

A bidirectional associative memory model with almost periodic endogenous and exogenous stimuli

Homero G. Díaz-Marín[†], Osvaldo Osuna[‡], Jose Geiser Villavicencio-Pulido^{§*}

[†]*Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana. Edif. Alfa, Ciudad Universitaria, C.P. 58040. Morelia, Michoacán, México*

[‡]*Instituto de Física y Matemáticas, Universidad Michoacana. Ciudad Universitaria, C.P. 58040. Morelia, Michoacán, México*

[§]*División de Ciencias Biológicas y de la Salud, Depto. de Ciencias Ambientales, Universidad Autónoma Metropolitana Unidad Lerma, Av. Hidalgo Poniente No. 46, Col. La Estación, 52006 Lerma de Villada, Edo. de México, México*

Email(s): hdiaz@umich.mx, osvaldo.osuna@umich.mx, j.villavicencio@correo.ler.uam.mx

Abstract. In this work, a two-neuron model that describes a module of a neuronal network is analyzed. Unlike other studies, all the rates involved in the model are assumed to be almost periodic functions. Assuming an almost periodicity in the neuronal mechanisms offers advantages because the endogenous and exogenous stimuli received by the neuron are not necessarily periodic or constant. Analysis of the model showed that it is associated with a unique stable almost periodic solution when some conditions on the parameters of the model are satisfied. Numerical simulations of the solutions of the model show that the neuronal state variable of both neurons can be underestimated or overestimated depending on whether the neuronal dynamics is modeled by periodic or almost periodic functions. Such estimation errors can lead to failure in forecasting the time in which neurons must synchronize.

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1 Introduction

Biological oscillators are found in a wide variety of phenomena and are at all levels of organization. In this direction, rhythmic phenomena appear in unicellular and multicellular organisms. Despite the ubiquity of biological clocks, the mechanism that induce rhythmic oscillations often remains unclear.

Rhythmic oscillations appear not only in living systems but also in chemical reactions. Depending on the nature of the phenomenon, periods associated with the biological oscillator range from fractions of a

*Corresponding author

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second to years. For example, neural rhythms are produced during fractions of a second while biennial plants complete their life cycle in two years.

On the other hand, from a theoretical perspective, the mathematical background of biological oscillations begins with an analysis of simultaneous reactions performed by Lotka [29–31]. In particular, in 1920 and 1925, Lotka analyzed oscillatory reactions through a kinetic model. Volterra, in 1926, proposed a similar model to analyze the ecological dynamics among predators and their prey [37].

Biological oscillatory phenomena have been described throughout the last two centuries from different perspectives. From an experimental perspective, Fecher, in 1829, showed that an electrochemical cell produces an oscillating current. In 1899, Ostwald concluded that the rate of chromium dissolution in acid exhibits a oscillatory behavior [10].

Other important results were proposed by Hodgkin and Huxley in 1952. They proposed a neurobiological model to analyze the action potential of excitable cells [18]. An important model that is derived from the Hodgkin-Huxley model is called the Fitzhugh-Nagumo model [11, 12, 32]. From a theoretical perspective, Lefever and coworkers proposed a model, called the brusselator, that extends the results of oscillatory reactions [25]. To learn about the mathematics of oscillatory biological systems, [13] is recommended.

The brain activity, at a macroscopic level, shows synchronous phenomena in electroencephalograms. It is possible that the synchronicity plays a relevant role in brain functions like perception as well as even in Parkinson's disease and epilepsy. At the level of a single neuron, specific spiking-bursting in the cat visual cortex is related to synchronous activity in the visual stimulation. In animal models of Parkinsons disease, bursting neurons exhibit synchronicity. Furthermore, two coupled-neurons extracted from a lobster show an asynchronicity phenomenon in spiking-bursting behavior [2].

An oscillatory system par excellence is presented in the functioning of neuronal networks. However, in this system, there is an implicit complexity due to the number of interconnected neurons. A first step to understand the mechanisms behind this biological oscillator is to understand each neuron individually.

Neurons are the fundamental blocks of the central nervous system that are responsible for memory, perception and motor control, among other things. In broad terms, a neuron comprises three essential parts, which are described below. Dendrites are the input stage of a neuron. Dendrites receive synaptic input from other neurons. The *soma* contains cellular machinery. Finally, the axon is the output stage. A single neuron may receive an input signal from other neurons (convergence), and may transmit a signal to other neurons.

