Journal of Mathematical Modeling
Vol. 13, No. 1, 2025, pp. 201-208. Research Article

JMM

Lower bounds of spatial analyticity radius for Benjamin-Bona-Mahony equation on the circle

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Abstract. It is shown that the radius of spatial analyticity $\sigma(t)$ of the solution $u(t)$ for the Benjamin-Bona-Mahony equation on the circle does not decay faster than $c|t|^{-2/3}$ (for some constant $c > 0$) as $|t| \to \infty$. This improves the work [A. A. Himonas, G. Petronilho, Evolution of the radius of spatial analyticity for the periodic Benjamin-Bona-Mahony Equation, Proc. Amer. Math. Soc. 148 (2020) 2953–2967], where the authors obtained a decay rate of order ct^{-1} for large t. The proof of our main theorems is based on a modified Gevrey space, Cauchy-Schwartz inequality, a method of almost conservation law and Sobolev embedding.

Keywords: Periodic BBM Equation, radius of analyticity of solutions, modified Gevrey spaces, lower bound for the radius.
AMS Subject Classification 2010: 35A01, 35B40.

1 Introduction

Consider the initial value problem (IVP) associated with the Benjamin-Bona-Mahony (BBM) equation on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

$$\begin{cases} \partial_t u - \partial_x^2 \partial_t u + \partial_x u + \frac{1}{2}\partial_x(u^2) = 0, & (x,t) \in \mathbb{T}\times\mathbb{R}, \\ u(x,0) = \phi(x), & x \in \mathbb{T}, \end{cases} \tag{1}$$

where $u := u(x,t)$ is a real-valued function. The BBM equation (1) was introduced in [2] to study the dynamics of small-amplitude surface water waves propagating unidirectionally. The well-posedness and ill-posedness of (1) in the Sobolev spaces $H^s(\mathbb{T})$ have been extensively studied (see for instance, [2, 3, 5, 15] and references therein).

Besides the well-posedness theory in the L^2-based Sobolev spaces, another fundamental question for nonlinear dispersive partial differential equations (PDEs) is about the asymptotic lower bound on the radius of spatial analyticity of solution in the Gevrey spaces. According to the Paley-Wiener Theorem

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Received: 19 August 2024 / Revised: 26 October 2024 / Accepted: 5 November 2024
DOI: 10.22124/jmm.2024.28211.2490

[14], the radius of spatial analyticity of a function can be characterized by decay property of its Fourier transform. It is therefore natural to take initial data in the Gevrey space $G^{\sigma,s} := G^{\sigma,s}(\mathbb{T})$ equipped with the norm

$$\|f\|^2_{G^{\sigma,s}} = \sum_{k\in\mathbb{Z}} \exp(2\sigma|k|)\langle k\rangle^{2s}|\widehat{f}(k)|^2, \quad \text{for} \quad \sigma > 0 \text{ and } s \in \mathbb{R},$$

where $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$ and $\widehat{f}$ is the Fourier transform of f given by

$$\widehat{f}(k) := \mathscr{F}_x[f](k) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{T}} \exp(-ikx) f(x)\, dx, \quad k \in \mathbb{Z},$$

and its inverse Fourier transform is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp(ikx)\widehat{f}(k).$$

It is clear that $G^{0,s} = H^s$, where $H^s := H^s(\mathbb{T})$ denotes the L^2-based Sobolev space of order s equipped with the norm (see the discussion in [1])

$$\|f\|^2_{H^s} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}|\widehat{f}(k)|^2,$$

while for $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{T})$ has a radius of analyticity of at least $\sigma > 0$ at each point $x \in \mathbb{T}$. Of course, this fact is contained in the Paley-Wiener Theorem which is stated as follows.

