Journal of Mathematical Modeling
Vol. 13, No. 1, 2025, pp. 105-120. Research Article

JMM

Improved feasible value constraint for multiobjective optimization problems

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Abstract. In this paper, we focus on the utilization of the feasible value constraint technique to address multiobjective optimization problems (MOPs). It is attempted to overcome certain drawbacks associated with this method, such as restrictions on functions and weights, inflexibility in constraints, and challenges in assessing proper efficiency. To accomplish this, we propose an improved version of the feasible value constraint technique. Then, by incorporating approximate solutions, we establish connections between ε-(weakly, properly) efficient points in a general MOP and ε-optimal solutions to the scalarization problem.

Keywords: Multiobjective optimization problem, feasible value constraint technique, scalarization techniques, ε-(weakly, properly) efficient solutions.
AMS Subject Classification 2020: 90C20, 90C29.

1 Introduction

The MOPs form a branch of mathematical programming that involves minimizing multiple objective functions over a given set of decisions. Due to the inherent conflicts among the objectives, it is difficult to find a single solution that optimizes all objective functions simultaneously. Therefore, the goal of an MOP is to discover the best possible trade-off among these criteria. Points satisfying a trade-off among conflicting objective functions are referred to as efficient points. The set consisting of the images of these points in the objective space is called an efficient frontier. In [5, 12, 14–16, 20, 27, 28] applications of MOPs in various real-world problems have been studied. Moreover, theory and methodology for solving MOPs have witnessed significant advancements, as evidenced by a range of works found in the literature (see for instance [1,4,7,31]). Scalarization techniques are commonly used strategies for solving MOPs. These approaches entail transforming MOP into a single objective problem, which may involve the incorporation of parameters or additional constraints (see [4, 7, 8] for more details).

*Corresponding author
Received: 29 July 2024 / Revised: 12 September 2024 / Accepted: 25 September 2024
DOI: 10.22124/jmm.2024.28044.2469

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Certain scalarization techniques allow for obtaining a reliable approximation of the efficient frontier by modifying the parameters. Notable examples within this category include the weighted constraint approach [3], the feasible value constraint technique [2], and the objective constraint method [22]. In the feasible value constraint technique, the optimization focuses on a specific k^{th} objective, while the remaining weighted objectives are incorporated as constraints.

An important purpose of the scalarization techniques is to demonstrate relevant conditions that connect the optimal solutions of the scalarization problem with the efficient solutions of the MOP. A renowned illustration of such conditions is the characterization of weakly and properly efficient solutions in convex MOPs, which are identified as the optimal solutions of the weighted sum method with nonnegative and positive weights (for more details we refer the reader to [1,4]). In [2], some necessary and/or sufficient conditions for (weakly) efficient solutions of the feasible value constraint technique have been gained. However, it should be emphasized that this scalarization technique does not yield results regarding proper efficiency. In this paper, we use the idea introduced in [6,21] and present a novel version of the feasible value constraint technique. To do this, we aim to determine various necessary and sufficient conditions for different types of efficient solutions in MOPs without limitations. On the other hand, there has been a growing interest in the past few decades towards approximate efficient solutions of MOPs. There are several valuable reasons for this trend. Firstly, numerical algorithms commonly provide approximate solutions instead of exact ones. Furthermore, in certain circumstances where assumptions such as compactness or boundedness may lead to an empty efficient solution set for the MOP, whereas the approximate efficient solutions set remains nonempty. This is advantageous because it often requires fewer assumptions or weaker hypotheses, making it more applicable in practical situations. The notion of approximate solution was initially introduced by Kutateladze [18]. In [19], Loridan extended this concept, and later, White proposed six types of the approximate solutions in the framework of MOPs [30]. Subsequently, many researchers developed into studying the properties of these approximate solutions, and provided various necessary and sufficient conditions for ε-(weakly, properly) efficient solutions in MOPs (see [9,10,17,21,24]). From a practical perspective, it is worth mentioning the works of Shao and Ehrgott, which employ the application of approximate efficient solutions in the field of radiation therapy process [25,26]. Considering the previous discussions, the study of approximate efficiency for MOPs is significant. It is important to note that the results obtained in this work are applicable and do not depend on any specific convexity assumptions.

The remaining part of this paper is organized in the following manner. Section 2, presents the essential preliminaries are required for the subsequent sections. In Section 3, a comprehensive formulation of the feasible value constraint technique is proposed, and its properties are studied. Section 4, establishes necessary and sufficient conditions for ε-(weakly, properly) efficient solutions in the general MOP. Finally, the paper is closed with some conclusions in the last section.

2 Prerequisites

A general MOP can be formulated as

$$\min_{x\in X} f(x) = (f_1(x),\dots,f_p(x)), \tag{1}$$

where each f_i, for $1 \leqslant i \leqslant p$ denotes a real-valued function defined on $\mathbb{R}^n$ (the decision space), and $X \subset \mathbb{R}^n$ is a non-empty feasible set. Let us express some fundamental definitions for a better understanding.

Definition 1 ([4]). Let $f(x),\ f(\hat{x}) \in \mathbb{R}^p$, where $x,\ \hat{x} \in X$. Then

(1) $f(x) \leqq f(\hat{x})$ if and only if $f_i(x) \leqslant f_i(\hat{x})$ for all $i = 1, \dots, p$,

(2) $f(x) < f(\hat{x})$ if and only if $f_i(x) < f_i(\hat{x})$ for all $i = 1, \dots, p$,

(3) $f(x) \leq f(\hat{x})$ if and only if $f(x) \leqq f(\hat{x})$ and $f(x) \neq f(\hat{x})$.

Definition 2 ([13]). The Hadamard product of two vectors $U, V \in \mathbb{R}^p$ is defined by $U \circ V = (u_1 v_1, \dots, u_p v_p)^T$.

Definition 3 ([4]). Consider the MOP (1). The feasible solution $\hat{x} \in X$ is called

(1) Weakly efficient solution if there is no another $x \in X$ such that $f(x) < f(\hat{x})$,

(2) Efficient solution if there is no another $x \in X$ such that $f(x) \leq f(\hat{x})$.

