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An improved inertial subgradient extragradient algorithm for pseudomonotone equilibrium problems and its applications

Bochra Zeghad[†][*], **Nourreddine Daili**[‡]

[†]*Laboratory of Fundamental and Numerical Mathematics LMFN, Department of Mathematics, Ferhat Abbas University Setif-1, Setif, Algeria*
[‡]*Department of Mathematics, Ferhat Abbas University Setif-1, Setif, Algeria*
Email(s): bochra.zeghad@univ-setif.dz, nourreddine.daili@univ-setif.dz

Abstract. This paper presents an improvement of the inertial subgradient extragradient algorithm by using two non-monotonic step size criterion for pseudomonotone equilibrium problems in real Hilbert spaces. A strong convergence theorem of the suggested algorithm is proved under suitable assumptions on the equilibrium bifunction and the control parameters. Finally, application and numerical example are given, which demonstrate the advantages and efficiency of the proposed algorithm.

Keywords: Equilibrium problem, variational inequality, inertial method, strong convergence, subgradient extragradient method.
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1 Introduction

Consider the equilibrium problem (*EPb)* stated by Blum and Oettli [14]:

$$\begin{cases} \text{find } v \in \mathscr{S} \ \text{ such that} \\ B(v,t) \geq 0, \forall t \in \mathscr{S}, \end{cases} \tag{1}$$

where B is a bifunction : $\mathscr{S} \times \mathscr{S} \to \mathbb{R}$ and $\mathscr{S}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathscr{H}$. The problem (1) also called as the Ky Fan inequality problem in preliminary studies [4]. The *EPb* is a fundamental and crucial concept in various disciplines, providing a unifying framework for a wide range of theoretical and applied problems such as economics, game theory, optimization theory, fixed point problems and variational inequality problems (see e.g., [2, 9, 10]). Many authors have considered equilibrium problems and their generalizations, such as invex equilibrium, hemiequilibrium

[*]Corresponding author

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problems, and quasi-equilibrium problems (see, for instance, [3, 11, 15, 16]). They have also paid significant attention to the study of the *EPb*, not only from a theoretical perspective, focusing on the existence and/or uniqueness of solutions (see, e.g., [8]), but also to developing iterative methods and improving their convergence analysis in Hilbert spaces. Some of these methods involve the proximal point method ([5, 13]), extragradient methods [12, 23] and subgradient extragradient methods ([1, 7, 19, 20]).
In recent years, the inertial method (see e.g., [6, 21]) has gained significant importance in solving *EPb* due to its ability to accelerate convergence rates and improve computational efficiency. This method incorporates inertial terms to enhance the iterative process.
D.V. Thong et al [24] proposed a new iterative scheme with monotonically decreasing step sizes $\{\rho_m\}$ to solve the *EPb* as follows:

$$\begin{cases} s_m = v_m + \gamma_m(v_m - v_{m-1}), \\ t_m = \underset{t\in\mathscr{S}}{\arg\min}\left(\rho_m B(s_m,t) + \frac{1}{2}\|t - s_m\|^2\right) = J_{\rho_m B(s_m,.)}(s_m), \\ z_m = \underset{t\in\mathscr{T}_m}{\arg\min}\left(\rho_m B(t_m,t) + \frac{1}{2}\|t - s_m\|^2\right) = J_{\rho_m B(s_m,.)}(t_m), \\ v_{m+1} = (1 - \delta_m - \theta_m)v_m + \theta_m z_m, \end{cases}$$

where $\rho_1 > 0$, $\gamma > 0$, $\zeta \in (0,1)$, $\theta_m \subset (1-\delta_m)$, $\lim\limits_{m\to\infty} \delta_m = 0$ and $\sum\limits_{m=1}^{\infty} \delta_m = \infty$ and $0 \le \gamma_m \le \widetilde{\gamma}_m$ such that

$$\widetilde{\gamma}_m = \begin{cases} \min\left\{\frac{\gamma}{2}, \frac{\varepsilon_m}{\|v_m - v_{m-1}\|}\right\} & \text{if } v_m \neq v_{m-1}, \\ \frac{\gamma}{2} & \text{else.} \end{cases}$$

Also $\mathscr{T}_m = \{t \in \mathscr{H} : \langle s_m - \rho_m u_m - t_m, t - t_m\rangle \le 0\}$, $u_m \in \partial B(s_m, t_m)$ and

$$\rho_{m+1} = \begin{cases} \min\left\{\frac{\zeta\left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2M}, \ \rho_m\right\} & \text{if } M > 0, \\ \rho_m & \text{otherwise,} \end{cases} \tag{2}$$

where $M = B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})$. Under several conditions, the authors showed that the sequences strongly converge to some solution of the *EPb*. The solution set of the (*EPb*) represent by $EQ(B, \mathscr{S})$ (see [24]).

Based on the works mentioned above, the purpose of this paper is to suggest a new modified subgradient extragradient algorithm strongly convergent under some appropriate conditions, without the need of the Lipschitz constants or line-search technique. Rather, we use two non-monotonic step sizes criterion allowing them to work well. Our results generalize and extend some related results in the literature.

The rest of this paper is organized as follows. Section 2 provides key definitions and related lemmas used throughout the paper. Section 3 introduces an iterative algorithm for solving problem (1) and examines its convergence. Section 4 presents the application of the main results to variational inequality problems. Section 5 includes numerical experiments to illustrate the computational performance of the proposed algorithm on a test problem and compare it with other algorithms. Finally, Section 6 concludes the paper with a brief summary.

2 Basic definitions and lemmas

Throughout this paper, let $\mathscr{S}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathscr{H}$. The inner product is denoted by $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|.\|$. The weak convergence and strong convergence of v_m to v are represented by $\rightharpoonup$ and $\longrightarrow$, respectively. For every $t\in\mathscr{H}$, the metric projection $P_{\mathscr{S}}(t)$ is defined by

$$P_{\mathscr{S}}(t)=\arg\min_{s\in\mathscr{S}}\|t-s\|.$$

Definition 1. *A subset of $\mathscr{S}$ at $t\in\mathscr{S}$ defined by*

$$N_{\mathscr{S}}(t):=\{t^*\in\ \mathscr{H}:<t^*,s-t>\leq 0,\quad \forall s\in\mathscr{S}\},$$

is called the normal cone.

