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A numerical method for solving boundary optimal control problem modeled by heat transfer equation, in the presence of a scale invariance property

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Abstract. In this paper, we present a computational approach for solving a boundary optimal control problem modeled by heat transfer equation with two-point boundary conditions, in the presence of a scale invariance property under dilation. First, we establish a scale-invariant solution. Indeed, the dependence of this solution towards a scale invariance factor naturally leads to an optimal control problem. Second, we propose a numerical approach to solve this problem. The idea consists in transforming the problem into an optimal control problem modeled by a system of ordinary differential equations invariant under dilation using the finite difference approximation. Therefor, the minimum principle of Pontryagin is applied to derive the necessary optimality conditions that are solved by the vartiational iteration method to get an approximate scale-invariant solutions for the optimal control law. Finally, to show the efficiency of this approach, a numerical example is illustrated and comparison with another method is performed.

Keywords: Optimal control, heat-transfer equation, scale invariance, iteration variational method, minimum principle of Pontryagin.
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1 Introduction and problem formulation

Optimal control problems modeled by partial differential equation (PDEs) play an important role in a range of application areas including in both science and engineering, such as reaction-diffusion PDEs via

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weak variations, diffusion and structural vibrations, control thermoelastic plate governed by the thermal equation and underground water flow, etc. (see, e.g. [2, 21, 24, 29, 37]). The objective of optimal control theory modeled by PDEs is to determine the control signals that will cause a process to satisfy the physical constraints governed by PDEs and at the same time maximize or minimize some performance criterion.

In this area, several numerical approaches have been developed to solve optimal control problem modeled by PDEs. These approaches can be divided into two classes. First, the direct approach which consists of discretizing the optimal control problem as it is posed and converting into a static optimization problem which can be solved by deterministic or stochastic optimization methods, for details see [1, 3, 9, 18, 24, 25, 31–33] . Second, the indirect approach which consists of solving the problem indirectly by converting the optimal control problem into a boundary value problem [4, 12, 22, 23]. The indirect approach goes through two steps: optimization then discretization. The optimization step consists of deriving the necessary conditions of optimality using the minimum principle of Pontryagin [28], on the other hand the discretization step consists of approaching the solution of the necessary conditions of optimality obtained in the form of a boundary value problem, called the Hamilton-Jacobi-Bellman (HJB) equation [5]. Then, the HJB equation can be easily solved using suitable numerical methods [13, 22].

For optimal control problem described by PDEs, obtaining the solution of the necessary optimality conditions is difficult due to the complexity of the equations to be manipulated. To overcome this issue, these systems are generally approached by optimal control systems described by ordinary differential equations (ODEs) using among others: the semi-discretization technique [6, 9] parameterization technique by orthogonal functions [11].

In recent years, semi-analytical methods for solving differential equations have been proposed in the literature, see e.g. [19, 20, 38]. These methods give successive approximations which quickly converge to the exact solution if it exists. In the case where the exact solution is difficult to calculate or cannot be obtained using elementary processes, these methods give an approximate solution in the form of a truncated series with very high precision. The solution is obtained using iterative processes by choosing an initial approximation determined by considering the initial conditions or boundary conditions of the problem considered. The variational iteration method (VIM) developed by [14] has been in the last two decades, one of the most used semi-analytical methods to provide an approximate analytical solution of differential equations using an iterative scheme [14–17, 30].

The VIM is widely used to solve efficiently and accurately a large class of optimal control problems modeled by both ODEs and PDEs by determining Lagrange multipliers in order to obtain an approximate analytical solution in the form of an infinite series [22, 23, 38]. The terms of the series are determined using correction functional by introducing the Lagrange multiplier [14–16], which can be identified using the calculus of variations theory. Moreover, the VIM method starts by considering an initial solution, which is chosen so that the boundary conditions are verified to ensure a rapid convergence. Indeed, several studies reported in the literature reveal good results on the convergence of VIM, see e.g. [26, 34–36, 39]. However, these methods are still not applied for optimal control problem governed by PDEs in the presence of scale invariance property due to its complexity, which motivates this work.

Scale invariance is a property that we encounter in many branches of physics, from physical system behavior, particularly in the phase transition, to high energy physics, see e.g. [10, 40]. This property generally applies to physical systems modeled by PDEs such as the heat-transfer equation [6–8, 27]. It is well known that the heat-transfer equation is scale invariant under the following transformations: spatial translations, differentiation, integration, and dilation. However, we will focus on a dilation for the 1D-heat transfer equation. This property appears when the equation owns a particularly behavior during dilation in time and space which allows indeed to considerably simplify the state equation which no longer depends on a single parameter. Moreover, this scale invariance also has consequences on the properties of dynamic system.