Definition 4 ([4]). A feasible solution $\hat{x} \in X$ is said to be a properly efficient solution of the MOP (1), if it is an efficient solution and there exists a positive constant M such that for each $1 \leqslant i \leqslant p$ and for any $x \in X$ with $f_i(x) < f_i(\hat{x})$, there exists $1 \leqslant j \leqslant p$ such that $f_j(x) > f_j(\hat{x})$, and the following inequality holds

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leqslant M.$$

In the rest of the paper, we will denote weakly efficient solutions, efficient solutions, and properly efficient solutions as X_{wE}, X_E and X_{pE}, respectively.

Definition 5 ([29]). Let $\varepsilon \geqslant 0$. Consider a real-valued function h defined on $X \subseteq \mathbb{R}^n$. A feasible point $\hat{x} \in X$ is referred to as an ε-optimal solution for the problem $\min_{x \in X} h(x)$, if $h(\hat{x}) - \varepsilon \leqslant h(x)$ for all $x \in X$.

Definition 6 ([21]). Consider $\varepsilon \in \mathbb{R}^p_{\geqq} = \{x \in \mathbb{R}^p \mid x \geqq 0\}$. A feasible point $\hat{x} \in X$ for the MOP (1) is called

(1) ε-Weakly efficient solution if there is no other $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$,

(2) ε-Efficient solution if there is no other $x \in X$ such that $f(x) \leq f(\hat{x}) - \varepsilon$.

Definition 7 ([21]). A feasible point $\hat{x} \in X$ is called ε-properly efficient solution for the MOP (1) if it is ε-efficient solution and there exists a positive constant M such that for each $1 \leqslant i \leqslant p$ and any $x \in X$ satisfying $f_i(x) < f_i(\hat{x}) - \varepsilon_i$, there exists $1 \leqslant j \leqslant p$ such that $f_j(x) > f_j(\hat{x}) - \varepsilon_j$ and the following inequality holds

$$\frac{f_i(\hat{x}) - f_i(x) - \varepsilon_i}{f_j(x) - f_j(\hat{x}) + \varepsilon_j} \leqslant M.$$

In the rest of this paper, the sets of all ε-weakly efficient, ε-efficient, and properly ε-efficient solutions will be referred to as $X_{\varepsilon wE}, X_{\varepsilon E}$ and $X_{\varepsilon pE}$, respectively.

One approach for solving the MOP (1) is the feasible value constraint technique. It can be formulated using the following scalarization model [2]

$$\begin{aligned} \min \quad & f_k(x) \\ \text{s.t.} \quad & w_i f_i(x) \leqslant w_k f_k(\hat{x}), \qquad i = 1,\ldots,p, \quad i \neq k, \\ & x \in X. \end{aligned} \tag{2}$$

Here, $\min\limits_{i=1,\ldots,p} \left\{ \inf\limits_{x\in X} f_i(x) \right\} > 0$, and the weights w_i are defined as $w_i = \dfrac{1/f_i(\hat{x})}{\sum_{j=1}^{p} 1/f_j(\hat{x})}$ for $1 \leqslant i \leqslant p$.

3 Improved scalarization problem

In this part, based on the ideas presented in [6,21], we introduce a generalized form of problem (2). This formulation aims to enhance the results and provide a characterization of (properly) efficient solutions for this model. The general form can be written as follows

$$\begin{aligned} \min \quad & f_k(x) - \sum_{i\neq k} \gamma_i s_i^+ + \sum_{i\neq k} \mu_i s_i^- \\ s.t. \quad & w_i f_i(x) + s_i^+ - s_i^- \leqslant \alpha_i, \qquad i = 1,\ldots,p, \quad i \neq k, \\ & x \in X,\ s_i^+, s_i^- \geqslant 0, \qquad i = 1,\ldots,p, \quad i \neq k, \end{aligned} \tag{3}$$

where the weights w_i, μ_i, and γ_i for $1 \leqslant i \leqslant p$ and $i \neq k$ are non-negative, and each α_i denotes an arbitrary upper bound for the function f_i. In addition, s_i^+ and s_i^- are assumed to be the components of the i^{th} vectors of s^+ and s^-, respectively. Note that for a given feasible solution (x, s^+, s^-) of problem (3), $x \in \mathbb{R}^n$ and $(s^+, s^-) \in \mathbb{R}^{p-1} \times \mathbb{R}^{p-1}$. By the next Lemma, we can assume that $\mu - \gamma \geqq 0$.

Lemma 1. *Suppose there exists $1 \leqslant i \leqslant p$ and $i \neq k$ such that $\mu_i - \gamma_i < 0$. Then problem (3) is unbounded, otherwise there exists a partition $I \cup \bar{I}$ of $1 \leqslant i \leqslant p$ and $i \neq k$ such that $s_i^+ = 0$ for all $i \in I$, and $s_i^- = 0$ for all $i \in \bar{I}$.*

Proof. The proof follows a similar approach to the proof of [6, Lemma 5.1]. □

The following lemma describes the first characteristic of the scalarization model (3).

Lemma 2. *Suppose that $\gamma \geqq 0$, and let the optimal solutions set for problem (3) be nonempty. Then there exists an optimal solution $(\hat{x}, \hat{s}^+, \hat{s}^-)$ of problem (3) such that*

$$w_j f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- = \alpha_j, \tag{4}$$

for some $1 \leqslant i \leqslant p$, $i \neq k$. In case $\gamma > 0$, all constraints are active for each optimal solution of problem (3).