Theorem (Paley-Wiener Theorem)**.** Let $\sigma > 0$ and $s \in \mathbb{R}$. A 2π-periodic function $f(x)$ is in $G^{\sigma,s}(\mathbb{T})$ if and only if it is the restriction to the real line of a function $F(x+iy)$ which is 2π-periodic in x, holomorphic in the strip

$$S_\sigma = \{x+iy \in \mathbb{C} : \; |y| < \sigma\},$$

and satisfies the bound

$$\sup_{|y|<\sigma} \|F(\cdot+iy)\|_{H^s(\mathbb{T})} < \infty.$$

Since the work of Kato and Masuda [13], several papers looked at spatial analyticity (see, e.g, [6–10, 12, 17] and references therein). Coming back to (1), Himonas and Petronilho [11] obtained the algebraic decay rate of order t^{-1} for large time t. In fact, the authors in [11] used the conservation of energy functional

$$E(t) = \int\limits_{\mathbb{T}} \left(u^2 + (\partial_x u)^2\right) dx, \tag{2}$$

to obtain the result. For the non-periodic IVP of the BBM equation, Bona and Grujić [4] obtained the algebraic decay rate of order t^{-1} for large t. Latter, in [18], Wang improved the result of Bona and Grujić, and obtained a decay rate of order $t^{-2/3}$.

The main interest in this paper is to improve the algebraic decay rate of spatial analyticity to solutions of (1) obtained in [11]. A class of analytic function spaces suitable for our analysis is the modified Gevrey class $H^{\sigma,s} := H^{\sigma,s}(\mathbb{T})$ (introduced in [6]), which is endowed with the norm

$$\|f\|^2_{H^{\sigma,s}} = \sum_{k\in\mathbb{Z}} \cosh^2(\sigma|k|)\langle k\rangle^{2s}|\widehat{f}(k)|^2.$$

Following that the fact $\cosh(\sigma|k|) \sim \exp(\sigma|k|)$, the norms $\|\cdot\|_{G^{\sigma,s}}$ and $\|\cdot\|_{H^{\sigma,s}}$ are equivalent. The idea of definition of $H^{\sigma,s}$ is connected to the decay rate of exponential weight of $G^{\sigma,s}$-norm. The desired decay rate of order $t^{-1/\beta}$ (for $0<\beta\le 1$) of the radius of analyticity σ is obtained from the estimate

$$1-\exp(-\sigma|k|) \le (\sigma|k|)^{\beta}.$$

Since the hyperbolic cosine function $\cosh(\sigma|k|)$ satisfies the estimate

$$1-\operatorname{sech}(\sigma|k|) \le (\sigma^2|k|)^{\beta}, \quad 0<\beta\le 1,$$

an application of our method in the new space $H^{\sigma,s}$ can yield a decay rate of order $t^{-1/(2\beta)}$ for some $0<\beta\le 1$ provided that the nonlinear estimates of the approximate conservation law can dissolve the weight $|k|^{2\beta}$. In this manuscript, we managed to obtain the decay rate of order $t^{-2/3}$ for the IVP (1) (this result corresponds to $\beta=3/4$).

Observe that the modified Gevrey spaces satisfy the following embedding property:

$$H^{\sigma,s} \subset H^{\sigma',s'} \subset H^{s'}, \quad \forall\, 0<\sigma'<\sigma \text{ and } s,s'\in\mathbb{R}. \tag{3}$$

As a consequence of this property and the existing well-posedness theory in H^1, IVP (1) with initial data $\phi\in H^{\sigma_0,1}$ for all $\sigma_0>0$ has a unique and smooth solution for all time.

Before leaving the introduction, we state our main result as follows.

Theorem 1 (Improved lower bounds of spatial analyticity radius)**.** *Assume that $\phi\in H^{\sigma_0,1}(\mathbb{T})$ with $\sigma_0>0$. Then, the global C^∞ solution u, of (1) satisfies $u(t)\in H^{\sigma(t),1}$, $\forall\, t\in\mathbb{R}$, with the radius of analyticity $\sigma:=\sigma(t)$ satisfying an asymptotic lower bound*

$$\sigma \ge c|t|^{-2/3}, \tag{4}$$

where $c>0$ is a constant depending on $\|\phi\|_{H^{\sigma_0,1}}$.