In this paper, we focus on an optimal control problem modeled by 1D-heat transfer equation with two-point boundary conditions, in the presence of a scale invariance property under dilation, which can be formulated as follows

$$\min_{u_0,u_\ell} J(u_0,u_\ell) = \int_0^{\alpha^2 t_f}\int_0^{\alpha\ell} T^2(x,t,\alpha)\,dx\,dt + \int_0^{\alpha^2 t_f}\left[q_0\,u_0^2(t,\alpha) + q_\ell\,u_\ell^2(t,\alpha)\right]dt, \tag{1}$$

subject to the 1D-heat transfer equation invariant under dilation

$$\frac{\partial T}{\partial t}(x,t,\alpha) = \frac{\partial^2 T}{\partial x^2}(x,t,\alpha) \qquad \text{in Q}, \tag{2}$$

with the initial and final conditions

$$T(x,0,\alpha) = T_0(x,\cdot) \qquad \text{in } \Omega\times\mathbb{R}_+^\star, \tag{3}$$

$$T(x,t_f,\alpha) = T_{t_f}(x,\cdot) \qquad \text{in } \Omega\times\mathbb{R}_+^\star, \tag{4}$$

and with two-points boundary conditions

$$T(0,t,\alpha) = u_0(t,\cdot) \qquad \text{on } \Sigma, \tag{5}$$

$$T(\ell,t,\alpha) = u_\ell(t,\cdot) \qquad \text{on } \Sigma, \tag{6}$$

where $Q = \Omega\times(0,t_f)\times\mathbb{R}_+^\star$, $\Sigma = \partial\Omega\times(0,t_f)\times\mathbb{R}_+^\star$ with $\Omega = (0,\ell)$ is a segment of the real axis, q_0 and q_ℓ are positive weighting factors, and $\alpha > 0$ is the scale factor. $T_0(x,\alpha)$ and $T_{t_f}(x,\alpha)$ are the initial and final scale-invariant temperature at the time $t = 0$ and $t = t_f$, respectively. The function $u_0(t,\alpha)$ and $u_\ell(t,\alpha)$ are the scale-invariant control variables on the two-point boundary conditions.

The objective is to determine the optimal value of scale-invariant control $u_0(t,\alpha)$ and $u_\ell(t,\alpha)$ used at the boundaries to reach the final scale-invariant temperature $T_{t_f}(x,\alpha)$, while minimizing the square of temperature over the rod, in the presence of scale invariance property under dilation.

To solve the above optimal control problem, first we propose to transform the problem into a control problem modeled by ODEs [12]. By using the finite difference approximation, we discretize the scale-invariant PDE (2) in space direction to obtain a system of ODEs. Then, we apply the trapezoidal rule for the performance index (1) to obtain a weighting quadratic performance index constituted by state and control variables. Second, in order to get the solution of the our optimal control problem, we use the minimum principle of Pontryagin [28] to deduce the necessary conditions of optimality which can be solved easily by the VIM [14, 22, 23] to obtain the optimal scale-invariant control law $u_0(t,\alpha)$ and $u_\ell(t,\alpha)$.

The rest of our paper is structured as follows. In Section 2 we first construct a scale-invariant solutions for the 1D-heat transfer equation. In Section 3 we present our numerical approach of solving boundary optimal control problem modeled by 1D heat transfer equation, in the presence of scale invariance property under dilation. Furthermore, a numerical example is illustrated in Section 4 to support our proposed approach. Section 5 consists the conclusion.

2 Scale-Invariant solutions of the heat transfer equation

It is well known that the scale invariance appears when the PDE has a particular behavior during a dilation in time and space. More particularly, the 1D-heat transfer equation has a natural scaling invariance [10]. Therefore, the scale invariance we deal with reads as follows.
We consider the Cauchy problem of 1D-heat transfer equation

$$\begin{cases} \dfrac{\partial T}{\partial t}(x,t) = \dfrac{\partial^2 T}{\partial x^2}(x,t), & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+^{\star}, \\ T(x,0) = T_0(x), & \forall x \in \mathbb{R}, \end{cases} \tag{7}$$

where $T_0(x)$ is the initial temperature at the time $t = 0$.

Let now α be a strictly positive real number. Moreover, under the dilation scaling

$$x \mapsto \alpha x, \ t \mapsto \alpha^2 t,$$

the 1D-heat transfer equation (7) is unchanged. More precisely, if $T(t,x)$ is an analytical solution of (7), then for any scale factor $\alpha > 0$, the mapping

$$(x,t) \mapsto T(x,t,\alpha) = \alpha T(\alpha^2 t, \alpha x),$$

is also a solution by dilation.

The scale-invariant solution $T(x,t,\alpha)$, which depends on the space variable x, the time variable t, and the scale factor α, is given by Benalia et al. [6] as follows :

$$T(x,t,\alpha) = T(x,t) + \varepsilon \sum_{j=1}^{N_0} \frac{1}{\alpha} T\left(\frac{x}{\alpha^j}, \frac{t}{\alpha^{2j}}\right), \text{ with } \quad \varepsilon \in \{-1,+1\}, N_0 \in \mathbb{N}^{\star}.$$

Then, we have

$$\frac{\partial T}{\partial t}(x,t,\alpha) = \frac{\partial T}{\partial t}(x,t) + \frac{\varepsilon}{\alpha} \sum_{j=1}^{N_0} \frac{\partial}{\partial t}\left[T\left(\frac{x}{\alpha^j}, \frac{t}{\alpha^{2j}}\right)\right].$$

Consequently, the dependence of this solution to the scale factor α, naturally leads to a control problem modeled by 1D-heat transfer equation invariant under dilation with two-points boundary conditions as defined by (1)–(6).

In the following section, a numerical approach for solving optimal control problem (1)–(6) is proposed.

3 Proposed approach

The proposed approach for solving optimal control problem (1)–(6) can be detailed as follows.