Proof. Let $(\hat{x}, \hat{s}^+, \hat{s}^-)$ be an optimal solution of problem (3), and for some $j \neq k$, we have $w_j f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- < \alpha_j$. Suppose that $v > 0$, such that $w_j f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- + v = \alpha_j$. Define new variables as follows

$$s_i^+ = \begin{cases} \hat{s}_i^+, & i \neq j,k \\ \hat{s}_i^+ + v, & i = j. \end{cases}$$

It can be verified that $(\hat{x}, s^{+}, \hat{s}^{-})$ is a feasible solution for problem (3) with the following property

$$\begin{aligned} f_k(\hat{x}) - \sum_{i\neq k} \gamma_i s_i^{+} + \sum_{i\neq k} \mu_i \hat{s}_i^{-} &= f_k(\hat{x}) - \sum_{i\neq k} \gamma_i \hat{s}_i^{+} - \gamma_j v + \sum_{i\neq k} \mu_i \hat{s}_i^{-} \\ &\leqslant f_k(\hat{x}) - \sum_{i\neq k} \gamma_i \hat{s}_i^{+} + \sum_{i\neq k} \mu_i \hat{s}_i^{-}. \end{aligned} \tag{5}$$

Equation (5) indicates that if $\gamma_j > 0$, then the solution $(\hat{x}, s^{+}, \hat{s}^{-})$ yields a better objective value for problem (3) compared to $(\hat{x}, \hat{s}^{+}, \hat{s}^{-})$, and it is equivalent to $(\hat{x}, \hat{s}^{+}, \hat{s}^{-})$ when $\gamma_j = 0$. □

The next theorem states the relationships between optimal solutions of the scalarization model (3) and efficient solutions of the MOP (1).

Theorem 1. *Assume that* $w \geqq 0$.

(i) Let $(\hat{x}, \hat{s}^{+}, \hat{s}^{-})$ *be an optimal solution of the scalarization model (3) for some* $1 \leqslant k \leqslant p$*. Then,* $\hat{x}$ *is a weakly efficient solution of the MOP (1).*

(ii) Let $(\hat{x}, \hat{s}^{+}, \hat{s}^{-})$ *be an optimal solution of the scalarization model (3) for all* $1 \leqslant k \leqslant p$*. Then,* $\hat{x}$ *is an efficient solution of the MOP (1).*

Proof. (i) Consider an arbitrary $k \in \{1, \ldots, p\}$ and assume that $\hat{x}$ is not a weakly efficient solution of the MOP (1). This implies that there exists a feasible point $x \in X$ such that $f_i(x) < f_i(\hat{x})$ for all $1 \leqslant i \leqslant p$. Consequently, one obtains

$$\begin{aligned} w_i f_i(x) + \hat{s}_i^{+} - \hat{s}_i^{-} &\leqslant w_i f_i(\hat{x}) + \hat{s}_i^{+} - \hat{s}_i^{-} \\ &\leqslant \alpha_i, \end{aligned}$$

where $1 \leqslant i \leqslant p, \quad i \neq k$. Obviously, $(x, \hat{s}^{+}, \hat{s}^{-})$ is a feasible point for the scalarization model (3), and

$$f_k(x) - \sum_{i\neq k} \gamma_i \hat{s}_i^{+} + \sum_{i\neq k} \mu_i \hat{s}_i^{-} < f_k(\hat{x}) - \sum_{i\neq k} \gamma_i \hat{s}_i^{+} + \sum_{i\neq k} \mu_i \hat{s}_i^{-}.$$

This contradicts the optimality of $(\hat{x}, \hat{s}^{+}, \hat{s}^{-})$. Therefore, we conclude that $\hat{x}$ is a weakly efficient solution of the MOP (1).

(ii) If $\hat{x}$ is not an efficient solution of the MOP (1), then there exists a feasible point $x \in X$ such that $f_i(x) \leqslant f_i(\hat{x})$ for all $1 \leqslant i \leqslant p$, and $f_j(x) < f_j(\hat{x})$ for some $1 \leqslant j \leqslant p$. Similar to part (1), one deduces $(x, \hat{s}^{+}, \hat{s}^{-})$ is a feasible point of problem (3) so that

$$f_j(x) - \sum_{i\neq j} \gamma_i \hat{s}_i^{+} + \sum_{i\neq j} \mu_i \hat{s}_i^{-} < f_j(\hat{x}) - \sum_{i\neq j} \gamma_i \hat{s}_i^{+} + \sum_{i\neq j} \mu_i \hat{s}_i^{-}.$$

This leads to contradiction with the optimality of $(\hat{x}, \hat{s}^{+}, \hat{s}^{-})$ for the scalarization model (3) with $k = j$. □

If $w > 0$, then we can get an additional sufficient condition for obtaining efficient solutions of the MOP (1) using the scalarization model (3).

Theorem 2. *Assume that $w > 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of the scalarization model (3) for some $1 \leqslant k \leqslant p$, and either $\gamma > 0$ or $\mu > 0$ with $\hat{s}^- > 0$, then $\hat{x}$ is an efficient solution of the MOP (1).*

Proof. Let $\gamma > 0$ (the case $\mu > 0$ and $\hat{s}^- > 0$ is similar) and suppose that $\hat{x}$ is not an efficient solution of the MOP (1), hence there exists a feasible point $x \in X$ satisfying $f_i(x) \leqslant f_i(\hat{x})$ for all $1 \leqslant i \leqslant p$, and $f_j(x) < f_j(\hat{x})$ for some $1 \leqslant j \leqslant p$. Since $w_i > 0$, for all $1 \leqslant i \leqslant p$, we have

$$\begin{aligned} w_i f_i(x) + \hat{s}_i^+ - \hat{s}_i^- &\leqslant w_i f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- \\ &\leqslant \alpha_i, \end{aligned}$$

for all $1 \leqslant i \leqslant p$ with $i \neq k$. We need to consider the following two cases.
If $j = k$, then similar to part (ii) of Theorem 1, we have a contradiction. We now turn to the case $j \neq k$, where

$$\begin{aligned} w_j f_j(x) + \hat{s}_j^+ - \hat{s}_j^- &< w_j f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- \\ &\leqslant \alpha_j. \end{aligned}$$

Consider $0 < v \leqslant w_j f_j(\hat{x}) - w_j f_j(x)$ and define

$$s_i^+ = \begin{cases} \hat{s}_i^+, & i \neq j, \\ \hat{s}_i^+ + v, & i = j. \end{cases}$$

It can be easily verified that $(x, s^+, \hat{s}^-)$ is a feasible point for problem (3) and also

$$\begin{aligned} f_k(x) - \sum_{i \neq k} \gamma_i s_i^+ + \sum_{i \neq k} \mu_i \hat{s}_i^- &= f_k(x) - \sum_{i \neq k} \gamma_i \hat{s}_i^+ - \gamma_j v + \sum_{i \neq k} \mu_i \hat{s}_i^- \\ &< f_k(x) - \sum_{i \neq k} \gamma_i \hat{s}_i^+ + \sum_{i \neq k} \mu_i \hat{s}_i^-. \end{aligned}$$

This contradicts the optimality of $(\hat{x}, \hat{s}^+, \hat{s}^-)$, leading to a contradiction. □

In continuation, we prove that any efficient solution can be viewed as an optimal solution for the scalarization model (3).