Although it is difficult to describe the functioning of a biological neuronal network, progress has been made in this subject using mathematical or computational models. To do this, there are models that emulate the brain, considering that neurons are configured in a specific way. For example, Kosko analyzed bidirectional associative memory networks, that emulate the complex dynamics of the brain [22, 23]. In [14], Gopalsamy and Xe analyzed the qualitative properties of the solutions of a continuous bidirectional associative memory network with a discrete delay. In this model, each neuron may receive a constant exogenous input. In addition, the decay rate of each neuron is equal to one, while the amplification rate is considered nonnegative. The authors proved the existence and stability of equilibrium points of the model. The model of Gopalsamy and Xe generalizes the Hopfield circuit model [19] and it is a special case from the model of Cohen and Grossberg [6]. These types of models have been intensively analyzed because of the wide variety of applications in which they are used, such as classification problems, associative memory, pattern recognition, signal processing [1, 7, 17, 20]. Gopalsamy and He proved that there is a global attractor of a bidirectional associative memory networks which describes a unique

pattern of memory for each external input. This globally stable solution is associated to a remember in which irrespective of the initial condition, the brain returns to a state in which it has been before. To wit, solutions go to the stable equilibrium for all initial condition [15]. Most of the models mentioned consider that the parameters of each model are constant. In other words, it is assumed that the endogenous and exogenous stimuli that each neuron receives are averaged. It is of paramount importance to underline that considering that activation rates are constant is very restrictive since biological systems are far from having such simple behavior. In this direction, recent studies have shown that there are neurons that are highly sensitive to changes in the synchronicity of their excitatory and inhibitory inputs [5].

In literature on neuronal networks, there are other generalizations of continuous bidirectional associative memory networks that consider non-constant parameters. In particular, in some results on the existence and stability of almost periodic solutions, the authors do not assume that all rates are almost periodic functions [3, 26–28, 36, 38, 39]. Although this scenario is more realistic than a scenario with constant rates, because the firing rates must describe variations in the stimuli, it is more plausible to assume that all parameters of the model are almost periodic.

Understanding the biological mechanisms that induce oscillatory behaviors is undoubtedly very complex. However, to understand the phenomenology behind this complexity, it is useful to assume that a biological process is compounded of modules such that each one describes separable autonomous functions, and when the modules are interconnected, higher-level functions are constructed. Each module must be analyzed isolated, and then they can be interconnected with other modules [16, 24].

In the modularity approach, it is important to underline that each module describes a particular function in a robust manner. In addition, changes in the connections among modules alter the functions of the biological system. In other words, changes in weak regulatory linkages affect highly conserved core processes [21].

In this work, two neurons that are connected are analyzed using an upstream system that is input-output connected to a downstream one. The results obtained can be used to understand a neuronal network if each pair of neurons is defined as a node of the network. The model proposed analyzes changes in the neuronal response when all the parameters of the model are almost periodic functions.

2 The almost periodic model

Models that describe the operation function of neuronal networks assume that each neuron is activated in terms of average firing rates. Typically, each neuron modeled has two states, which are characterized by the output N_i of the neuron having the values N_i^0 or N_i^1 . These values are often 0 and 1, respectively. The neuron receives stimuli from two resources: external inputs, denoted by I_i and inputs from other neurons. So, the inputs receive from other neurons depend on the synaptic interconnection strength among neurons. These models describe the evolution of the states in time in terms of stochastic evolution. So, each neuron changes its state according to a threshold rule which is used at random times when the neuron samples its inputs. A classic generalization of models that describe the function on neuronal networks is based on continuous variables and responses. In these models, the output variable, N_i , takes values in the interval $N_i^0 \leq N_i \leq N_i^1$. Commonly, the output variable is a differentiable monotonic increasing real valued function on $(-\infty, \infty)$ of the instantaneous input u_i to neuron i . A function used in models that describe a neuronal output variable is given by

$$S(x) = \tanh(x), \quad x \in (-\infty, \infty). \quad (1)$$

Notice that, $S(x)$, given in (1), is a sigmoid function with asymptotes -1 and 1 ; see Figure 1 (a).

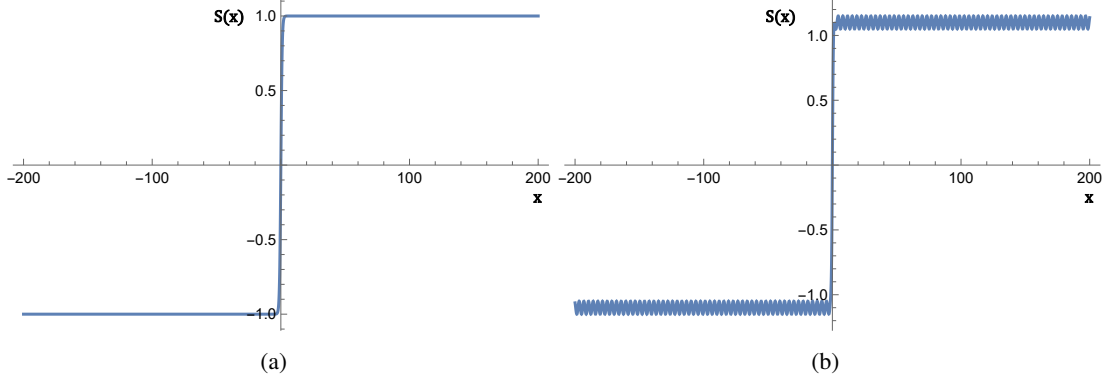


Figure 1: In case (a), the function $S(x) = \tanh(x)$ is plotted. In case (b), the function $S(x) = (1 + 0.05 \sin(\sqrt{3}x)) \tanh(\sqrt{2}x)$ is shown.