3.1 Finite difference and trapezoidal approximation

To transform problem (1)–(6) into an optimal control problem modeled by a system of ODEs invariant under dilation, we discretize the 1D-heat transfer equation (2) in the spatial direction x. By dividing the interval $[0,\alpha\ell]$ into N intervals of length $h=\frac{\alpha\ell}{N}$, the discrete points are given as

$$x_k = kh, \quad k=0,\dots,N. \tag{8}$$

If we denote $T(x,t,\alpha)=\alpha T(x_k,\alpha^2 t)=T_k(\tau)$ with $\tau=\alpha^2 t$, and by using the central difference technique for $\frac{\partial^2 T_k(\tau)}{\partial x^2}$, we obtain

$$\frac{\partial^2 T_k(\tau)}{\partial x^2}=\frac{1}{h^2}(T_{k+1}(\tau)-2T_k(\tau)+T_{k-1}(\tau)), \quad k=1,\dots,N. \tag{9}$$

Next, we approximate the first derivative with respect to t, $\frac{\partial T(x_k,\alpha^2 t)}{\partial t}$ by $\dot{T}_k(\tau)$, then the PDE (2) becomes as

$$\dot{T}_k(\tau)=\frac{1}{h^2}(T_{k+1}(\tau)-2T_k(\tau)+T_{k-1}(\tau)), \quad k=1,\dots,N, \quad \tau=\alpha^2 t, \tag{10}$$

which yields the following system of ODEs under dilation

$$\begin{cases} \dot{T}_1(\tau)=\frac{1}{h^2}(T_2(\tau)-2T_1(\tau)+T_0(\tau)), \\ \dot{T}_k(\tau)=\frac{1}{h^2}(T_{k+1}(\tau)-2T_k(\tau)+T_{k-1}(\tau)), \qquad k=2,\dots,N-2, \\ \dot{T}_{N-1}(\tau)=\frac{1}{h^2}(T_N(\tau)-2T_{N-1}(\tau)+T_{N-2}(\tau)), \end{cases} \tag{11}$$

with the initial and final conditions (3)–(4) becomes as

$$T(x_k,0)=T_k(0), \tag{12}$$

$$T(x_k,\alpha^2 t_f)=T_k(\alpha^2 t_f), \tag{13}$$

and the two-points boundary conditions (5)–(6) becomes as

$$T_0(\tau)=u_0(\tau), \tag{14}$$

$$T_N(\tau)=u_\ell(\tau). \tag{15}$$

Moreover, by using the trapezoidal approximation to the double integral (11) in the x-direction, the performance index (1) becomes as

$$J(u_0,u_\ell)=(\frac{h}{2}+q_0)\int_0^{\alpha^2 t_f} u_0^2(\tau)d\tau+h\sum_{k=1}^{N-1}\int_0^{\alpha^2 t_f} T_k^2(\tau)d\tau+(\frac{h}{2}+q_l)\int_0^{\alpha^2 t_f} u_\ell^2(\tau)d\tau. \tag{16}$$

In summary, by using (11)–(16), the initial optimal control problem (1)–(6) can be reduced to an optimal control problem modeled by a system of ODEs invariant under dilation, which is given as follows :

$$\min_{u_0,u_\ell} J(u_0,u_\ell)=(\frac{h}{2}+q_0)\int_0^{\alpha^2 t_f} u_0^2(\tau)d\tau+h\sum_{k=1}^{N-1}\int_0^{\alpha^2 t_f} T_k^2(\tau)d\tau+(\frac{h}{2}+q_l)\int_0^{\alpha^2 t_f} u_\ell^2(\tau)d\tau. \tag{17}$$

subject to the system of ODEs invariant under dilation

$$\begin{cases} \dot{T}_1(\tau) = \dfrac{1}{h^2}(T_2(\tau) - 2T_1(\tau) + u_0(\tau)), \\ \dot{T}_k(\tau) = \dfrac{1}{h^2}(T_{k+1}(\tau) - 2T_k(\tau) + T_{k-1}(\tau)), \qquad k = 2, \ldots, N-2, \\ \dot{T}_{N-1}(\tau) = \dfrac{1}{h^2}(u_\ell(\tau) - 2T_{N-1}(\tau) + T_{N-2}(\tau)), \end{cases} \tag{18}$$

with boundary conditions

$$T(x_k, 0) = T_k(0), \;\; T(x_k, \alpha^2 t_f) = T_k(\alpha^2 t_f), \qquad k = 1, \ldots, N-1. \tag{19}$$

3.2 Minimum principle of Pontryagin

The minimum principle of Pontryagin developed in [28] is a method that allows deriving the necessary optimality conditions for an extremum of the optimal control problem. According to the minimum principle of Pontryagin, the solution of the optimal control problem (18)–(19) is determined by minimizing the Hamiltonian [28] which is defined by

$$H(T(\tau), u(\tau), p(\tau)) = (\frac{h}{2} + q_0)\, u_0^2(\tau) + h\sum_{k=1}^{N-1} T_k^2(\tau) + (\frac{h}{2} + q_l)\, u_\ell^2(\tau) + p(\tau)^T f(T(\tau), u(\tau)), \tag{20}$$

where $p(\tau) \in \mathbb{R}^{N-1}$ is the adjoint vector and $f : \mathbb{R}^{N-1} \times \mathbb{R}^2 \to \mathbb{R}^{N-1}$ is a vector function. System (18) can be written as

$$\dot{T}(\tau) = f(T(\tau), u(\tau)), \;\; T(\tau) \in \mathbb{R}^{N-1}, \; u(\tau) \in \mathbb{R}^2.$$