Theorem 3. *If $\hat{x}$ is an efficient solution of the MOP (1), then there exist $w \geq 0$, $(\gamma, \mu) \geqq 0$ and $(\alpha, \hat{s}^+, \hat{s}^-)$ such that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of problem (3) for all $1 \leqslant k \leqslant p$.*

Proof. Let $\gamma = \mu = \hat{s}^- = \hat{s}^+ = 0$, and assume that the feasible point (x, s^+, s^-) is an optimal solution of problem (3), hence $f_k(x) < f_k(\hat{x})$ for some $1 \leqslant k \leqslant p$. Define $\alpha_i = w_i f_i(\hat{x}) - s_i^-$ for all $1 \leqslant i \leqslant p$ with $i \neq k$. Therefore

$$\begin{aligned} w_i f_i(x) \leqslant w_i f_i(x) + s_i^+ &\leqslant \alpha_i + s_i^- \\ &= w_i f_i(\hat{x}), \end{aligned} \tag{6}$$

for all $i \neq k$. From (6) and the assumption $w_i > 0$ for all $i \neq k$, we arrive at $f_i(x) \leqslant f_i(\hat{x})$ for all $i \neq k$. These inequalities with $f_k(x) < f_k(\hat{x})$ imply that $f(x) \leq f(\hat{x})$. This leads to a contradiction with the efficiency of $\hat{x}$. □

To verify the existence of properly efficient solutions, we express the following lemma which depicts a connection between the properly efficient solutions of the MOP (1) while the feasible set X is replaced by the feasible set of the scalarization model (3).

Lemma 3. *Assume that $f(X)$ is bounded and $w_i > 0$ for all $1 \leqslant i \leqslant p$ with $i \neq k$. If $\hat{x}$ is a properly efficient solution of the MOP (1) with the feasible set problem (3) and there exists a partition $I \cup \bar{I}$ of $\{1,2,\ldots,p\} - \{k\}$ such that $w_i f_i(\hat{x}) < \alpha_i$ for all $i \in I$, and $w_i f_i(\hat{x}) > \alpha_i$ for all $i \in \bar{I}$, then $\hat{x}$ is a properly efficient solution of the MOP (1) for the feasible set X.*

Proof. The proof is similar to the proof of [21, Lemma 3.3]. □

Theorem 4. *Let $f(X)$ be bounded and $(w,\gamma,\mu) > 0$. If $(\hat{x},\hat{s}^+,\hat{s}^-)$ is an optimal solution of the scalarization model (3) for some $1 \leqslant k \leqslant p$, and there exists a partition $I \cup \bar{I}$ of $1 \leqslant i \leqslant p$ and $i \neq k$ such that $\hat{s}_i^+ > 0$, $\hat{s}_i^- = 0$ for all $i \in I$ and $\hat{s}_i^- > 0$, $\hat{s}_i^+ = 0$ for all $i \in \bar{I}$, then $\hat{x}$ is a properly efficient solution of the MOP (1).*

Proof. Our proof involves looking at Lemma 2. Hence, $w_i f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- = \alpha_i$ for all $i \neq k$. Moreover, $w_i f_i(\hat{x}) < \alpha_i$ for all $i \in I$ and $w_i f_i(\hat{x}) > \alpha_i$ for all $i \in \bar{I}$. Thus

$$f_k(\hat{x}) - \sum_{i\neq k}\gamma_i\hat{s}_i^+ + \sum_{i\neq k}\mu_i\hat{s}_i^- = f_k(\hat{x}) - \sum_{i\in I}\gamma_i\hat{s}_i^+ + \sum_{i\in\bar{I}}\mu_i\hat{s}_i^- \tag{7}$$

$$= f_k(\hat{x}) - \sum_{i\in I}\gamma_i(\alpha_i - w_i f_i(\hat{x})) + \sum_{i\in\bar{I}}\mu_i(w_i f_i(\hat{x}) - \alpha_i) \tag{8}$$

$$= f_k(\hat{x}) + \sum_{i\in I}\gamma_i w_i f_i(\hat{x}) + \sum_{i\in\bar{I}}\mu_i w_i f_i(\hat{x}) - \sum_{i\in I}\gamma_i\alpha_i - \sum_{i\in\bar{I}}\mu_i\alpha_i. \tag{9}$$

Define

$$\lambda_i = \begin{cases} \gamma_i w_i, & i \in I, \\ \mu_i w_i, & i \in \bar{I}, \\ 1, & i = k. \end{cases}$$

Since the term $-\sum_{i\in I}\gamma_i\alpha_i - \sum_{i\in\bar{I}}\mu_i\alpha_i$ is constant, $\hat{x}$ is an optimal solution of the weighted sum problem

$$\min\Big\{\sum_{i=1}^{p}\lambda_i f_i(x) \mid \ w_i f_i(x) < \alpha_i,\ i\in I, w_i f_i(x) > \alpha_i,\ \ i \in \bar{I}\Big\}, \tag{10}$$

where $\lambda > 0$. By Geoffrion's Theorem [11], $\hat{x}$ is a properly efficient solution of the MOP (1) with respect to a feasible set of problem (10). Now, from Lemma 3, it can be concluded that $\hat{x}$ is also a properly efficient solution of the MOP (1) with the feasible set X. □

The following example illustrates how model (3), in contrast to model (2), can effectively identify properly efficient solutions for MOP (1).