Definition 2. *The subdifferential of convex function $h:\mathscr{S}\to\mathbb{R}$ at $t\in\mathscr{S}$ is defined by*

$$\partial h(t):=\{u\in\ \mathscr{H}:h(s)-h(t)\geq<u,s-t>,\quad \forall s\in\mathscr{S}\}.$$

Definition 3. *The proximal operator $J_{\tau h}$ of a proper, convex and lower semicontinuous function $h:\mathscr{S}\to\mathbb{R}$, with a parameter $\tau>0$ at $t\in\mathscr{H}$ is given by*

$$J_{\tau h}(t):=\arg\min_{s\in\mathscr{S}}\left(\tau h(s)+\frac{1}{2}\|s-t\|^2\right),\quad t\in\mathscr{H}.$$

Lemma 1 ([25,27])**.** *Suppose $\mathscr{S}$ is a nonempty convex subset of $\mathscr{H}$. Consider $h:\mathscr{S}\to\mathbb{R}\cup\{+\infty\}$ as a convex function that is subdifferentiable and lower semicontinuous. Then, t^* is a solution to the following convex optimization problem:*

$$\min\{h(t):t\in\mathscr{S}\},$$

if and only if

$$0\in\partial h(t^*)+N_{\mathscr{S}}(t^*),$$

where $\partial h(t^*)$, $N_{\mathscr{S}}(t^*)$ are the subdifferential of h and the normal cone of $\mathscr{S}$ at t^*, respectively.

Lemma 2 ([17])**.** *Let $\{a_m\},\{b_m\}$ and $\{c_m\}$ be positive sequences such that*

$$a_{m+1}\leq\ a_m b_m+c_m,\ \ \forall m\in\mathbb{N}.$$

If $\{b_m\}\subset[1,\infty)$, $\sum\limits_{m=1}^{\infty}(b_m-1)<\infty$ and $\sum\limits_{m=1}^{\infty}c_m<\infty$, then $\lim\limits_{m\to\infty}a_m$ exists.

Lemma 3 ([28])**.** *Let $\{J_m\}\subset[0,+\infty)$ and $\{L_m\}\subset\mathbb{R}$ be sequences satisfying*

$$J_{m+1}\leq(1-\ \delta_m)J_m+\ \delta_m L_m,\ \ \forall m\in\mathbb{N},$$

where $\{\ \delta_m\}\subset(0,1)$, $\sum\limits_{m=1}^{+\infty}\ \delta_m=+\infty$. If $\limsup\limits_{m\to+\infty}L_m\leq 0$ for every subsequence $\{J_{m_k}\}$ of $\{J_m\}$ such that

$$\liminf_{k\to\infty}(J_{m_k+1}-J_{m_k})\geq 0,$$

then $\lim\limits_{m\to\infty}J_m=0$.

3 Algorithm and convergence analysis

In this section, we propose a modified subgradient extragradient algorithm with two non-monotonic step size criterion for solving *EPb* (1). Let B be a bifunction: $\mathscr{H} \times \mathscr{H} \to \mathbb{R}$ with $EQ(B, \mathscr{S})$ denotes the solution set of an equilibrium problem over the set $\mathscr{S}$. Denote by $\longrightarrow$ and $\rightharpoonup$ the weak convergence and strong convergence, respectively. For the strong convergence theorem assume the following conditions.

(H_1) The bifunction B is pseudomonotone on $\mathscr{S}$, i.e.,

$$B(v,t) \geq 0 \Rightarrow B(t,v) \leq 0, \qquad \forall v,t \in \mathscr{S},$$

(H_2) B is Lipschitz type continuous on $\mathscr{H}$ with two positive constants L_1 and L_2, i.e.,

$$B(v,t)+B(t,w) \geq B(v,w) - L_1 \|v-t\|^2 - L_2 \|t-w\|^2, \ \forall t,v,w \in \mathscr{S},$$

(H_3) $B(v,\cdot)$ is convex and subdifferentiable on $\mathscr{H}$ for each fixed $v \in \mathscr{S}$,

(H_4) $\limsup_{m\to\infty} B(v_m,t) \leq B(v,t)$, for every weakly convergent $\{v_m\} \subset \mathscr{S}$ to $v \in \mathscr{H}$ and $t \in \mathscr{S}$,

(H_5) let $\{\varepsilon_m\}$ be a positive sequence such that $\lim_{m\to\infty} \frac{\varepsilon_m}{\delta_m} = 0$, where $\{\delta_m\} \subset (0,1)$ satisfies $\lim_{m\to\infty} \delta_m = 0$ and $\sum_{m=1}^{\infty} \delta_m = \infty$. Also let $\{\sigma_m\} \subset [0,\infty[$ and $\{\omega_m\} \subset [1,\infty[$ such that $\sum_{m=1}^{\infty} \sigma_m < \infty$ and $\sum_{m=1}^{\infty} (\omega_m - 1) < \infty$.

Remark 1. *From (H_1) and (H_2), we have $B(v,v) = 0, \forall v \in \mathscr{S}$ (see [26]).*

The solution set $EQ(B, \mathscr{S})$ of *EPb* (1) is convex and closed under the conditions (H_1)-(H_4) ([23]). We propose the following algorithm for solving *EPb* (1).

Remark 2. *The observation presented below was extracted from Algorithm 1.*

- *If we choose ($\delta_m = 0$, $\gamma = 2\theta$, $\sigma_m = 0$) or ($\delta_m = 0$, $\gamma_m = 0$, $\rho_m = \rho$, $\mu = 1$), the algorithm reduces to the Algorithm 3.4 in [22] and the standard extragradient algorithm in [23], respectively.*
- *Notice that, the sequence $\{\rho_m\}$ defined by (4) is non-monotonic step size, independent to the Lipschitz constants and does not need any Armijo line-search technique.*

Remark 3. *From the expression (3), is apparent that $\lim_{m\to+\infty} \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| = 0$. Indeed, $\gamma_m \leq \frac{\varepsilon_m}{\|v_m - v_{m-1}\|}$ and $\lim_{m\to\infty} \frac{\varepsilon_m}{\delta_m} = 0$, implies*

$$\lim_{m\to\infty} \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| \leq \lim_{m\to\infty} \frac{\varepsilon_m}{\delta_m} \|v_m - v_{m-1}\| = 0.$$

In order to establish the strong convergence of Algorithm 1, our initial steps involves proving the following basic results.