Then, the optimal control law is given by the minimisation of the Hamiltonian (22) as follows :

$$\begin{cases} \dfrac{\partial H}{\partial u_0}(T(\tau), u(\tau), p(\tau)) = 0, \\ \dfrac{\partial H}{\partial u_\ell}(T(\tau), u(\tau), p(\tau)) = 0, \end{cases} \tag{21}$$

which yields

$$\begin{cases} u_0(\tau) = -\dfrac{p_1(\tau)}{h^3 + 2h^2 q_0}, \\ u_\ell(\tau) = -\dfrac{p_{N-1}(\tau)}{h^3 + 2h^2 q_\ell}. \end{cases} \tag{22}$$

Substituting the expression of the optimal control law (22) into the Hamiltonian (20) and using the minimum principle of Pontryagin, the Hamilton-Pontryagin equations are given as

$$\begin{cases} \dot{T}(\tau) = \dfrac{\partial H}{\partial p(\tau)}(T(\tau), p(\tau)), \\ \dot{p}(\tau) = -\dfrac{\partial H}{\partial T(\tau)}(T(\tau), p(\tau)), \end{cases} \tag{23}$$

which yields

$$\begin{cases} \dot{T}_1(\tau) = \dfrac{1}{h^2}\left(T_2(\tau) - 2T_1(\tau) - \dfrac{p_1(\tau)}{h^3 + 2h^2 q_0}\right), \\ \dot{T}_k(\tau) = \dfrac{1}{h^2}\left(T_{k+1}(\tau) - 2T_k(\tau) + T_{k-1}(\tau)\right), \qquad k = 2, \ldots, N-2, \\ \dot{T}_{N-1}(\tau) = \dfrac{1}{h^2}\left(-\dfrac{p_N(\tau)}{h^3 + 2h^2 q_\ell} - 2T_{N-1}(\tau) + T_{N-2}(\tau)\right), \end{cases} \tag{24}$$

$$\begin{cases} \dot{p}_1(\tau) = -\left(2hT_1(\tau) - 2\dfrac{p_1(\tau)}{h^2} + \dfrac{p_2(\tau)}{h^2}\right), \\ \dot{p}_k(\tau) = -\left(2hT_k(\tau) + \dfrac{p_{k-1}(\tau)}{h^2} - 2\dfrac{p_k(\tau)}{h^2} + \dfrac{p_{k+1}(\tau)}{h^2}\right), \qquad k = 2, \ldots, N-2, \\ \dot{p}_{N-1}(\tau) = -\left(2hT_{N-1}(\tau) + \dfrac{p_{N-2}(\tau)}{h^2} - 2\dfrac{p_{N-1}(\tau)}{h^2}\right). \end{cases} \tag{25}$$

3.3 Application of He's VIM

3.3.1 A brief description

Before applying VIM, it is important to illustrate its basic principle. Consider the following differential equation, written in operator form

$$\mathscr{L}T(\tau) + \mathscr{N}T(\tau) = g(\tau), \tag{26}$$

where $\mathscr{L} = \dfrac{d^m}{d\tau^m}$, $m \in \mathbb{N}$ is a linear operator, $\mathscr{N}$ is a nonlinear operator and $g(\tau)$ is the inhomogeneous term.

To obtain the solution of equation (26), we construct the following correction functional [14, 16]

$$T^{n+1}(\tau) = T^n(\tau) + \int_0^\tau \lambda(s)\left(\mathscr{L}T^n(s) + \mathscr{N}\tilde{T}^n(s) - g(s)\right)ds, \tag{27}$$

where $\lambda(s)$ is the Lagrange multiplier [14–16], which can be determined using the stationary conditions [16], and $\tilde{T}^n(s)$ is a restricted variation, which means that $\delta\tilde{T}^n(s) = 0$.
By imposing the variational with respect to T^n in (27), we obtain

$$\delta T^{n+1}(\tau) = \delta T^n(\tau) + \delta\left(\int_0^\tau \lambda(s)\mathscr{L}T^n(s)\,ds\right). \tag{28}$$

In our case, we take $\mathscr{L} = \dfrac{d}{d\tau}(\cdot)$. Then (28) becomes as follows

$$\delta T^{n+1}(\tau) = \delta T^n(\tau) + \delta\left(\int_0^\tau \lambda(s)\dot{T}^n(s)\,ds\right). \tag{29}$$

By using integration by parts for (29), we obtain the following stationary conditions

$$\begin{cases} \dot{\lambda}(s)|_{s=\tau} = 0, \\ 1 + \lambda(s)|_{s=\tau} = 0, \end{cases} \tag{30}$$

which yields

$$\lambda(s) = -1. \tag{31}$$

Generally, for mth order ODE

$$\lambda(s) = \frac{(-1)^m}{(m-1)!}(s-\tau)^{m-1}, \quad \text{for } m \geq 1. \tag{32}$$

Once the Lagrange multiplier $\lambda(s)$ is determined, and by choosing a zeroth approximation $T^0(\tau)$, the successive approximations $T^n(\tau), n > 0$, of the solution $T(\tau)$ are determined using the correction functional (22). Consequently, the solution of (26) is given by

$$T(\tau) = \lim_{n\to\infty} T^n(\tau). \tag{33}$$

3.3.2 Convergence results

In this section, we present the convergence results of VIM series solution for the system of ODEs [26]. The necessary optimality conditions (24) of our optimal control problem can be formulated in operator form as follows

$$\begin{cases} \dot{T}_1(\tau) + \mathcal{N}_1(T_1, T_2, \ldots, T_n) = g_1(\tau), \\ \dot{T}_2(\tau) + \mathcal{N}_2(T_1, T_2, \ldots, T_n) = g_2(\tau), \\ \qquad\qquad\vdots \\ \dot{T}_n(\tau) + \mathcal{N}_2(T_1, T_2, \ldots, T_n) = g_n(\tau), \end{cases} \tag{34}$$

subject to the initial conditions

$$T_i(0) = c_i, \quad i = 1,\, 2, \ldots, n, \tag{35}$$

where $n \in \mathbb{N}$, $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n$ are nonlinear operators, $g_1(\tau), g_2(\tau), \ldots, g_n(\tau)$ are known analytic functions, and c_i's are real numbers.