Example 1. Consider the following problem [23]

$$\begin{array}{ll} \min & ((x_1-5)^2+(x_2-5)^2+x_3^2, (x_1-6)^2+(x_2-6)^2+(x_3-6)^2) \\ s.t. & x_1+x_2+x_3 \leqslant 5, \\ & x_1,x_2,x_3 \geqslant 0. \end{array} \tag{11}$$

Table 1: The results of model (3) correspond to MOP (11).

$\hat{x}_1$	$\hat{x}_2$	$\hat{x}_3$	$\hat{s}_2^+$	$\hat{s}_2^-$
2.2222	2.2222	0.5556	0.0000	58.1852
2.0833	2.0833	0.8333	0.0000	57.3750
2.0000	2.0000	1.0000	0.0000	57.0000
1.9444	1.9444	1.1111	0.0000	56.7963
1.9048	1.9048	1.1905	0.0000	56.6735
1.8750	1.8750	1.2500	0.0000	56.5938
1.8519	1.8519	1.2963	0.0000	56.5391
1.8333	1.8333	1.3333	0.0000	56.5000
1.8182	1.8182	1.3636	0.0000	56.4711
1.8056	1.8056	1.3889	0.0000	56.4491
1.7949	1.7949	1.4103	0.0000	56.4320
1.7857	1.7857	1.4286	0.0000	56.4184
1.7778	1.7778	1.4444	0.0000	56.4074
1.7708	1.7708	1.4583	0.0000	56.3984
1.7647	1.7647	1.4706	0.0000	56.3910
1.7593	1.7593	1.4815	0.0000	56.3848
1.7544	1.7544	1.4912	0.0000	56.3795
1.7500	1.7500	1.5000	0.0000	56.3750
1.7460	1.7460	1.5079	0.0000	56.3711

As is mentioned in [23], the properly efficient set is the segment joining the points $(\frac{5}{2},\frac{5}{2},0)$ and $(\frac{5}{3},\frac{5}{3},\frac{5}{3})$.

Table 1 shows the results obtained from model (3), with the parameters $\alpha_2 = 0$, $w_2 = \gamma_2 = 1$, and $1 \leqslant \mu_2 \leqslant 20$. As indicated in Table 1, the proper efficiency of all points can be identified using Theorem 4, whereas model (2) fails to detect the proper efficiency of these points.

When $f(X)$ is unbounded, the result of Theorem 4 in general may not be correct.

Example 2. Consider the following MOP [6]

$$\begin{array}{ll} \min & (f_1(x), f_2(x)) = (x_1, x_2) \\ s.t. & \quad x_1x_2 = 1, \\ & -1 \leqslant x_1 < 0. \end{array}$$

It is evident that $X_E = X$. Now, we display that $X_{pE} = \emptyset$. Let us consider an arbitrary solution $\hat{x} = (\hat{x}_1, \hat{x}_2)$. We define $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon) = (-\varepsilon, -\frac{1}{\varepsilon})$, where $0 < \varepsilon < -\hat{x}_1$. Thus, $x^\varepsilon \in X$. Consequently, we have $f_1(x^\varepsilon) > f_1(\hat{x})$, and $f_2(x^\varepsilon) < f_2(\hat{x})$. According to Definition 4, we achieve

$$\frac{f_2(\hat{x}) - f_2(x^\varepsilon)}{f_1(x^\varepsilon) - f_1(\hat{x})} = \frac{\frac{1}{\hat{x}_1} + \frac{1}{\varepsilon}}{-\varepsilon - \hat{x}_1} = -\frac{1}{\varepsilon \hat{x}_1}. \tag{12}$$

As $\varepsilon \to 0$, there is no exists positive constant M in which the result in Eq. (12) is less than or equal to M. Hence, $\hat{x} \in X_{pE}$. Since $\hat{x}$ is an arbitrary solution, we conclude that $X_{pE} = \emptyset$. Consequently, an optimal solution of the scalarization model (3) cannot be a properly efficient solution. Let us assume $k = w_2 = 1, \gamma_2 = 4, \mu_2 = 5$, and $\alpha_2 = -\frac{1}{2}$. Then, the scalarization model (3) possesses an optimal solution $(\hat{x}_1, \hat{x}_2) = (-1,-1)$ and $(\hat{s}_2^+, \hat{s}_2^-) = (\frac{1}{2}, 0)$.

At the end, we are going to verify that any properly efficient solution of MOP (1) can be looked at as an optimal solution of the scalarization model (3) with $w > 0$.

Theorem 5. *Let $\hat{x}$ be a properly efficient solution of the MOP (1). Then for all $1 \leqslant k \leqslant p$, there exist $w > 0$, $(\gamma, \hat{\mu}) \geqq 0$, and $(\alpha, \hat{s}^+, \hat{s}^-)$ such that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of problem (3) for all $\mu \geqq \hat{\mu}$.*

Proof. The proof is taken from [21, Theorem 3.6]. Now, let $k \in \{1, 2, \dots, p\}$. Putting $\gamma = 0$, $\hat{s}^+ = 0$ and $\alpha_i = w_i f_i(\hat{x})$, where the weights w_i for $1 \leqslant i \leqslant p$ with $i \neq k$ are positive constants. Let $\hat{s}_i^- = 0$ for all $i \neq k$. Since $\hat{x}$ is a properly efficient solution of the MOP (1), there exists $M > 0$ such that for all $1 \leqslant i \leqslant p$ and $x \in X$ with $f_i(x) < f_i(\hat{x})$. Hence there exists $j \in \{1, \dots, p\}$ such that $f_j(\hat{x}) < f_j(x)$ and

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leqslant M.$$

Define $\hat{\mu}_i = \frac{M}{w_i}$ for all $1 \leqslant i \leqslant p$ with $i \neq k$. Let (x, s^+, s^-) be a feasible point of problem (3). Since $\gamma = 0$, we can put $s_i^+ = 0$, and so $s_i^- \geqslant \max\{0, w_i f_i(x) - \alpha_i\} = \max\{0, w_i f_i(x) - w_i f_i(\hat{x})\}$, for all $i \neq k$. We have to show that

$$f_k(x) + \sum_{i \neq k} \mu_i s_i^- \geqslant f_k(\hat{x}) + \sum_{i \neq k} \mu_i \hat{s}_i^- = f_k(\hat{x}). \tag{13}$$