Lemma 4. *The sequence $\{\rho_m\}$ created by (4) is well defined and $\lim_{m\to+\infty} \rho_m$ exists.*

Algorithm 1

Initialization: Given $v_0, v_1 \in \mathscr{S}$, $\rho_1 > 0$, $\gamma > 0$, $\zeta \in (0,1)$ and $\mu \in \left(0, \dfrac{2}{(1+\zeta)}\right)$. Select the sequences $\{\delta_m\}, \{\sigma_m\}$ and $\{\omega_m\}$ to satisfy (H_5).
Step 1: Compute

$$s_m = (1-\delta_m)(v_m + \gamma_m(v_m - v_{m-1})),$$

where $0 \le \gamma_m \le \widetilde{\gamma}_m$ such that

$$\widetilde{\gamma}_m = \begin{cases} \min\left\{\gamma, \frac{\varepsilon_m}{\|v_m - v_{m-1}\|}\right\} & \text{if } v_m \neq v_{m-1}, \\ \gamma & \text{else,} \end{cases} \tag{3}$$

Step 2: Compute

$$t_m = \arg\min_{t \in \mathscr{S}} \left(\rho_m B(s_m, t) + \frac{1}{2}\|t - s_m\|^2\right) = J_{\rho_m B(s_m,\cdot)}(s_m).$$

If $s_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.
Step 3:

$$v_{m+1} = \arg\min_{t \in \mathscr{T}_m} \left(\mu\rho_m B(t_m, t) + \frac{1}{2}\|t - s_m\|^2\right) = J_{\mu\rho_m B(t_m,\cdot)}(s_m),$$

where the half-space $\mathscr{T}_m$ is given by

$$\mathscr{T}_m = \{t \in \mathscr{H} : \langle s_m - \rho_m u_m - t_m, t - t_m\rangle \le 0\}, u_m \in \partial B(s_m, t_m),$$

and

$$\rho_{m+1} = \begin{cases} \min\left\{\frac{\zeta\left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2M}, \omega_m\rho_m + \sigma_m\right\} & \text{if } M > 0, \\ \omega_m\rho_m + \sigma_m & \text{otherwise,} \end{cases} \tag{4}$$

where $M = B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})$.

Proof. Since B fulfills (H_2), it follows that

$$\begin{aligned} \frac{\zeta\left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2\left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})\right)} &\ge \frac{\zeta\left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2\left(L_1\|s_m - t_m\|^2 + L_2\|v_{m+1} - t_m\|^2\right)} \\ &\ge \frac{\zeta}{2\max\{L_1, L_2\}}. \end{aligned}$$

This, in addition to the expression (4), gives $\rho_{m+1} \ge \min\left\{\frac{\zeta}{2\max\{L_1,L_2\}}, \rho_m\right\}$. Moreover, $\rho_m \ge \min\left\{\frac{\zeta}{2\max\{L_1,L_2\}}, \rho_1\right\}$. In contrast, it becomes clear from expression (4) that

$$\rho_{m+1} \le \omega_m\rho_m + \sigma_m, \ \forall m \ge 1.$$

It follows from condition (H_5) and Lemma 2 that $\lim_{m\to+\infty} \rho_m$ exists. Since $\min\left\{\frac{\zeta}{2\max\{L_1,L_2\}}, \rho_1\right\}$ is the lower boundary of $\{\rho_m\}$, then $\lim_{m\to+\infty}\rho_m := \rho > 0$. □

Lemma 5. *Let $\{s_m\}$, $\{t_m\}$ and $\{v_m\}$ represent the sequences generated by Algorithm 1. Then*

(i) $< s_m - t_m, t - t_m > \leq \rho_m \left(B(s_m,t) - B(s_m,t_m)\right), \forall t \in \mathscr{S}$,

(ii) *if $t_m = s_m$, then $t_m \in EQ(B, \mathscr{S})$,*

(iii) *for all $r \in EQ(B, \mathscr{S})$, the following inequality is holds:*

$$\|v_{m+1} - r\|^2 \leq \|s_m - r\|^2 - \mu_m^* \left(\|v_{m+1} - t_m\|^2 + \|s_m - t_m\|^2\right), \tag{5}$$

where

$$\mu_m^* = \begin{cases} \mu\left(1 - \frac{\zeta\rho_m}{\rho_{m+1}}\right) & \text{if} \quad \mu \in (0,1), \\ 2 - \mu - \frac{\zeta\mu\rho_m}{\rho_{m+1}} & \text{if} \quad \mu \in [1, \dfrac{2}{(1+\zeta)}). \end{cases}$$

Proof. **(i)** According to Lemma 1 and $t_m = J_{\rho_m B(s_m,.)}(s_m)$, we have

$$0 \in \partial\left(\rho_m B(s_m,t) + \frac{1}{2}\|t - s_m\|^2\right)(t_m) + N_{\mathscr{S}}(t_m).$$

Then, there exists $u_m \in \partial B(s_m,t_m)$ and $\vartheta \in N_{\mathscr{S}}(t_m)$, such that

$$\rho_m u_m + t_m - s_m + \vartheta = 0.$$

By the definition of $N_{\mathscr{S}}$, we get

$$< s_m - t_m, t - t_m >= \rho_m < u_m, t - t_m > + < \vartheta, t - t_m > \leq \rho_m < u_m, t - t_m >, \forall t \in \mathscr{S}.$$

Since $u_m \in \partial B(s_m,t_m)$, we have

$$< u_m, t - t_m > \leq B(s_m,t) - B(s_m,t_m), \forall t \in \mathscr{S}.$$

From the last two inequalities, we obtain

$$< s_m - t_m, t - t_m > \leq \rho_m \left(B(t_m,t) - B(s_m,t_m)\right), \forall t \in \mathscr{S}. \tag{6}$$

(ii) If $t_m = s_m$, then from inequalities (6) and $\rho_m > 0$, we find $B(t_m,t) \geq 0$, for all $t \in \mathscr{S}$. Thus $t_m \in EQ(B, \mathscr{S})$.

(iii) We have $v_{m+1} = J_{\mu\rho_m B(t_m,.)}(s_m)$, as similar arguments to the proof of (**i**), we obtain

$$< s_m - v_{m+1}, t - v_{m+1} > \leq \mu\rho_m \left(B(t_m,t) - B(t_m,v_{m+1})\right), \forall t \in \mathscr{T}_m. \tag{7}$$

In particular, substituting $t = v_{m+1}$ in (6) and $t = r$ in (7), we get

$$\begin{cases} \langle s_m - t_m, v_{m+1} - t_m \rangle \leq \rho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m) \right), \\ \langle s_m - v_{m+1}, r - v_{m+1} \rangle \leq \mu \rho_m \left(B(t_m, r) - B(t_m, v_{m+1}) \right). \end{cases}$$

So, from the pseudo monotonicity of B, we have $B(r, t_m) \geq 0$. Thus $B(t_m, r) \leq 0$. Then

$$\begin{aligned} 2\mu\rho_m & \left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1}) \right) \\ & \geq 2\mu < s_m - t_m, v_{m+1} - t_m > +2 < s_m - v_{m+1}, r - v_{m+1} > \\ & \geq \mu \left\| s_m - t_m \right\|^2 + \mu \left\| v_{m+1} - t_m \right\|^2 - \mu \left\| v_{m+1} - s_m \right\|^2 + \left\| v_{m+1} - s_m \right\|^2 \\ & \quad + \left\| v_{m+1} - r \right\|^2 - \left\| s_m - r \right\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| v_{m+1} - r \right\|^2 \leq & \left\| s_m - r \right\|^2 - \mu \left\| s_m - t_m \right\|^2 - \mu \left\| v_{m+1} - t_m \right\|^2 - (1 - \mu) \left\| v_{m+1} - s_m \right\|^2 \\ & + 2\mu\rho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1}) \right). \end{aligned} \tag{8}$$

