_2) \in \mathscr{H},\ b(t)\varphi_1 \in V\}. \tag{18}$$

while F and B are defined by

$$B(U(t)) = \begin{pmatrix} 0 \\ Du_t(t) \end{pmatrix} \qquad \text{and} \qquad F(U(t)) = \begin{pmatrix} 0 \\ \psi(u(t)) \end{pmatrix}$$

2. The operator B is continuous and linear, and the nonlinear function F is globally Lipschitz continuous on $\mathscr{H}$ with the Lipschitz constant η, based on the assumptions previously made on D and ψ.
3. For getting the desired results, we introduce the following time-dependent inner product on $\mathscr{H}$

$$\langle \varphi, w \rangle = b(t)\langle \nabla\varphi_1, \nabla w_1 \rangle + \langle \varphi_2, w_2 \rangle,$$

for $\varphi = (\varphi_1, \varphi_2)^T$ and $w = (w_1, w_2)^T$.

Proposition 1. *The time independent operator $A(t)$ given by (17) satisfies the hypothesis (H_1), (H_2) and (H_3), for $t \in [0, T]$ with $T > 0$.*

Proof. Let us check if the hypothesis (H_1), (H_2) and (H_3) are satisfied.

Firstly, according to (17), the domain of $A(t)$ is independent on time, which means $D(A(t)) = D(A(0))$.

Secondly, the linear operator $A(t)$ generates a linear C_0-semi-group on $\mathscr{H}$. Let us start by proving that $A(t)$ is dissipative. For all $\varphi = (\varphi_1, \varphi_2)^T \in D(A(t))$, we have

$$\langle A(t)\varphi, \varphi\rangle = b(t)\,\langle \nabla\varphi_2, \nabla\varphi_1\rangle + b(t)\,\langle \Delta\varphi_1, \varphi_2\rangle - \langle CC^*\varphi_2, \varphi_2\rangle,$$

and using the Green's formula we arrive at

$$\langle \Delta\varphi_1, \varphi_2\rangle = -\,\langle \nabla\varphi_1, \nabla\varphi_2\rangle\,.$$

Consequently, for all $t \geq 0$, it result in

$$\langle A(t)\varphi, \varphi\rangle \leq -\|C^*\varphi_2\|^2. \tag{19}$$

So, $A(t)$ is dissipative operator. Next, we show that $\lambda I - A(t)$ is surjective, for $\lambda > 0$. Indeed, for $k = (k_1, k_2) \in \mathscr{H}$, there exists $\varphi = (\varphi_1, \varphi_2) \in D(A(t))$ that satisfies

$$(\lambda I - A(t))\,\varphi = k. \tag{20}$$

which is equivalent to

$$\begin{cases} \lambda\varphi_1 - \varphi_2 = k_1, \\ \lambda\varphi_2 - a(t)\Delta\varphi_1 + CC^*\varphi_2 = k_2. \end{cases} \tag{21}$$

The first equation of (21) gives

$$\varphi_2 = \lambda\varphi_1 - k_1, \tag{22}$$

and by substituting in the second equation of (21), we get

$$\lambda^2\varphi_1 - a(t)\Delta\varphi_1 + \lambda CC^*\varphi_1 = k_2 + \lambda k_1 + CC^*k_1. \tag{23}$$

Thus, we need to solve the equation (23). Indeed, we take the duality brackets $\langle .,.\rangle_{V',V}$, with $\phi \in V$:

$$\left\langle \lambda^2\varphi_1 - b(t)\Delta\varphi_1 + \lambda CC^*\varphi_1, \phi\right\rangle_{V',V} = \left\langle \tilde{k}, \phi\right\rangle_{V',V}$$

where $\tilde{k} = k_2 + k_1 + CC^*k_1$. By using Green's formula, we get

$$\left\langle \lambda^2\varphi_1, \phi\right\rangle_{V',V} + \langle b(t)\nabla\varphi_1, \nabla\phi\rangle_{V',V} + +\lambda\,\langle C^*\varphi_1, C^*\phi\rangle_{V',V} = \left\langle \tilde{k}, \varphi\right\rangle_{V',V}. \tag{24}$$