We have the following convergence result.

Theorem 1. *Let us consider the system (34)–(35). Then, the variational iteration solution*

$$(T_1(\tau), T_2(\tau), \ldots, T_n(\tau)) = \left(\sum_{k=0}^{\infty} v_{1,k}(\tau), \sum_{k=0}^{\infty} v_{2,k}(\tau), \ldots, \sum_{k=0}^{\infty} v_{n,k}(\tau) \right), \tag{36}$$

obtained using the following iterative formula

$$\begin{cases} v_{i,0}(\tau) = c_i, \\ v_{i,k+1}(\tau) = -1 \displaystyle\int_0^{\tau} \left(\frac{d}{d\tau}[v_{i,0} + \ldots + v_{i,k}](s) \right. \\ \qquad \left. + \mathcal{N}[(v_{1,0} + \ldots + v_{1,k}), (v_{2,0} + \ldots + v_{2,k}), \ldots, (v_{n,0} + \ldots + v_{n,k})](s) - g_i(s)) \right) ds, \end{cases} \tag{37}$$

for $i = 1,\, 2, \ldots, n$, converges to a solution of system (34)–(35) if $\exists\, 0 < \gamma_1, \gamma_2, \ldots, \gamma_n < 1$ such that

$$\| v_{i,k+1} \| \leq \| v_{i,k} \|, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{38}$$

Proof. We refer to the proof of [26, Theorem 1 and Theorem 2]. □

Note that Theorem 1 is a special case of result developed by Odibat [26], as a sufficient condition for convergence of VIM for nonlinear differential equations.

3.3.3 Solving Hamilton-Pontryagin equations (24)–(25) using VIM

Here, we use the VIM to solve the Hamilton-Pontryagin equations (24)–(25). Consequently, the correction functional is given as

$$\begin{cases} T_1^{n+1}(\tau) = T_1^n(\tau) + \int_0^\tau \lambda_{1T}\, \dot{T}_1^n(s)\, ds - \int_0^\tau \lambda_{1T} \left[\frac{1}{h^2} \left(\tilde{T}_2^n(s) - 2\tilde{T}_1^n(s) + \frac{\tilde{p}_1^n(s)}{h^3 + 2h^2 q_0} \right) \right] ds, \\ T_k^{n+1}(\tau) = T_k^n(\tau) + \int_0^\tau \lambda_{kT}\, \dot{T}_k^n(s)\, ds \\ \qquad - \int_0^\tau \lambda_{kT} \left[\frac{1}{h^2} \left(\tilde{T}_{k+1}^n(s) - 2\tilde{T}_k^n(s) + \tilde{T}_{k-1}^n(s) \right) \right] ds, \qquad k = 2, \ldots, N-2, \\ T_{N-1}^{n+1}(\tau) = T_{N-1}^n(\tau) + \int_0^\tau \lambda_{N-1T}\, \dot{T}_{N-1}^n(s)\, ds \\ \qquad + \int_0^\tau \lambda_{N-1T} \left[\frac{1}{h^2} \left(\frac{\tilde{p}_{N-1}^n(s)}{h^3 + 2h^2 q_\ell} + 2\tilde{T}_{N-1}^n(s) - \tilde{T}_{N-2}^n(s) \right) \right] ds, \end{cases} \tag{39}$$

$$\begin{cases} p_1^{n+1}(\tau) = p_1^n(\tau) + \int_0^\tau \lambda_{1p}\, \dot{p}_1^n(s)\, ds + \int_0^\tau \lambda_{1p} \left[2h\tilde{T}_1^n(s) - 2\frac{\tilde{p}_1^n(s)}{h^2} + \frac{\tilde{p}_2^n(s)}{h^2} \right] ds, \\ p_k^{n+1}(\tau) = p_k^n(\tau) + \int_0^\tau \lambda_{kp}\, \dot{p}_k^n(s)\, ds \\ \qquad + \int_0^\tau \lambda_{kp} \left[2h\tilde{T}_k^n(s) + \frac{\tilde{p}_{k-1}^n(s)}{h^2} - 2\frac{\tilde{p}_k^n(s)}{h^2} + \frac{\tilde{p}_{k+1}^n(s)}{h^2} \right] ds, \qquad k = 2, \ldots, N-2, \\ p_{N-1}^{n+1}(\tau) = p_{N-1}^n(\tau) + \int_0^\tau \lambda_{N-1p}\, \dot{p}_{N-1}^n(s)\, ds \\ \qquad + \int_0^\tau \lambda_{N-1p} \left[2h\tilde{T}_{N-1}^n(s) + \frac{\tilde{p}_{N-2}^n(s)}{h^2} - 2\frac{\tilde{p}_{N-1}^n(s)}{h^2} \right] ds, \end{cases} \tag{40}$$

where $\lambda_{kT} = \lambda_{kp} = -1_{N+1}$, with $k = 0, \ldots, N-1$ are general Lagrange multipliers, which are obtained by using the variational theory, and $\tilde{T}(\tau)$ and $\tilde{p}(\tau)$ are restricted variations that means $\delta\tilde{T}(\tau) = 0$, $\delta\tilde{p}(\tau) = 0$. By substituting the value of the general Lagrange multipliers into (39)–(40), the successive approximations $T_k^{n+1}(\tau), p_k^{n+1}(\tau)$, $k = 1, \ldots, N-1$ of the solution $T_k(\tau)$ and $p_k(\tau)$, respectively will follow immediately.