Since the BBM equation given in (1) is invariant under the change of variables $(x,t)\to(-x,-t)$, we may restrict ourselves to positive times $t\ge 0$. Following this, the rest of the paper is organized as follows. In Section 2, we discuss on the local-in-time well-posedness result of the IVP (1). In Section 3, an almost conservation law for a modified energy functional associated with $Iu := \mathscr{F}_x[\cosh(\sigma|k|)\widehat{u}]$ is derived. Finally, our main result which is stated in Theorem 1 is proved in Section 4 by combining the local result and the almost conservation law.

Notation 1. For any positive numbers a and b, the notation $a\lesssim b$ is used when there exists a positive constant C such that $a\le Cb$. We also write $a\sim b$ when $a\lesssim b\lesssim a$ hold.

2 Local well-posedness in $H^{\sigma_0,1}$

We outline the argument in [11] that enables the authors to obtain the local well-posedness result for (1) in $G^{\sigma_0,1}$ for $\sigma_0>0$.

Now, denote

$$\varphi(D) := \partial_x\left(1-\partial_x^2\right)^{-1}. \tag{5}$$

By applying $\left(1-\partial_x^2\right)^{-1}$ to (1), we obtain

$$\begin{cases} \partial_t u + \varphi(D)(u + \frac{1}{2}u^2) = 0, \\ u(x,0) = \phi(x). \end{cases} \tag{6}$$

Then, Duhamel's integral formula allows us to write IVP (6) in the equivalent integral equation of the form

$$u(t) = \phi - \int_0^t \varphi(D)(u(t') + \frac{1}{2}u^2(t'))dt'. \tag{7}$$

Now, applying the contraction mapping argument in the space $C\left([0,T];H^{\sigma,1}\right)$ to the integral equation (7), and then using the bilinear estimate (Its proof can be found in [18, Lemma 2.1].)

$$\|uv\|_{H^{\sigma,1}} \leq C\|u\|_{H^{\sigma,1}}\|v\|_{H^{\sigma,1}}, \quad \forall\, \sigma > 0 \text{ and } u,v \in H^{\sigma,1}(\mathbb{T}), \tag{8}$$

the fact $\||k|(1+|k|^2)^{-1} \leq 1$, the Parseval's identity

$$\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)},$$

and Sobolev embedding yields the following local result.

Theorem 2. *Let $\phi \in H^{\sigma_0,1}$ for $\sigma_0 > 0$. Then, there exists a time $T > 0$ and a unique solution u of (1) on the time interval $(0,T)$ such that $u \in C\left([0,T];H^{\sigma_0,1}\right)$. Moreover, the solution depends continuously on the initial data ϕ, and the existence time is given by*

$$T \sim \left(1 + \|\phi\|_{H^{\sigma_0,1}}\right)^{-1}. \tag{9}$$

Furthermore,

$$\|u\|_{L_T^\infty H^{\sigma_0,1}} \lesssim \|\phi\|_{H^{\sigma_0,1}}. \tag{10}$$

3 Almost conservation law

To prove an almost conservation law for the model (1), we follow the argument used in [18]. To do this, we first define a Fourier multiplier operator

$$\mathscr{F}_x[Iu](k) = m(k)\mathscr{F}_x[u](k), \tag{11}$$

where

$$m(k) = \cosh(\sigma|k|), \quad k \in \mathbb{T},\ \sigma > 0. \tag{12}$$

It is clear that

$$\|\cosh(\sigma|D|)u\|_{H^s} \sim \|u\|_{H^{\sigma,s}} \sim \|Iu\|_{H^s}. \tag{13}$$

Using the Taylor expansion

$$e^{\sigma|k|} = \sum_{n=0}^{\infty} \frac{(\sigma|k|)^n}{n!}. \tag{14}$$

Since $\cosh(\sigma|k|) = \frac{1}{2}(e^{\sigma|k|} + e^{-\sigma|k|})$, we have

$$m^2(k) = 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2\sigma|k|)^n}{(2n)!}. \tag{15}$$

Define a modified energy functional associated with Iu by

$$E_\sigma(t) = \frac{1}{2}\int_{\mathbb{T}} \left((Iu)^2 + (\partial_x Iu)^2\right) dx.$$

Observe that for $\sigma = 0$, we have $Iu = u$, and therefore the energy is conserved, i.e., $E_0(t) = E_0(0)$ for all time t. But, this fails to hold for $\sigma > 0$. In what follows, we will prove the following almost conservation law.