The left hand of (24) is bilinear, coercive and continuous form, then

$$\left\langle \lambda^2\varphi_1 - b(t)\Delta\varphi_1 + \lambda CC^*\varphi_1, \phi\right\rangle_{V',V} \leq c\|\varphi_1\|_{V'}\|\phi\|_V.$$

For $\phi = \varphi_1 \in V$,

$$\left|\left\langle \lambda^2\varphi_1 - b(t)\Delta\varphi_1 + \lambda CC^*\varphi_1, \varphi_1\right\rangle_V\right| \geq \lambda^2\,|\langle\varphi_1, \varphi_1\rangle_V| = \lambda^2\|\varphi_1\|_V^2.$$

Applying the Lax-Milgram's theorem, we know that the equation (23) has a unique solution for $\varphi_1 \in V$. Once φ_1 is determined, we can find φ_2 from the equation (22). Thus, $\lambda I - A(t)$ is surjective.

Finally, (19) and (20) affirm that $A(t)$ is maximal monotone operator. Then, using Lummer-Phillips's theorem ([29], Theorem I.4.6), we deduce that $A(t)$ generates a C_0-semi-group of contraction on $\mathscr{H}$.

Lastly, let $\varphi = (\varphi_1, \varphi_2)^T \in D(A(0))$. Then

$$\frac{d}{dt}A(t)\varphi = \begin{pmatrix} 0 \\ b'(t)\Delta\varphi_1 \end{pmatrix}$$

Since $b \in W^{1,\infty}(\mathbb{R}_+)$, we know that b' is essentially bounded. Therefore, the operator $b'(t)\Delta$ acting on φ_1 is also bounded, and so

$$\frac{d}{dt}A(t)\varphi \in L^\infty([0,T], \mathscr{B}(D(A(0)), \mathscr{H}).$$

Consequently, the proof of Proposition 1 is completed. Thus, by using Theorem 1, we assure the existence of a unique evolution system $\{\mathscr{U}(t,s)\}$, $0 \le s \le t \le T$ with $T > 0$. □

Proposition 2. *Let b satisfies*

$$\exists\theta > 0 \quad \textit{such that} \quad b'(t) \le -\theta b(t), \quad \forall t \ge 0. \tag{25}$$

Then, the evolutionary family $\{\mathscr{U}(t,s)\}_{0\le s\le t}$ generated by $A(t)$ given in (17) is (uniformly) exponentially stable.

Proof. To demonstrate that the evolutionary family $\{\mathscr{U}(t,s)\}_{0\le s\le t}$ generated by $A(t)$ is uniformly exponentially stable, we will examine the decay of the energy function E. The goal is to show that E, the total energy of the linear and non delayed part of the system (16), decays exponentially as time increases, indicating that the system is stable.

The linear and non delayed part of the system (16) is given as follows

$$\begin{cases} u_{tt}(x,t) - b(t)\Delta u(x,t) + CC^* u_t(x,t) = 0, & x \in \Omega \quad t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \ge 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1. \end{cases} \tag{26}$$

The energy function E is given by

$$E(t) = \frac{1}{2}b(t)\|\nabla u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2, \quad \forall t \in \mathbb{R}_+. \tag{27}$$

To investigate the stability of the system, we need to calculate the time derivative of the energy function. We differentiate E with respect to time,

$$\frac{d}{dt}E(t) = \frac{1}{2}b'(t)\|\nabla u(t)\|^2 + b(t)\langle\nabla u(t), \nabla u_t(t)\rangle + \langle u_{tt}(t), u_t(t)\rangle.$$