From the definition of ρ_m, we have

$$2\mu\rho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1}) \right) \leq \frac{\mu\zeta\rho_m}{\rho_{m+1}} \left(\left\| s_m - t_m \right\|^2 + \left\| v_{m+1} - t_m \right\|^2 \right). \tag{9}$$

Substituting (9) into (8), we obtain

$$\begin{aligned} \left\| v_{m+1} - r \right\|^2 \leq & \left\| s_m - r \right\|^2 - \mu \left\| s_m - t_m \right\|^2 - \mu \left\| v_{m+1} - t_m \right\|^2 - (1 - \mu) \left\| v_{m+1} - s_m \right\|^2 \\ & + \frac{\mu\zeta\rho_m}{\rho_{m+1}} \left(\left\| s_m - t_m \right\|^2 + \left\| v_{m+1} - t_m \right\|^2 \right) \\ \leq & \left\| s_m - r \right\|^2 - (1 - \mu) \left\| v_{m+1} - s_m \right\|^2 \\ & - \mu \left(1 - \frac{\zeta\rho_m}{\rho_{m+1}} \right) \left(\left\| s_m - t_m \right\|^2 + \left\| v_{m+1} - t_m \right\|^2 \right). \end{aligned} \tag{10}$$

If $\mu \in (0, 1)$, then

$$\left\| v_{m+1} - r \right\|^2 \leq \left\| s_m - r \right\|^2 - \mu \left(1 - \frac{\zeta\rho_m}{\rho_{m+1}} \right) \left(\left\| v_{m+1} - t_m \right\|^2 + \left\| s_m - t_m \right\|^2 \right).$$

Note that

$$\left\| v_{m+1} - s_m \right\|^2 \leq \left(\left\| v_{m+1} - t_m \right\| + \left\| s_m - t_m \right\| \right)^2 \leq 2 \left(\left\| v_{m+1} - t_m \right\|^2 + \left\| s_m - t_m \right\|^2 \right),$$

which yields that

$$-(1 - \mu) \left\| v_{m+1} - s_m \right\|^2 \leq -2(1 - \mu) \left(\left\| v_{m+1} - t_m \right\|^2 + \left\| s_m - t_m \right\|^2 \right), \quad \forall \mu \geq 1.$$

From expression (10), we get

$$\left\| v_{m+1} - r \right\|^2 \leq \left\| s_m - r \right\|^2 - \left(2 - \mu - \frac{\zeta\mu\rho_m}{\rho_{m+1}} \right) \left(\left\| v_{m+1} - t_m \right\|^2 + \left\| s_m - t_m \right\|^2 \right), \quad \forall \mu \geq 1.$$

This completes the proof. □

Remark 4. *It easy to cheek that* $\lim_{m\to\infty} \mu_m^* > 0$. *Using Lemma 4, we have*

$$\lim_{m\to\infty} \mu_m^* = \begin{cases} \mu\,(1-\zeta) & if \quad \mu \in (0,1), \\ 2-\mu\,(1+\zeta) & if \quad \mu \in [1, \dfrac{2}{(1+\zeta)}). \end{cases}$$

Moreover, there exists $m_0 \geq 0$ *such that* $\mu_m^* > 0$ *for all* $m \geq m_0$.

Lemma 6. *The sequence* $\{v_m\}$ *generated by Algorithm 1 is bounded. Consequently,* $\{s_m\}$ *and* $\{t_m\}$ *are bounded.*

Proof. From the definition of s_m, we have

$$\begin{aligned} \|s_m - r\| &= \|(1-\delta_m)(v_m + \gamma_m(v_m - v_{m-1})) - r\| \\ &= \|(1-\delta_m)(v_m - r) + (1-\delta_m)\gamma_m(v_m - v_{m-1}) - \delta_m r\| \\ &\leq (1-\delta_m)\|v_m - r\| + (1-\delta_m)\gamma_m\|v_m - v_{m-1}\| + \delta_m\|r\| \\ &\leq (1-\delta_m)\|v_m - r\| + \delta_m I_1, \end{aligned} \tag{11}$$

where

$$(1-\delta_m)\frac{\gamma_m}{\delta_m}\|v_m - v_{m-1}\| + \|r\| \leq I_1.$$

From (5), we obtain

$$\|v_{m+1} - r\|^2 \leq \|s_m - r\|^2, \forall m \in \mathbb{N} \tag{12}$$

Using (11), then

$$\begin{aligned} \|v_{m+1} - r\| &\leq (1-\delta_m)\|v_m - r\| + \delta_m I_1 \\ &\leq \max\{\|v_m - r\|, I_1\} \qquad\qquad (\forall m \geq m_0) \\ &\;\;\vdots \\ &\leq \max\{\|v_0 - r\|, I_1\}. \end{aligned}$$

Thus, we conclude that $\{\|v_m - r\|\}$ is bounded sequence which implies that $\{v_m\}$ is bounded. Hence $\{s_m\}$ and $\{t_m\}$ are also bounded. □

Theorem 1. *Suppose that the the conditions* (H_1)-(H_5) *hold and* $EQ(B, \mathscr{S}) \neq \emptyset$. *Then the sequence* $\{v_m\}$ *generated by Algorithm 1 converges in norm to r, where* $\|r\| = \min\{\|q\| : q \in EQ(B, \mathscr{S})\}$, *i.e.,* $r = P_{EQ(B,\,\mathscr{S})}(0)$.

Proof. First, we show that the sequence $\{v_m\}$ and $\{s_m\}$ generated by Algorithm 1 achieves the following:

$$\begin{cases} J_{m+1} \leq (1-\delta_m)J_m + \delta_m L_m, \forall m \geq m_0, \\ \limsup_{m\to\infty} L_m \leq 0, \end{cases}$$

where

$$\begin{cases} J_m = \|v_m - r\|^2, \\ L_m = \gamma_m \|v_m - v_{m-1}\| \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| + 2(1- \delta_m) \|v_m - r\| \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| \\ \qquad + 2\|r\| \|s_m - v_{m+1}\| + 2\langle r, r - v_{m+1}\rangle . \end{cases}$$

Indeed, according to the inequality (12) and the definition of s_m, we have

$$\begin{aligned} \|v_{m+1} - r\|^2 &\le \|(1-\delta_m)(v_m - r) + (1-\delta_m)\gamma_m(v_m - v_{m-1}) - \delta_m r\|^2 \\ &= \|(1-\delta_m)(v_m - r) + (1-\delta_m)\gamma_m(v_m - v_{m-1})\|^2 + \| \delta_m r\|^2 \\ &\quad + 2 \delta_m \langle -r, s_m + r\rangle + 2\delta_m \|r\|^2 \\ &\le (1-\delta_m)^2 \|v_m - r\|^2 + (1-\delta_m)^2 \gamma_m^2 \|v_m - v_{m-1}\|^2 \\ &\quad + 2\gamma_m (1-\delta_m)^2 \|v_m - r\| \|v_m - v_{m-1}\| \\ &\quad + 2 \delta_m \langle -r, s_m - v_{m+1}\rangle + 2 \delta_m \langle -r, v_{m+1} + r\rangle . \end{aligned}$$