If $f_k(x) \geqslant f_k(\hat{x})$, then inequality (13) holds else if $f_k(x) < f_k(\hat{x})$ define $J = \{1 \leqslant j \leqslant p,\ j \neq k \mid f_j(\hat{x}) < f_j(x)\}$. The set J is nonempty because $\hat{x}$ is a properly efficient solution. Since $f_k(x) < f_k(\hat{x})$, there exists $j \in J$ such that $f_k(\hat{x}) - f_k(x) \leqslant M(f_j(x) - f_j(\hat{x}))$, and also

$$\begin{aligned}
f_k(x) + \sum_{i \neq k} \hat{\mu}_i s_i^- &\geqslant f_k(x) + \sum_{i \neq k} \hat{\mu}_i \max\{0, w_i f_i(x) - w_i f_i(\hat{x})\} \\
&\geqslant f_k(x) + \sum_{i \in J} \hat{\mu}_i (w_i f_i(x) - w_i f_i(\hat{x})) \\
&\geqslant f_k(x) + \hat{\mu}_j w_j (f_j(x) - f_j(\hat{x})) \\
&= f_k(x) + M(f_j(x) - f_j(\hat{x})) \\
&\geqslant f_k(\hat{x}).
\end{aligned}$$

Consequently, we deduce that inequality (13) holds for all $\mu \geqq \hat{\mu}$. □

4 ε-weakly and ε-properly efficient solutions

In this part, we present several sufficient conditions that can be applied to characterize approximate (weakly and properly) efficient solutions of the general MOP (1) through the utilization of the scalarization model (3).

The following theorem establishes some sufficient conditions for the ε-weakly efficient solution.

Theorem 6. *Let $\varepsilon \geqslant 0$, and $\varepsilon \geqq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an ε-optimal solution of problem (3) for some $k \in \{1,2,\ldots,p\}$, and also*

(i) $\varepsilon \leqslant \min_{i=1,\ldots,p} \varepsilon_i$, *then $\hat{x}$ is an ε-weakly efficient solution of the MOP (1).*

(ii) $\varepsilon \leqslant \sum_{i\neq k} \gamma_i w_i \varepsilon_i$, *then $\hat{x}$ is an ε-weakly efficient solution of the MOP (1).*

(iii) $\varepsilon \leqslant \sum_{i\neq k} \mu_i w_i \varepsilon_i$ *and* $w \circ \varepsilon \leqq \hat{s}^-$, *then $\hat{x}$ is an ε-weakly efficient solution of the MOP (1).*

Proof. Here, we present the proof of part (iii). The proofs for parts (i) and (ii) are similar and will be omitted.

(iii) Suppose that $\hat{x}$ is not an ε-weakly efficient solution of the MOP (1), then there exists $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$. Therefore, $w_i f_i(x) \leqslant w_i f_i(\hat{x}) - w_i \varepsilon_i$ for $i \neq k$. This implies that

$$\begin{aligned} w_i f_i(x) + \hat{s}_i^+ - \hat{s}_i^- + w_i \varepsilon_i &\leqslant w_i f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- \\ &\leqslant \alpha_i, \end{aligned}$$

for $i \neq k$. Set $s_i^- = \hat{s}_i^- - w_i \varepsilon_i$ for all $i \neq k$. Therefore, $(x, \hat{s}^+, s^-)$ is a feasible point of problem (3) such that

$$\begin{aligned} f_k(x) - \sum_{i\neq k} \gamma_i \hat{s}_i^+ + \sum_{i\neq k} \mu_i s_i^- &= f_k(x) - \sum_{i\neq k} \gamma_i \hat{s}_i^+ + \sum_{i\neq k} \mu_i \hat{s}_i^- - \sum_{i\neq k} \mu_i w_i \varepsilon_i \\ &< f_k(\hat{x}) - \sum_{i\neq k} \gamma_i \hat{s}_i^+ + \sum_{i\neq k} \mu_i \hat{s}_i^- - \varepsilon. \end{aligned}$$

This is a contradiction to the ε-optimality of $(\hat{x}, \hat{s}^+, \hat{s}^-)$. □

By utilizing positive parameters and weights, we can derive several sufficient conditions for the ε-efficient solution of the MOP (1).

Theorem 7. *Let $\varepsilon \geqslant 0$, and $\varepsilon \geqq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an ε-optimal solution of problem (3) for some $k \in \{1,2,\ldots,p\}$ such that*

(i) $(w, \gamma) > 0$ *and* $\varepsilon \leqslant \sum_{i\neq k} \gamma_i w_i \varepsilon_i$, *then $\hat{x}$ is an ε-efficient solution of the MOP (1).*

(ii) $(w, \mu) > 0$, $w \circ \varepsilon \leqq \hat{s}^-$ *and* $\varepsilon \leqslant \sum_{i\neq k} \mu_i w_i \varepsilon_i$, *then $\hat{x}$ is an ε-efficient solution of the MOP (1).*

Proof. (i) Assume that $\hat{x}$ is not an ε-efficient solution of the MOP (1), then there exists $x \in X$ such that $f_i(x) \leqslant f_i(\hat{x}) - \varepsilon_i$ for all i, and $f_j(x) < f_j(\hat{x}) - \varepsilon_j$ for some j. Hence, for all $i \neq k$ we have

$$\begin{aligned} w_i f_i(x) + \hat{s}_i^+ - \hat{s}_i^- &\leqslant w_i f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- - w_i \varepsilon_i \\ &\leqslant \alpha_i, \end{aligned}$$

and

$$\begin{aligned} w_j f_j(x) + \hat{s}_j^+ - \hat{s}_j^- &< w_j f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- - w_j \varepsilon_j \\ &\leqslant \alpha_j, \end{aligned}$$

for some $j \neq k$. Let us consider $v_j > 0$ such that $v_j \leqslant w_j(f_j(\hat{x}) - f_j(x) - \varepsilon_j)$. It follows that $w_j f_j(x) + \hat{s}_j^+ - \hat{s}_j^- + v_j + w_j\varepsilon_j \leqslant \alpha_j$. Putting

$$s_i^+ = \begin{cases} \hat{s}_i^+ + w_i\varepsilon_i, & i \neq j,\ k \\ \hat{s}_i^+ + v_i + w_i\varepsilon_i, & i = j. \end{cases}$$