From the first equation of (26), we have

$$\langle u_{tt}(t), u_t(t)\rangle = b(t)\langle\Delta u(t), u_t(t)\rangle - \langle CC^* u_t(t), u_t(t)\rangle.$$

Using Green's formula and inserting this last in the derivative of E, we conclude that

$$\frac{d}{dt}E(t) = \frac{1}{2}b'(t)\|\nabla u(t)\|^2 - \|C^* u_t(t)\|^2 \le \frac{1}{2}b'(t)\|\nabla u(t)\|^2.$$

and so

$$\frac{d}{dt}E(t) \le \frac{b'(t)}{2b(t)} b(t)\|\nabla u(t)\|^2 \le \frac{b'(t)}{b(t)} E(t).$$

By a simple integration over 0 and t, we obtain

$$E(t) - E(0) \le \int_0^t \frac{b'(s)}{b(s)} E(s) ds.$$

Using the fact that b is exponential stable; the inequality (25) and applying Gronwall's inequality yield

$$E(t) \le E(0) e^{-\int_0^t \theta ds}, \quad \forall t \ge 0. \tag{28}$$

Thus, the evolutionary family $\{\mathscr{U}(t,s)\}_{0\le s\le t}$ satisfies the assumption (6) with $M = 1$ and $\omega = \theta$. □

Based on the previous results, the solution to (16) exists globally and decays exponentially. The exponential decay rate is ensured by the fact that b has an exponential decay rate, which is provided in the following.

Theorem 4. *For $(u_0, u_1, f_0) \in V \times H \times C([-\tau(0), 0]; H)$, the nonautonomous delayed wave equation (16) admits a unique global solution $U \in C([0, +\infty[; \mathscr{H})$ and if (25) and*

$$\left(\|\alpha\|_\infty \|D\| \frac{e^{\theta\tau_2}}{1-d} + \eta \right) < \theta,$$

hold. Thus, the solution of (16) is exponentially stable.

Remark 3. *The stability result obtained in Theorem (4), is very general where it is established without any conditions between the dissipative non-delayed and delayed term while in the [13, 19, 23, 24], they have been considered to guaranty the dissipation of energy and thus the stability result. Moreover, the autonomous and the constant delay cases fall within the scope of our abstract model, see [25, 28].*

4.2 Nonautonomous Petrovsky equation

Let us consider the next nonautonomous Petrovsky system which can be described within the framework of our abstract model

$$\begin{cases} u_{tt}(x,t) - b(t)\Delta^2 u(x,t) + CC^* u_t(x,t) = \alpha(t) D u_t(x, t-\tau(t)) + \psi(u(x,t)), & x \in \Omega \quad t > 0, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega, \quad t \ge 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \\ D u_t(t) = f_0(t) \quad t \in [-\tau(0), 0], \end{cases} \tag{29}$$

where b is a given function which belongs to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}_+^*)$ and Δ. In this case, we consider $V = H^4(\Omega) \cap H_0^2(\Omega)$ and by following the same arguments in the previous application under some technical modifications, we get the exponential stability.

4.3 Nonautonomous viscoelastic wave equation

Furthermore, this viscoelastic nonautonomous wave equation is compatible with our abstract setting, for $x \in \Omega$ and $t > 0$,

$$u_{tt}(x,t) - b(t)\Delta u(x,t) + \int_0^\infty g(s)c(t)\Delta u(x,t-s)\, ds = \alpha(t)Du_t(x,t-\tau(t)) + \psi(u(x,t)),$$

with

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ u(x,-t) = u_0(t), \quad u_t(x,0) = u_1, \quad x \in \Omega, \quad t \geq 0 \\ Du_t(t) = f_0(t) \quad t \in [-\tau(0), 0], \end{cases} \tag{30}$$

where b, c are given functions of class $C^1(\mathbb{R}_+, \mathbb{R}_+^*)$. The decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ represents the kernel of the viscoelastic term. The stability result is established under some conditions on b, c and g, for more details, we refer to [1, 13].

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