Since $\delta_m \subset (0,1)$, for all $m \ge m_0$, the above expression yields that

$$\begin{aligned} \|v_{m+1} - r\|^2 \le &(1- \delta_m) \|v_m - r\|^2 + \delta_m[\gamma_m \|v_m - v_{m-1}\| \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| \\ &+ 2(1- \delta_m) \|v_m - r\| \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| + 2\|r\| \|s_m - v_{m+1}\| \\ &+ 2\langle r, r - v_{m+1}\rangle]. \end{aligned} \tag{13}$$

The last inequality can be written as

$$J_{m+1} \le (1- \delta_m) J_m + \delta_m L_m, \forall m \ge m_0.$$

Due to Lemma 3, suppose that $\{J_{m_k}\}$ is a subsequence of $\{J_m\}$ satisfies

$$\liminf_{k\to\infty} \left(J_{m_{k+1}} - J_{m_k}\right) > 0.$$

Now, we demonstrate that $\limsup_{m\to\infty} L_m \le 0$. From (11), we have

$$\begin{aligned} \|s_m - r\|^2 &\le \|(1- \delta_m) \|v_m - r\| + \delta_m I_1\|^2 \\ &\le (1- \delta_m)^2 \|v_m - r\|^2 + \delta_m^2 I_1^2 + 2I_1(1- \delta_m) \delta_m \|v_m - r\| \\ &\le \|v_m - r\|^2 + \delta_m \left(\delta_m I_1^2 + 2I_1(1- \delta_m) \|v_m - r\|\right) \\ &\le \|v_m - r\|^2 + \delta_m I_2, \text{ for all } m \ge 1 \end{aligned} \tag{14}$$

where $I_2 = \sup_{m\in} \left\{ \left(\delta_m I_1^2 + 2I_1(1- \delta_m) \|v_m - r\|\right) \right\}$.

It follows from (5) and (14) that

$$\mu_m^* \left(\|v_{m+1} - t_m\|^2 + \|s_m - t_m\|^2\right) \le \|v_m - r\|^2 - \|v_{m+1} - r\|^2 + \delta_m I_2, \forall m \ge m_0. \tag{15}$$

From (15), $\lim_{m\to\infty} \delta_m = 0$ and Remark 4, we get

$$\begin{aligned}\mu^*_{m_k}\left(\|v_{m_k+1}-t_{m_k}\|^2+\|s_{m_k}-t_{m_k}\|^2\right) &\leq \limsup_{k\to\infty}\left(J_{m_k}-J_{m_{k+1}}\right)+\limsup_{k\to\infty}\delta_{m_k}I_2,\\ &\leq -\liminf_{k\to\infty}\left(J_{m_{k+1}}-J_{m_k}\right)\\ &\leq 0.\end{aligned}$$

Then

$$\lim_{k\to\infty}\|v_{m_k+1}-t_{m_k}\|=0 \text{ and } \|s_{m_k}-t_{m_k}\|=0. \tag{16}$$

Consequently,

$$\lim_{k\to\infty}\|v_{m_k+1}-s_{m_k}\|=0 \text{ and } \lim_{k\to\infty}\|v_{m_k+1}-s_{m_k}\|\,\|r\|=0. \tag{17}$$

Furthermore,

$$\begin{aligned}\|v_{m_k}-s_{m_k}\| &= \|(1-\ \delta_{m_k})\gamma_{m_k}(v_{m_k}-v_{m_k-1})-\ \delta_m v_{m_k}\|\\ &\leq \|(1-\ \delta_{m_k})\gamma_{m_k}(v_{m_k}-v_{m_k-1})\|+\|\ \delta_m v_{m_k}\|\\ &=\ \delta_{m_k}\left[(1-\ \delta_{m_k})\frac{\gamma_{m_k}}{\delta_{m_k}}\|v_{m_k}-v_{m_k-1}\|+\|v_{m_k}\|\right],\end{aligned}$$

then, we deduce that

$$\lim_{k\to\infty}\|v_{m_k}-s_{m_k}\|=0. \tag{18}$$

Consequently

$$\lim_{k\to\infty}\|v_{m_k+1}-v_{m_k}\|=0. \tag{19}$$

Next we show that $\limsup_{k\to\infty}\langle r, r-v_{m_k+1}\rangle=0.$ Due to the reflexive property of the Hilbert space $\mathscr{H}$, the boundedness of $\{v_{m_k}\}$ guarantee the existence of a subsequence $\left\{v_{m_{k_j}}\right\}$ of $\{v_{m_k}\}$ converges weakly to q as $j\to\infty$. Moreover

$$\limsup_{k\to\infty}\langle r, r-v_{m_k}\rangle=\lim_{j\to\infty}\left\langle r, r-v_{m_{k_j}}\right\rangle=\langle r, r-q\rangle. \tag{20}$$

It follows from (16) and (18) that $t_{m_k}\rightharpoonup q$ and $s_{m_k}\rightharpoonup q$. By means of $t_{m_k}=J_{\rho_{m_k}B(s_{m_k},\cdot)}(s_{m_k})$, we have

$$\mu\rho_{m_k}B(t_{m_k},v_{m_k+1})+<s_{m_k}-v_{m_k+1},t-v_{m_k+1}>\leq \mu\rho_{m_k}B(t_{m_k},t),\ \text{for } t\in\mathscr{T}_m. \tag{21}$$

From (4), we get

$$\begin{aligned}2\mu\rho_{m_k}B(t_{m_k},v_{m_k+1}) &\geq 2\mu\rho_{m_k}\left(B(s_{m_k},v_{m_k+1})-B(s_{m_k},t_{m_k})\right)\\ &\quad-\frac{\mu\zeta\rho_{m_k}}{\rho_{m_k+1}}\left(\|s_{m_k}-t_{m_k}\|^2+\|v_{m_k+1}-t_{m_k}\|^2\right),\end{aligned} \tag{22}$$

Substituting (22) into (21), we obtain

$$\begin{aligned}\mu\rho_{m_k}B(t_{m_k},t) &\geq \mu\rho_{m_k}\left(B(s_{m_k},v_{m_k+1})-B(s_{m_k},t_{m_k})\right)-\frac{\mu\zeta\rho_{m_k}}{2\rho_{m_k+1}}\left(\|s_{m_k}-t_{m_k}\|^2+\|v_{m_k+1}-t_{m_k}\|^2\right)\\ &\quad+<s_{m_k}-v_{m_k+1},t-v_{m_k+1}>,\quad \text{for } t\in\mathscr{T}_m.\end{aligned}$$