3.4 Algorithm for the proposed approach

The proposed algorithm for solving the optimal control problem (1)–(6) can be summarized as followings:

Step (1). Transform the 1D-heat transfer equation (2) into a system of ODEs invariant under dilation by using the finite difference approximation.

Step (2). Transform the double integral (1) into a one integral, by using the trapezoidal approximation.
Step (3). Determine the necessary optimality conditions for the an approximation of $J(u_0(\tau), u_\ell(\tau))$, by using the minimum principle of Pontryagin.
Step (4). Solve Hamilton-Pontryagin equations (24)–(25), by using VIM [14] to determine the successive approximations

$$T_k^{n+1}(\tau), p_k^{n+1}(\tau), \ \ k = 1, \dots, N-1.$$

Step (5). Set $n = 0$, $T^0(\tau) = 0$ and $p^\tau = A$ is a vector which can be determined by using the boundary conditions.
Step (6). Determine the optimal control law $u_0^{n+1}(\tau)$ and $u_\ell^{n+1}(\tau)$ by using the relationships (21) and (22), respectively.
Step (7). Evaluate the performance index $J(u_0^{n+1}(\tau), u_\ell^{n+1}(\tau))$.
Step (8). Stopping criterion

if $|J(u_0^{n+1}(\tau), u_\ell^{n+1}(\tau)) - J(u_0^n(\tau), u_\ell^n(\tau))| \leq \varepsilon$, where ε is the desired threshold, Stop,
else, set n=n+1 and go to step (4).

We note that, the number of iterations n needed to get the approximate analytical solution is determined based on the objective function (1).

4 Numerical example, results and discussions

In this section, an illustrative numerical example is treated. Moreover, to show the efficiency of the our proposed approach, a comparison is made between the obtained results by the presented approach with those of the shooting method [18].

We take $\ell = \pi$, $t_f = \frac{3}{2}$, $q_0 = q_\ell = 0$, $T_0 = 0$ and $T_{t_f} = \frac{3}{2}$. Then, the boundary optimal control problem (1)–(6), can be written as:

$$\min_{u_0, u_\ell} J = \int_0^{\frac{3}{2}\alpha^2} \int_0^{\alpha\pi} T^2(x,t,\alpha)\, dx\, dt + \int_0^{\frac{3}{2}\alpha^2} \left[u_0^2(t,\alpha) + u_\ell^2(t,\alpha)\right] dt, \tag{41}$$

subject to the 1D-Heat transfer equation invariant under dilation

$$\frac{\partial T}{\partial t}(x,t,\alpha) = \frac{\partial^2 T}{\partial x^2}(x,t,\alpha), \qquad \text{in } Q, \tag{42}$$

with the initial and final conditions

$$T(x,0,\alpha) = T_0(x,\alpha) = 0, \qquad \text{in } \ \Omega \times \mathbb{R}_+^\star, \tag{43}$$

$$T(x,\frac{3}{2},\alpha) = T_{t_f=\frac{3}{2}}(x,) = \frac{3}{2}, \qquad \text{in } \ \Omega \times \mathbb{R}_+^\star, \tag{44}$$

and with two-points boundary conditions

$$T(0,t,\alpha) = u_0, \qquad \text{on } \ \Sigma, \tag{45}$$

$$T(\pi,t,\alpha) = u_\ell, \qquad \text{on } \ \Sigma. \tag{46}$$

In practice, we choose the number of discretizations $N = 5$, then we have $h = \frac{\alpha\pi}{5}$. Consequently, we obtain the following boundary optimal control modeled by a system of ODEs invariant under dilation

$$\min_{u_0,u_\ell} J = \frac{\alpha\pi}{10}\int_0^{\frac{3}{2}\alpha^2} u_0^2(\tau)\,d\tau + \frac{\alpha\pi}{5}\sum_{k=1}^{N-1}\int_0^{\frac{3}{2}\alpha^2} T_k^2(\tau)\,d\tau + \frac{\alpha\pi}{10}\int_0^{\frac{3}{2}\alpha^2} u_l^2(\tau)\,d\tau, \tag{47}$$

subject to

$$\begin{cases} \dot{T}_1(\tau) = \left(\frac{5}{\alpha\pi}\right)^2 \Big(T_2(\tau) - 2T_1(\tau) + u_0(\tau)\Big), \\ \dot{T}_k(\tau) = \left(\frac{5}{\alpha\pi}\right)^2 \Big(T_{k+1}(\tau) - 2T_k(\tau) + T_{k-1}(\tau)\Big), \qquad k = 2,\dots,N-2, \\ \dot{T}_{N-1}(\tau) = \left(\frac{5}{\alpha\pi}\right)^2 \Big(u_\ell(\tau) - 2T_{N-1}(\tau) + T_{N-2}(\tau)\Big), \end{cases} \tag{48}$$

with boundary conditions

$$T_k(0) = 0, \;\; T_k(\tfrac{3\alpha^2}{2}) = \tfrac{3}{2}, \;\; k = 1,\dots,N-1. \tag{49}$$