Theorem 3 (Almost conservation law). *Let $\phi \in H^{\sigma,1}$. Assume that $u \in C\left((0,T); H^{\sigma,1}\right)$ is the local-in-time solution to the IVP (1) that is constructed in Theorem 2. Then*

$$\sup_{t\in[0,T]} E_\sigma(t) \le E_\sigma(0) + C\sigma^{\frac{3}{2}}[E_\sigma(0)]^{\frac{3}{2}}.$$

Proof. Multiplying (1) with $2I^2u$ and integrating over $\mathbb{T}$, we obtain

$$\frac{d}{dt}E_\sigma(t) = -2(u\partial_x u, I^2 u), \tag{16}$$

where $(\cdot)$ denotes the inner product in $L^2(\mathbb{T})$ and we used the fact

$$(\partial_x u, I^2 u) = \int_{\mathbb{T}} \frac{1}{2}\partial_x (Iu)^2 dx = 0.$$

Here, we need to estimate $-2(u\partial_x u, I^2 u)$. By using integration by parts and the Parseval's identity, we obtain

$$-2(u\partial_x u, I^2 u) = \int_{\mathbb{T}} (\partial_x I^2 u)u^2 dx = \sum_{k_1+k_2+k_3=0} ik_1 m^2(k_1)\widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3). \tag{17}$$

Let $m_j = m(k_j)$ for $j = 1,2,3$. By symmetry and the Parseval's identity, we have (see [18, Eq. (3.6)])

$$-2(u\partial_x u, I^2 u) = \int_{\mathbb{T}} (\partial_x I^2 u)u^2 dx = \frac{i}{3}\sum_{k_1+k_2+k_3=0} (k_1 m_1^2 + k_2 m_2^2 + k_3 m_3^3)\widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3). \tag{18}$$

By inserting (15) into (18) and then using Lemma 3.3 of [18] and triangle inequality, we obtain

$$\begin{aligned}
-2(u\partial_x u, I^2 u) &= \frac{i}{6}\sum_{k_1+k_2+k_3=0}\sum_{n=1}^{\infty} \frac{(2\sigma|k|)^n}{(2n)!}(k_1^{2n+1} + k_2^{2n+1} + k_3^{2n+1})\widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3) \\
&\le C\sum_{k_1+k_2+k_3=0} \sigma^{\frac{3}{2}}|k_1k_2k_3|^{\frac{5}{6}} e^{\sigma(|k_1|+|k_2|+|k_3|)}\widehat{u}(k_1)\widehat{u}(k_2)\widehat{u}(k_3) \\
&\le C\sigma^{\frac{3}{2}}\sum_{k_1+k_2+k_3=0} |k_1k_2k_3|^{\frac{5}{6}} e^{\sigma|k_1|}e^{\sigma|k_2|}e^{\sigma|k_3|}|\widehat{u}(k_1)||\widehat{u}(k_2)||\widehat{u}(k_3)|.
\end{aligned} \tag{19}$$