Due to μ, $\rho_{m_k} > 0$, condition (H_4) $v_{m_k} \rightharpoonup q$ and (16), we have

$$0 \leq \limsup_{k\to\infty} B(t_{m_k}, t) \leq B(q,t), \forall t \in \mathscr{T}_m.$$

Since $\mathscr{S} \subset \mathscr{T}_m$, $B(q,t) \geq 0$, $\forall t \in \mathscr{S}$ and hence $q \in EQ(B, \mathscr{S})$. By using (20) and the definition of r, we obtain

$$\lim_{k\to\infty} \langle r, r - v_{m_k}\rangle = \langle r, r-q\rangle \leq 0.$$

This with (19) gives,

$$\limsup_{k\to\infty} \langle r, r - v_{m_k+1}\rangle \leq \limsup_{k\to\infty} \langle r, r - v_{m_k}\rangle + \limsup_{k\to\infty} \langle r, v_{m_k+1} - v_{m_k}\rangle \leq 0. \tag{23}$$

Moreover, from (17), (23) and $\lim_{m\to\infty} \frac{\gamma_m}{\delta_m} \|v_m - v_{m-1}\| = 0$, we concludes that

$$\limsup_{k\to\infty} L_{m_k} \leq 0.$$

Applying Lemma 3, we obtain $\lim_{m\to\infty} \|v_m - r\| = 0$, as desired. □

4 Application to variational inequality problem

In this section, we present an application of the strong convergence theorem (Theorem 1) to the variational inequality problem. Let $B(v,t) = \langle \Psi(v), t - v\rangle$, with $\Psi : \mathscr{S} \to \mathscr{S}$ be an operator. The *EPb* (1) turn to variational inequality problem as follows:

$$\begin{cases} \text{find } v \in \mathscr{S} \text{ such that} \\ \langle \Psi(v), t - v\rangle \geq 0, \quad \forall t \in \mathscr{S}. \end{cases} \tag{24}$$

Assume that the solution set of (24) (denoted by $VI(\Psi, \mathscr{S})$) is nonempty and the operator Ψ satisfies the following:

(H_1') Ψ is pseudomonotone on $\mathscr{S}$, i.e.,

$$\langle \Psi(v), t - v\rangle \geq 0 \Rightarrow \langle \Psi(t), v - t\rangle \leq 0, \quad \forall v, t \in \mathscr{S},$$

(H_2') Ψ is Lipschitz continuous on $\mathscr{H}$ with $L > 0$, i.e.,

$$\|\Psi(v) - \Psi(t)\| \leq L\|v - t\|, \quad \forall v, t \in \mathscr{S},$$

(H_3') $\limsup_{m\to\infty} \langle \Psi(v_m), t - v_m\rangle \leq \langle \Psi(v), t - v\rangle$ for every weakly convergent $\{v_m\} \subset \mathscr{S}$ to $v \in \mathscr{H}$ and $t \in \mathscr{S}$.

The sequence t_m rewritten as

$$\begin{aligned} t_m &= \arg\min_{t\in\mathscr{S}} \left(\rho_m B(s_m,t) + \frac{1}{2}\|t-s_m\|^2\right) \\ &= \arg\min_{t\in\mathscr{S}} \left(\rho_m \langle \Psi(s_m), t-s_m\rangle + \frac{1}{2}\|t-s_m\|^2\right) \\ &= \arg\min_{t\in\mathscr{S}} \left(\frac{1}{2}\|t-(s_m-\rho_m\Psi(s_m))\|^2\right) - \frac{1}{2}\|\rho_m\Psi(s_m)\|^2 \\ &= P_{\mathscr{S}}(s_m-\rho_m\Psi(s_m)). \end{aligned}$$

Similarly, $v_{m+1} = P_{\mathscr{T}_m}(s_m - \mu\rho_m\Psi(t_m))$.

Corollary 1. *Assume that the conditions* $(H_1^{'}) - (H_3^{'})$ *hold. Let* $\{v_m\}$ *and* $\{t_m\}$ *be two sequences created in the following way:*

(i) *Given* $v_0, v_1 \in \mathscr{S}$, $\rho_1 > 0$, $\gamma > 0$, $\zeta \in (0,1)$ *and* $\mu \in \left(0, \frac{2}{(1+\zeta)}\right)$. *Select the sequences* $\{\delta_m\}$, $\{\omega_m\}$ *and* $\{\sigma_m\}$ *to satisfy* (H_5) .

(ii) *Compute* $s_m = (1-\delta_m)(v_m + \gamma_m(v_m - v_{m-1}))$, *where* γ_m *is defined in (3).*

(iii) *Compute*

$$\begin{cases} t_m = P_{\mathscr{S}}(s_m - \rho_m\Psi(s_m)), \\ v_{m+1} = P_{\mathscr{T}_m}(s_m - \mu\rho_m\Psi(s_m)), \end{cases}$$

where $\mathscr{T}_m = \{t \in \mathscr{H} : \langle s_m - \rho_m\Psi(s_m) - t_m, t - t_m\rangle \le 0\}$, *and*

$$\rho_{m+1} = \begin{cases} \min\left\{ \frac{\zeta\left(\|s_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right)}{2\langle\Psi(s_m)-\Psi(t_m),v_{m+1}-t_m\rangle}, \omega_m\rho_m+\sigma_m\right\}, & \text{if } \langle \Psi(s_m)-\Psi(t_m), v_{m+1}-t_m\rangle > 0, \\ \omega_m\rho_m+\sigma_m, & \text{otherwise}. \end{cases}$$

Then, the sequence $\{v_m\}$ *converges in norm to r, for each* $r \in VI(\Psi,\mathscr{S}) \neq \emptyset$.

5 Numerical illustrations

In this section, we give numerical example to demonstrate the computational efficiency of the proposed algorithm compared to some related results. All the programs are implemented in MATLAB.