Therefore, $(x, s^+, \hat{s}^-)$ is a feasible point of problem (3) such that

$$\begin{aligned} f_k(x) - \sum_{i\neq k}\gamma_i s_i^+ + \sum_{i\neq k}\mu_i\hat{s}_i^- &= f_k(x) - \sum_{i\neq k}\gamma_i\hat{s}_i^+ - \sum_{i\neq k}\gamma_i w_i\varepsilon_i - \gamma_j v_j + \sum_{i\neq k}\mu_i\hat{s}_i^- \\ &< f_k(\hat{x}) - \sum_{i\neq k}\gamma_i\hat{s}_i^+ + \sum_{i\neq k}\mu_i\hat{s}_i^- - \varepsilon. \end{aligned}$$

This contradicts the ε-optimality of $(\hat{x}, \hat{s}^+, \hat{s}^-)$. The proof of part (ii) runs as part (i) and will be omitted. □

The next theorem finds out additional sufficient conditions for the ε-efficient solution of the MOP (1).

Theorem 8. *Let $\varepsilon > 0$, and $\varepsilon \geqq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an ε-optimal solution of problem (3) for some $k \in \{1,\ldots,p\}$, in which*

(i) $\varepsilon < \sum_{i\neq k}\gamma_i w_i\varepsilon_i$, then $\hat{x}$ is an ε-efficient solution of the MOP (1).

(ii) $\varepsilon < \sum_{i\neq k}\mu_i w_i\varepsilon_i$ and $w \circ \varepsilon \leqq \hat{s}^-$, then $\hat{x}$ is an ε-efficient solution of the MOP (1).

Proof. The proof is similar in spirit to that of Theorem 7. □

Finally, we demonstrate that under the conditions stated in Theorems 7 and 8, we can derive ε-properly efficient solutions for the MOP (1) using the scalarization problem (3).

Theorem 9. *Let $\varepsilon \geqslant 0$, and $\varepsilon \geqq 0$. Suppose that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an ε-optimal solution of problem (3) for some $k \in \{1,2,\ldots,p\}$. If*

(i) $(w,\gamma) > 0$ and $\varepsilon \leqslant \sum_{i\neq k}\gamma_i w_i\varepsilon_i$, then $\hat{x}$ is an ε-properly efficient solution of the MOP (1).

(ii) $(w,\mu) > 0, w \circ \varepsilon \leqq \hat{s}^-$ and $\varepsilon \leqslant \sum_{i\neq k}\mu_i w_i\varepsilon_i$, then $\hat{x}$ is an ε-properly efficient solution of the MOP (1).

Proof. Here, we provide the proof for part (i). The proof of the second part resembles to the first part and will be omitted. By part one of Theorem 7, $\hat{x}$ is an ε-efficient solution of the MOP (1). Let $\hat{x} \notin X_{\varepsilon pE}$. Thus, for all $M > 0$ there exists $l \in \{1,\ldots,p\}$, and $x \in X$ with $f_l(x) < f_l(\hat{x}) - \varepsilon_l$ such that

$$\frac{f_l(\hat{x}) - f_l(x) - \varepsilon_l}{f_j(x) - f_j(\hat{x}) + \varepsilon_j} > M, \tag{14}$$

for all j with $f_j(x) > f_j(\hat{x}) - \varepsilon_j$. For the index l, there exists $v > 0$ such that

$$w_l f_l(x) + v = w_l f_l(\hat{x}) - w_l \varepsilon_l. \tag{15}$$

Thus we arrive at

$$\begin{aligned} w_l f_l(x) + \hat{s}_l^+ + v + w_l \varepsilon_l - \hat{s}_l^- &= w_l f_l(\hat{x}) + \hat{s}_l^+ - \hat{s}_l^- \\ &\leqslant \alpha_l. \end{aligned}$$

Define $J = \{1 \leqslant j \leqslant p \mid f_j(x) > f_j(\hat{x}) - \varepsilon_j\}$. Since $\hat{x} \in X_{\varepsilon E}$, hence $J \neq \emptyset$. From (14) and (15) it follows that $f_j(x) < f_j(\hat{x}) + \frac{v}{Mw_l} - \varepsilon_j$, for all $j \in J$. Therefore, by the ε-optimality of $(\hat{x}, \hat{s}^+, \hat{s}^-)$ we get

$$\begin{aligned} w_j f_j(x) + \hat{s}_j^+ - \hat{s}_j^- &< w_j f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- \\ &\leqslant \alpha_j + \frac{v w_j}{M w_l} - w_j \varepsilon_j. \end{aligned}$$

for all $j \in J$. So, we attain

$$w_j f_j(x) + \hat{s}_j^+ + w_j \varepsilon_j - \hat{s}_j^- - \frac{v w_j}{M w_l} \leqslant \alpha_j.$$

On the other hand, if $i \notin J \cup \{l\}$, then $f_i(x) \leqslant f_i(\hat{x}) - \varepsilon_i$. Therefore, by the ε-optimality assumption for all $i \notin J \cup \{l\}$, we achieve

$$\begin{aligned} w_i f_i(x) + \hat{s}_i^+ + w_i \varepsilon_i - \hat{s}_i^- &\leqslant w_i f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- \\ &\leqslant \alpha_i. \end{aligned}$$

Define

$$s_i^+ = \begin{cases} \hat{s}_i^+ + v + w_i \varepsilon_i, & i = l, \\ \hat{s}_i^+ + w_i \varepsilon_i, & i \neq l, \end{cases}$$

and

$$s_i^- = \begin{cases} \hat{s}_i^- + \frac{v w_i}{M w_l}, & i \in J, \\ \hat{s}_i^-, & i \notin J. \end{cases}$$