Let $\widehat{w} = |\widehat{u}|$. Now, using the Parseval's identity and Sobolev embedding, we obtain

$$\begin{aligned}
-2(u\partial_x u, I^2 u) &\leq C\sigma^{\frac{3}{2}} \int_{\mathbb{T}} (|D|^{\frac{5}{6}} e^{\sigma|D|} w)^3 \, dx \\
&\leq C\sigma^{\frac{3}{2}} \left\| |D|^{\frac{5}{6}} e^{\sigma|D|} w \right\|_{L^3}^3 \\
&\leq C\sigma^{\frac{3}{2}} \left\| |D| e^{\sigma|D|} w \right\|_{L^2}^3 \\
&\sim C\sigma^{\frac{3}{2}} \left\| |k| e^{\sigma|k|} |\widehat{u}| \right\|_{L^2}^3 \\
&\sim C\sigma^{\frac{3}{2}} \| |k| \cosh(\sigma|k|) |\widehat{u}| \|_{L^2}^3 \\
&\leq C\sigma^{\frac{3}{2}} \|Iu\|_{H^1}^3 . \qquad (20)
\end{aligned}$$

Consequently, integrating (16) in time over the interval $(0,s)$ for $s \leq T$ and then plugging (20) and using Hölder's inequality in time gives

$$\begin{aligned}
E_\sigma(s) &= E_\sigma(0) + C\sigma^{\frac{3}{2}} \int_0^s \|Iu\|_{H^1}^3 \, dt \\
&\leq E_\sigma(0) + C\sigma^{\frac{3}{2}} T \|Iu\|_{L_T^\infty H^1}^3 \\
&\leq E_\sigma(0) + C\sigma^{\frac{3}{2}} \|Iu\|_{L_T^\infty H^1}^3 . \qquad (21)
\end{aligned}$$

From (10) and (13), we get

$$\|Iu\|_{L_T^\infty H^1} \sim \|u\|_{L_T^\infty H^{\sigma,1}} \leq C\|\phi\|_{H^{\sigma,1}} \sim C\|I\phi\|_{H^1}. \qquad (22)$$

On the other hand

$$\begin{aligned}
\|I\phi\|_{H^1}^2 &= \sum_{k\in\mathbb{Z}} \langle k\rangle^2 m^2(k) |\widehat{\phi}(k)|^2 \\
&= \sum_{k\in\mathbb{Z}} (m^2(k)|\widehat{\phi}(k)|^2 + k^2 m^2(k)|\widehat{\phi}(k)|^2) \\
&= \int_{\mathbb{T}} (I\phi(x))^2 + (\partial_x I\phi(x))^2) \, dx \\
&= 2E_\sigma(0). \qquad (23)
\end{aligned}$$

It follows from (22) and (23) that

$$\|Iu\|_{L_T^\infty H^1} \leq CE_\sigma^{\frac{1}{2}}(0). \qquad (24)$$

Finally, using (24) in (21) gives the desired estimate in Theorem 3. □

4 Proof of Theorem 1

Assume that $\phi \in H^{\sigma_0,1}$ for some $\sigma_0 > 0$. Then, $v_{\sigma_0}(0) = \cosh(\sigma_0|D|)\phi \in H^1$, and hence we have

$$E_{\sigma_0}(0) \sim \|v_{\sigma_0}(0)\|_{H^1}^2 < \infty. \qquad (25)$$

Now by Theorem 2 there is a unique solution u to (1) satisfying

$$u \in C\left((0,T); H^{\sigma_0,1}(\mathbb{T})\right),$$

with existence time T as in (9). Then, following the argument in [16, 17] we can construct a solution on $[0,T^*]$ for arbitrarily large time T^*. To do this, we apply the almost conservation law in Theorem 3 to repeat the above local result on successive short time intervals of size T to reach T^* by adjusting the strip width parameter σ according to the size of T^*. This strategy established the bound

$$\sup_{t\in[0,T^*]} E_\sigma(t) \leq 2E_\sigma(0),$$

for σ satisfying

$$\sigma \geq c(T^*)^{-\frac{2}{3}}.$$

This implies that $E_\sigma(t) < \infty$ for every $0 \leq t \leq T^*$, and hence

$$u(t) \in H^{\sigma(t),1} \quad \text{with } \sigma(t) := \sigma \geq c(T^*)^{-\frac{2}{3}},$$

for all $0 \leq t \leq T^*$, where $c > 0$ is a constant depending on the initial data norm.

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