Consider the Nash-Cournot oligopolistic equilibrium model in [23] where the bifunction B is defined as follows:

$$B(v,t) = \langle Pv + Qt + q, t - v\rangle,$$

where $q \in \mathbb{R}^n$ and $P, Q \in \mathbb{R}^{n\times n}$ are two matrices of order n such that Q is symmetric positive semidefinite and $Q-P$ is negative semidefinite with the Lipschitz-type constants $L_1 = L_2 = \frac{1}{2}\|Q-P\|$. It can be checked that all the conditions (H_1)-(H_4) are satisfied (for more details see [23]). Let $n = 5$ and the matrices P and Q (randomly generated) be

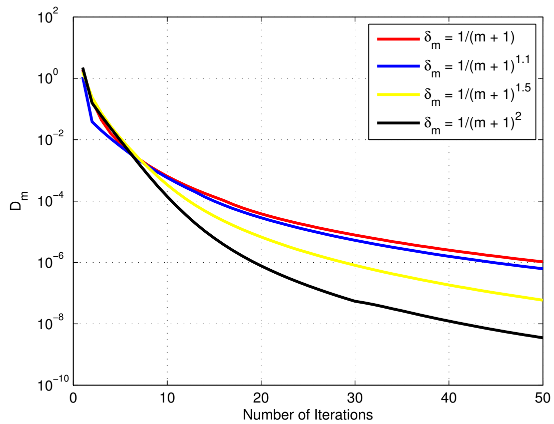

Figure 1: Numerical behavior of the proposed algorithm with different δ_m.

$$Q=\begin{pmatrix} 1.6 & 1 & 0 & 0 & 0\\ 1 & 1.6 & 0 & 0 & 0\\ 0 & 0 & 1.5 & 1 & 0\\ 0 & 0 & 1 & 1.5 & 0\\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},\ P=\begin{pmatrix} 3.1 & 2 & 0 & 0 & 0\\ 2 & 3.6 & 0 & 0 & 0\\ 0 & 0 & 3.5 & 2 & 0\\ 0 & 0 & 2 & 3.3 & 0\\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

with $q=(1,-2,-1,2,-1)^t$, and $\mathscr{S}$ given by

$$\mathscr{S} := \{v\in\mathbb{R}^n : -5\le v_i\le 5, i=1,...,5\},$$

The initial values v_0 and v_1 are randomly generated by MATLAB function $rand(5,1)$. We use the maximum number of iterations 50 as a common stopping criterion for all algorithms and $D_m = \|v_{m+1}-v_m\|^2$ is used to measure the error of the $m-th$ iteration step and CPU to represent the execution time of all algorithms in seconds. We compare the proposed Algorithms 1 (shortly, Our Alg. 1) with the Algorithm 3.1 suggested by Tan et al. [22] (shortly, BSJ Alg. 3.1) and the Algorithm 1 introduced by Panyanak et al. [18] (shortly, BCNN Alg. 1). The control parameters of all algorithms are choose as follows:

1. (Our Alg. 1): $\varepsilon_m=\frac{100}{(m+1)^2}$, $\delta_m=\frac{1}{20(m+1)^2}$, $\sigma_m=\frac{1}{(m+100)^3}$, $\omega_m=1+\frac{1}{20(m+1)^{1.1}}$.
2. (BCNN Alg. 1) in [18]: $\varepsilon_m=\frac{100}{(m+1)^2}$, $\omega_m=1+\frac{1}{20(m+1)^{1.1}}$, $\alpha_m=\frac{(1-\alpha)}{10}$, $\beta_m=\frac{1}{5m+2}$, $T(x)=\frac{x}{5}$.
3. (BSJ Alg. 3.1) in [22]: $\varepsilon_m=\frac{100}{(m+1)^2}$, $\alpha_m=1/(4m+1)$, $\beta_m=0.5$, $S(x)=x$, $\varphi(x)=0.5x$.

We first test the numerical behavior of the proposed algorithm with different parameter δ_m, as shown in Fig 1. and Table 1. Finally, the numerical results of all algorithms with different parameters are shown in Fig. 2-5 and Table 2.

Table 1: Numerical results of the proposed algorithm with different δ_m.

Our Algo. 1	D_n	CPU
$\delta_m = \dfrac{1}{(m+1)^2}$	$3.5E-9$	1.78
$\delta_m = \dfrac{1}{(m+1)^{1.5}}$	$5.91E-8$	1.80
$\delta_m = \dfrac{1}{(m+1)^{1.1}}$	$6.24E-7$	1.59
$\delta_m = \dfrac{1}{m+1}$	$1.04E-6$	1.62

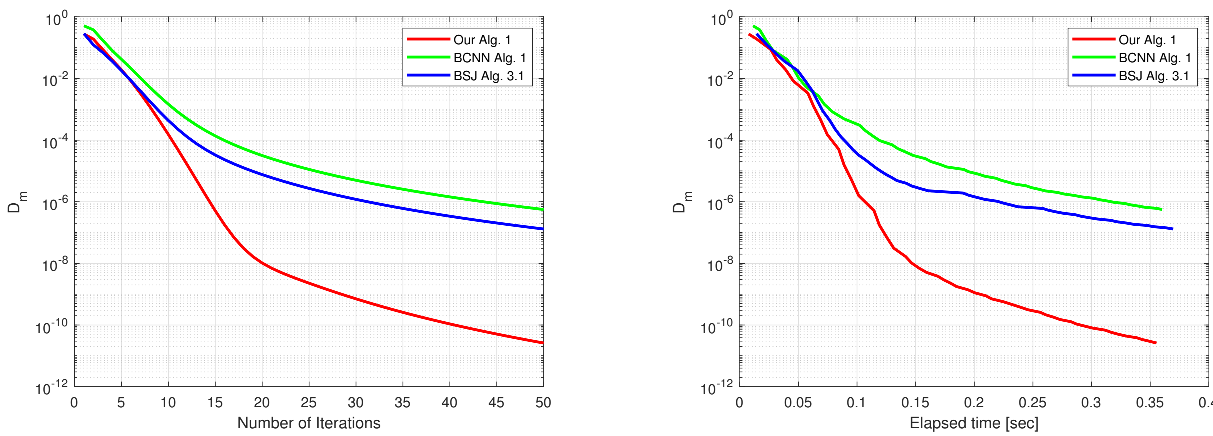

Figure 2: Numerical behavior of all algorithms with ($\gamma = 0.4$, $\zeta = 0.5$, $\mu = 1.25$, $\rho_0 = 0.1$)

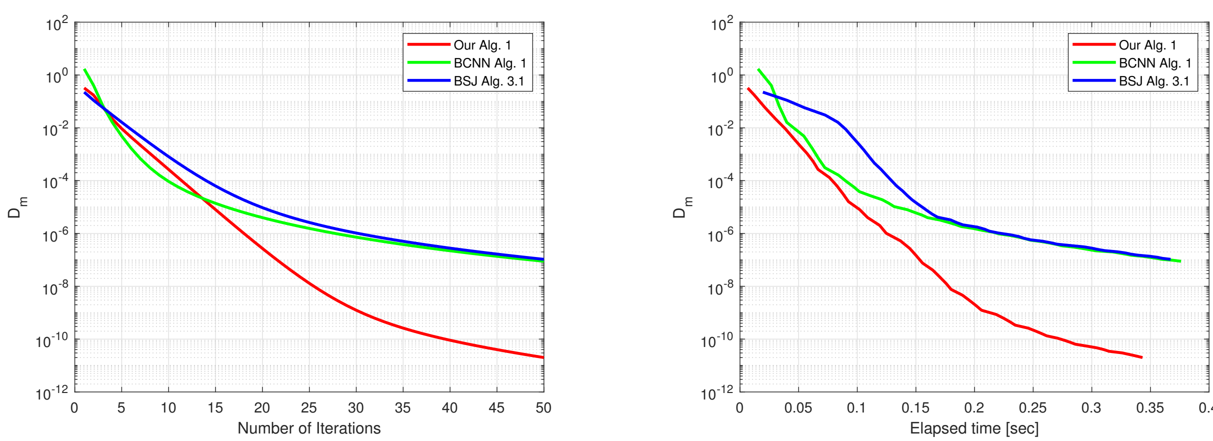

Figure 3: Numerical behavior of all algorithms with ($\gamma = 0.2$, $\zeta = 0.264$, $\mu = 0.5$, $\rho_0 = 0.36$).