According to VIM, the iterative scheme is given as

$$\begin{cases} T_1^{n+1}(\tau) = T_1^n(\tau) - \int_0^\tau \dot{T}_1^n(s)\,ds + \int_0^\tau \left(\frac{5}{\alpha\pi}\right)^2 \left(\tilde{T}_2^n(s) - 2\tilde{T}_1^n(s) - \frac{\tilde{p}_1^n(s)}{(\frac{\alpha\pi}{5})^3}\right) ds, \\ T_2^{n+1}(\tau) = T_2^n(\tau) - \int_0^\tau \dot{T}_2^n(s)\,ds + \int_0^\tau \left(\frac{5}{\alpha\pi}\right)^2 \left(\tilde{T}_3^n(s) - 2\tilde{T}_2^n(s) + \tilde{T}_1^n(s)\right) ds, \\ T_3^{n+1}(\tau) = T_3^n(\tau) - \int_0^\tau \dot{T}_3^n(s)\,ds + \int_0^\tau \left(\frac{5}{\alpha\pi}\right)^2 \left(\tilde{T}_4^n(s) - 2\tilde{T}_3^n(s) + \tilde{T}_2^n(s)\right) ds, \\ T_4^{n+1}(\tau) = T_4^n(\tau) - \int_0^\tau \dot{T}_4^n(s)\,ds + \int_0^\tau \left(\frac{5}{\alpha\pi}\right)^2 \left(\tilde{T}_3^n(s) - 2\tilde{T}_4^n(s) - \frac{\tilde{p}_4^n(s)}{(\frac{\alpha\pi}{5})^3}\right) ds, \end{cases} \tag{50}$$

$$\begin{cases} p_1^{n+1}(\tau) = p_1^n(\tau) - \int_0^\tau \dot{p}_1^n(s)\,ds - \int_0^\tau \left(\frac{2\alpha\pi}{5}\tilde{T}_1^n(s) - 2\frac{\tilde{p}_1^n(s)}{(\frac{\alpha\pi}{5})^2} + \frac{\tilde{p}_2^n(s)}{(\frac{\alpha\pi}{5})^2}\right) ds, \\ p_2^{n+1}(\tau) = p_2^n(\tau) - \int_0^\tau \dot{p}_2^n(s)\,ds - \int_0^\tau \left(\frac{2\alpha\pi}{5}\tilde{T}_2^n(s) + \frac{\tilde{p}_1^n(s)}{(\frac{\alpha\pi}{5})^2} - 2\frac{\tilde{p}_2^n(s)}{(\frac{\alpha\pi}{5})^2} + \frac{\tilde{p}_3^n(s)}{(\frac{\alpha\pi}{5})^2}\right) ds, \\ p_3^{n+1}(\tau) = p_3^n(\tau) - \int_0^\tau \dot{p}_3^n(s)\,ds - \int_0^\tau \left(\frac{2\alpha\pi}{5}\tilde{T}_3^n(s) + \frac{\tilde{p}_2^n(s)}{(\frac{\alpha\pi}{5})^2} - 2\frac{\tilde{p}_3^n(s)}{(\frac{\alpha\pi}{5})^2} + \frac{\tilde{p}_4^n(s)}{(\frac{\alpha\pi}{5})^2}\right) ds, \\ p_4^{n+1}(\tau) = p_4^n(\tau) - \int_0^\tau \dot{p}_4^n(s)\,ds - \int_0^\tau \left(\frac{2\alpha\pi}{5}\tilde{T}_4^n(s) + \frac{\tilde{p}_3^n(s)}{(\frac{\alpha\pi}{5})^2} - 2\frac{\tilde{p}_4^n(s)}{(\frac{\alpha\pi}{5})^2}\right) ds. \end{cases} \tag{51}$$

Using the initial conditions of the problem, the zeroth approximation can be selected as $T_k^0(\tau) = 0$ and $p_k^0(\tau) = a_k,\; k = 1,\dots,4$ which can be determined by imposing the boundary conditions. The obtained results can be summarized in Tables 1 and 2.

The obtained results show that the optimal control is determined by assuming a threshold $\varepsilon = 10^{-6}$. Hence we conclude that the our proposed approach converges after 12 iterations. Moreover, from the previous results we can say that our proposed approach gives approximations of the solution which quickly converges towards the exact solution of the problem.

Table 1: Iterations result with $\alpha = 1$.

n	value of a_1	value of a_2	value of a_3	value of a_4	$J(\alpha = 1)$
0	1.345454545	1.970707071	2.818282827	3.243434345	16.849174915
1	0.872727272	1.260606061	2.194949495	2.760606061	16.299193614
2	0.996969696	1.417171717	2.523232323	2.893656565	15.768233852
3	0.960717171	1.381212121	2.490373737	2.858282827	15.673149021
4	0.974343433	1.376252524	2.502727237	2.849090908	15.581137592
5	0.970292929	1.351313130	2.501414242	2.847979787	15.510974193
6	0.989017172	1.350171720	2.500373737	2.846181817	15.449297918
7	0.989161616	1.350292928	2.500484848	2.846272727	15.438176827
8	0.987834342	1.349919192	2.500171717	2.846010101	15.436721790
9	0.987907071	1.350018181	2.500292929	2.846161616	15.436444583
10	0.986262781	1.351919191	2.501553535	2.845151516	15.436424953
11	0.987070706	1.352323232	2.502020212	2.846464647	15.436423592
12	0.988484847	1.353535353	2.504040403	2.847373736	15.436423485

Table 2: Difference between two successive iterations for different values of α.