Thus, (x, s^+, s^-) is a feasible point of problem (3) and

$$\begin{aligned} f_k(x) - \sum_{i \neq k} \gamma_i s_i^+ + \sum_{i \neq k} \mu_i s_i^- &= f_k(x) - \sum_{i \neq l,k} \gamma_i (\hat{s}_i^+ + w_i \varepsilon_i) - \gamma_l (\hat{s}_l^+ + v + w_l \varepsilon_l) \\ &\quad + \sum_{i \in J - \{k\}} \mu_i (\hat{s}_i^- + \frac{v w_i}{M w_j}) + \sum_{i \notin J - \{k\}} \mu_i \hat{s}_i^- \\ &= f_k(x) - \sum_{i \neq k} \gamma_i \hat{s}_i^+ + \sum_{i \neq k} \mu_i \hat{s}_i^- + \sum_{i \in J - \{k\}} \mu_i \frac{v w_i}{M w_l} - \gamma_l w_l v - \sum_{i \neq k} \gamma_i w_i \varepsilon_i \\ &< f_k(x) - \sum_{i \neq k} \gamma_i \hat{s}_i^+ + \sum_{i \neq k} \mu_i \hat{s}_i^{\,-} - \varepsilon, \end{aligned}$$

Table 2: Summary of results for a solution $(\hat{x}, \hat{s}^+, \hat{s}^-)$ of problem (3).

Condition on the parameters	Implication for $\hat{x}$	Reference
Optimal for some k and $(w, \mu, \gamma) \geqq 0$	$\hat{x} \in X_{wE}$	Theorem 1 (1)
Optimal for all k and $(w, \mu, \gamma) \geqq 0$	$\hat{x} \in X_E$	Theorem 1 (2)
Optimal for some k, $\mu \geqq 0$ and $(w, \gamma) > 0$	$\hat{x} \in X_E$	Theorem 2
Optimal for some k, $\gamma \geqq 0$ and $(w, \mu, \hat{s}^-) > 0$	$\hat{x} \in X_E$	Theorem 2
Optimal for some k, $(w, \mu, \gamma) > 0$, $f(X)$ is bounded and all constraints are inactive	$\hat{x} \in X_{pE}$	Theorem 4
Form of solution	**Optimality for problem (3)**	
$\hat{x} \in X_E$	There exists $(\alpha, \hat{s}^+, \hat{s}^-)$ such that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is optimal for all k.	Theorem 3
$\hat{x} \in X_{pE}$	For all k, there exists $(\alpha, \hat{s}^+, \hat{s}^-)$ such that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is optimal.	Theorem 5

Table 3: Summary of results for an ε-optimal solution of problem (3) for some k.

Condition on the parameters	Implication for $\hat{x}$	Reference
$\varepsilon \leqslant \min_{i=1,\dots,p} \varepsilon_i$	$\hat{x} \in X_{\varepsilon wE}$	Theorem 6 (1)
$\varepsilon \leqslant \sum_{i \neq k} \gamma_i w_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon wE}$	Theorem 6 (2)
$\varepsilon \leqslant \sum_{i \neq k} \mu_i w_i \varepsilon_i,\ w \circ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon wE}$	Theorem 6 (3)
$(w, \gamma) > 0,\ \varepsilon \leqslant \sum_{i \neq k} \gamma_i w_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon E}$	Theorem 7 (1)
$(w, \mu) > 0,\ \varepsilon \leqslant \sum_{i \neq k} \mu_i w_i \varepsilon_i,\ w \circ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon E}$	Theorem 7 (2)
$0 < \varepsilon < \sum_{i \neq k} \gamma_i w_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon E}$	Theorem 8 (1)
$0 < \varepsilon < \sum_{i \neq k} \mu_i w_i \varepsilon_i,\ w \circ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon E}$	Theorem 8 (2)
$(w, \gamma) > 0,\ \varepsilon \leqslant \sum_{i \neq k} \gamma_i w_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 9 (1)
$(w, \mu) > 0,\ \varepsilon \leqslant \sum_{i \neq k} \mu_i w_i \varepsilon_i,\ w \circ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 9 (2)
$0 < \varepsilon < \sum_{i \neq k} \gamma_i w_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 10 (1)
$0 < \varepsilon < \sum_{i \neq k} \mu_i w_i \varepsilon_i,\ w \circ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 10 (2)

the last inequality holds under the assumption that

$$M > \frac{1}{\gamma_l w_l^2} \sum_{i \in J - \{k\}} \mu_i w_i.$$

This contrary to the ε-optimality of $(\hat{x}, \hat{s}^+, \hat{s}^-)$. $\square$

More sufficient conditions for the ε-properly efficient solution of the MOP (1) are as follows.

Theorem 10. *Let $\varepsilon > 0$, and $\varepsilon \geqq 0$. Assume that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an ε-optimal solution of problem (3) for some $k \in \{1, \dots, p\}$. If*

(i) $\varepsilon < \sum_{i \neq k} \gamma_i w_i \varepsilon_i$, then $\hat{x}$ is an ε-properly efficient solution of the MOP (1).

(ii) $\varepsilon < \sum_{i \neq k} \mu_i w_i \varepsilon_i$ *and* $w \circ \varepsilon \leqq \hat{s}^-$, *then* $\hat{x}$ *is an* ε*-properly efficient solution of the MOP (1).*

Proof. The proof is similar to that of Theorem 9. □

5 Conclusions

In this research, we introduced a general formulation of the feasible value constraint technique for effectively solving MOPs. By incorporating slack and surplus variables, the suggested model enables the characterization of both properly efficient and efficient solutions for a MOP. Regarding the significance of epsilon efficient solutions in addressing MOPs, we extracted necessary and sufficient conditions for ε-(weakly, properly) efficient solutions based on the proposed model. To provide a brief overview of the results obtained, Tables 2 and 3 portray a summary of key findings derived from the mentioned scalarization model.

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