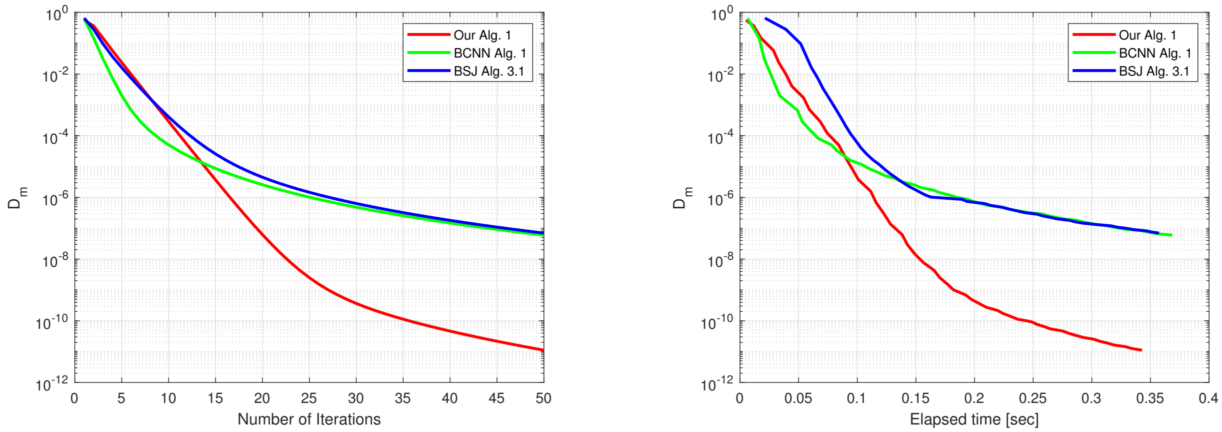

Figure 4: Numerical behavior of all algorithms with ($\gamma = 0.2$, $\zeta = 0.5$, $\mu = 0.5$, $\rho_0 = 0.5$).

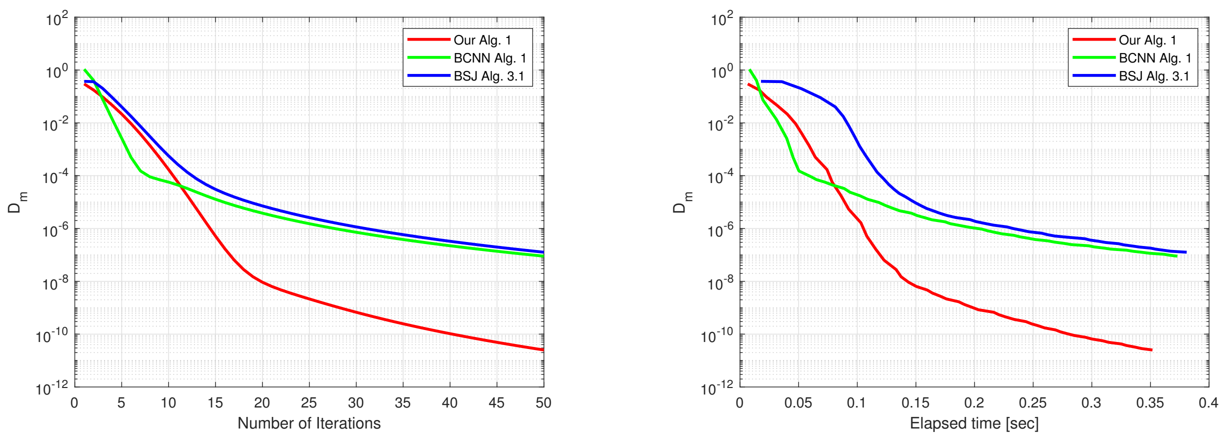

Figure 5: Numerical behavior of all algorithms with ($\gamma = 0.4$, $\zeta = 0.5$, $\mu = 0.5$, $\rho_0 = 0.3$).

Table 2: Numerical results of all algorithms with different parameters.

Algorithms	Our Alg. 1		BCNN Algo. 1		BSJ Algo 3.1	
	D_n	CPU	D_n	CPU	D_n	CPU
$\gamma = 0.4$, $\zeta = 0.5$, $\mu = 1.25$, $\rho_0 = 0.1$	$2.58E-11$	0.35	$5.6E-7$	0.36	$1.37E-7$	0.37
$\gamma = 0.2$, $\zeta = 0.264$, $\mu = 0.5$, $\rho_0 = 0.36$	$2.005E-11$	0.34	$8.58E-8$	0.38	$1.05E-7$	0.37
$\gamma = 0.2$, $\zeta = 0.5$, $\mu = 0.5$, $\rho_0 = 0.5$	$1.11E-11$	0.34	$5.97E-8$	0.36	$6.94E-8$	0.37
$\gamma = 0.4, \zeta = 0.5, \mu = 0.5$, $\rho_0 = 0.3$	$2.5E-11$	0.35	$8.97E-8$	0.37	$1.26E-7$	0.38

Remark 5. *Based on our numerical experiments, we have the following observation:*

- *Algorithm 1 demonstrates clear superiority over the other algorithms in terms of convergence* *speed and accuracy.*

- *Algorithm 1 converges faster to lower error levels even as the number of iterations increases, reaching an error tolerance of* 10^{-11} *is reached after a few iterations, while the BCNN Alg. 1 and BSJ Alg. 3.1 require significantly more iterations to achieve comparable results.*
- *Compared to BCNN Alg. 1 and BSJ Alg. 3.1, Algorithm 1 performs consistently well across with different parameters. It achieves the highest precision with a reasonable computation time, making it more efficient for large-scale problems.*

6 Conclusions

An algorithm with a simple and uncomplicated structure is presented, which combines the subgradient extragradient method with inertial terms for pseudomonotone equilibrium problems in real Hilbert space. It can be interpreted as an extension of the Algorithm 2.1 in ([21]). The main advantage of this algorithm is using two non-monotonic step size criterion that can work adaptively without the Lipschitz constants and line search technique. The strong convergence is established under appropriate conditions of the equilibrium bifunction B and the control parameters. Application and numerical example are presented to demonstrate the advantages of the proposed algorithm.

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