n	$\|J(u_0^{n+1}(\tau), u_\ell^{n+1}(\tau)) - J(u_0^n(\tau), u_\ell^n(\tau))\|$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 10$
0	–	–	–
1	0.549981301	0.89289566	1.67698260
2	0.530959761	0.52335187	1.04756755
3	0.095084831	0.08661081	0.10156208
4	0.092011429	0.09145199	0.10723897
5	0.070163399	0.06856165	0.08039717
6	0.061676275	0.02453983	0.02877604
7	0.011121091	0.01062745	0.01246203
8	0.001455037	0.00342660	0.00409854
9	0.000277207	0.00014821	0.00083735
10	0.000019630	0.00009170	0.00009973
11	0.000001361	0.00000877	0.00000952
12	0.000000107	0.00000053	0.00000073

In Figures 1, 2 and 3, the approximate solutions obtained from the proposed method and those obtained using the shooting method are plotted which show that the results are very close, (see e.g. [18] for a more detailed discussion of shooting method).

A comparison in terms of convergence between the results obtained by the our presented method with those of the shooting method is illustrated in Table 3.

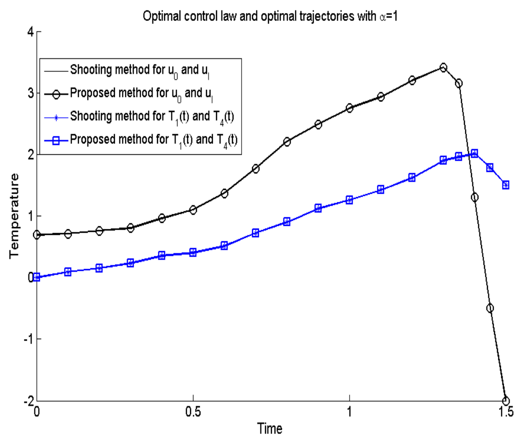

Figure 1: Optimal control law and optimal trajectories with $\alpha = 1$.

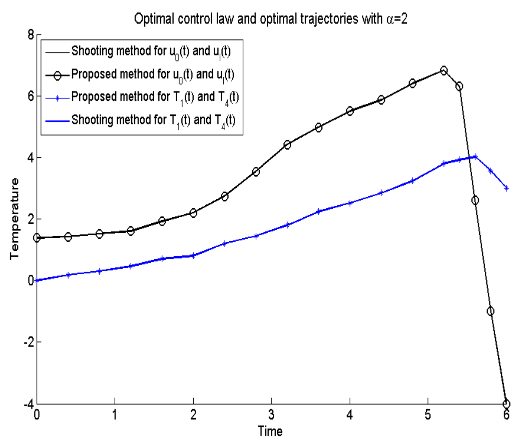

Figure 2: Optimal control law and optimal trajectories with $\alpha = 2$.

From the results obtained, we clearly conclude that the both methods are efficient in terms of convergence. Nevertheless, the advantage of the proposed method lies in the use of the variational iteration method which starts which starts by considering an initial solution, which is chosen such that the boundary conditions are verified to ensure rapid convergence. Unlike the shooting method which suffers from difficulties in finding a rough initial estimate under unspecified conditions. Indeed, the solution is often very sensitive to small changes in the unspecified boundary conditions.

5 Conclusion and perspective

In this work, a new approach based on VIM is employed successfully to determine an approximate solution for boundary optimal control modeled by heat transfer equation with two-point boundary conditions, in the presence of scale invariance property under dilation. Applying the finite difference approximation and trapezoidal integral rule the original problem is transformed to an optimal control problem modeled

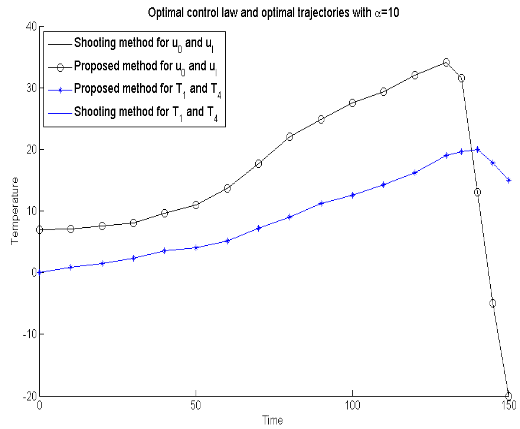

Figure 3: Optimal control law and optimal trajectories with $\alpha = 10$.

Table 3: Difference between two successive iterations for both methods.

n	$\lvert J(u_0^{n+1}, u_\ell^{n+1}) - J(u_0^n, u_\ell^n)\rvert$ with $\alpha = 10$	
	Shooting method	Proposed method
0	–	–
1	1.67698266	1.67698260
2	1.04756759	1.04756755
3	0.10156211	0.10156208
4	0.10723899	0.10723897
5	0.08039719	0.08039717
6	0.02877606	0.02877604
7	0.01246205	0.01246203
8	0.00409855	0.00409854
9	0.00083736	0.00083735
10	0.00009973	0.00009973
11	0.00000952	0.00000952
12	0.00000073	0.00000073

by a system of ODEs invariant under dilation. Then, in order to obtain a scale-invariant approximate analytical solution for the resulting problem, the VIM is adapted to solve the necessary optimality conditions derived by the minimum Principle of Pontryagin.

The proposed approach is illustrated on a numerical example. In fact, the calculated results are in excellent agreement with those obtained directly using the shooting method.

In perspective, the work presented in this paper can be extended for multidimensional heat transfer equation and other nonlinear partial differential equations, in the presence of a scale invariance property.

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