Models describing neural networks often consider constant rates. However, these models do not describe changes in the external inputs due to changes in the environment. Recent studies have shown that neurons are highly affected by changes in spike timing. So, in more general models, the activation of neurons through a phenomenon of synchronicity is considered. Notice that, the last type of models considers the information carried by the spikes. In this work, it is modelled synchronous and asynchronous phenomena. To do this, we model that the spike timing is almost periodic since modeling neurons as having periodic behaviour is very restrictive. Thus, the model proposed is a generalization of the models analyzed in [3, 26–28, 38, 39]. To achieve this, it is proposed to analyze a two-neuron model that describes a module of a continuous bidirectional associative neuronal network, assuming that all rates involved in the model are almost periodic functions. In particular, to model an almost neuronal output variable, we use a function as shown in Figure 1 (b). We also consider that the rates associated with decay terms in neurons when isolated are almost periodic functions. Then, the proposed model is given by

$$\begin{aligned} \frac{du_1}{dt} &= J_1(t) + a_{12}(t) \tanh(b_{12}(t)u_2) - c_1(t)u_1, \\ \frac{du_2}{dt} &= J_2(t) + a_{21}(t) \tanh(b_{21}(t)u_1) - c_2(t)u_2, \end{aligned} \quad (2)$$

where $J_i(t), a_{ij}(t), c_i(t), b_{ij}(t)$ are non-negative almost periodic functions.

In model (2), for $i = 1, 2$, it is considered that the neuron i receives an exogenous input given by J_i that activates the neuron. The terms $c_i(t)$ are decay terms. Observe that $c_i(t)$ represents the rate at which the i -th neuron will reset its potential to the resting state in isolation when disconnected from the other neuron and there are not any external inputs. The endogenous feedback term for the neuron i is given by $a_{ij}(t) \tanh(b_{ij}(t)u_j)$, for $j = 1, 2$. In the activation function $a_{ij}(t) \tanh(b_{ij}(t)u_j)$, a_{ij} is the synaptic weight while $b_{i,j}$ is the amplifier gain. Notice that, the input-output relation $a_{ij}(t) \tanh(b_{ij}(t)u_j)$ is an increasing sigmoid function (as a function of u_j). The term $a_{ij}(t) \tanh(b_{ij}(t)u_j)$ converts the membrane potential of $u_j(t)$ into the firing rate of $u_i(t)$. Also, u_i can describe the neuronal state variable from its excitatory and inhibitory inputs.

3 Almost periodic functions in cooperative systems

In this section we review some basic aspects about almost periodic functions, as well as about cooperative systems, we refer the reader to [4, 8, 34, 35] for a detailed description about these topics.

Definition 1. We say that a continuous function $\phi \in C^0(\mathbb{R})$ is almost periodic if it can be approximated in the sup-norm by trigonometric polynomials.

For every $\phi \in \text{AP}(\mathbb{R}) \subset C^0(\mathbb{R})$ in the subspace of almost periodic functions, we have the Fourier series:

$$\phi \sim \sum_{n \in \mathbb{N}} c(\phi, \lambda_n) e^{i\lambda_n t}.$$

as well as a mean

$$\mathcal{M}(\phi) := c_0(\phi, 1) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi(t) dt,$$

such that $\phi \geq 0$ implies $\mathcal{M}[\phi] \geq 0$. The analogous Parseval's Theorem for almost periodic functions says that

$$\mathcal{M}[|\phi|^2] = \sum_{n \in \mathbb{N}} |c(\phi, \lambda_n)|^2.$$

We consider a two-dimensional system

$$\begin{aligned} x'(t) &= u(t, x(t), y(t)), \\ y'(t) &= v(t, x(t), y(t)), \end{aligned} \tag{3}$$

in a domain $(x, y) \in D \subseteq \mathbb{R}^2$. We say that the C^1 functions, $u(t, x, y)$ and $v(t, x, y)$ are *uniformly almost periodic with respect to* $(x, y) \in C$ for every compact $C \subseteq D$, if f (resp. g) has generalized Fourier series,

$$u(t, x, y) \sim \bar{u}(x, y) + \sum_{n \in \mathbb{N}} a(u, \lambda_n; (x, y)) \cos(\lambda_n t) + b(u, \lambda_n, (x, y)) \sin(\lambda_n t),$$

where the uniform mean $\bar{u}(x, y) := \mathcal{M}[u(t, x, y)]$ as well as the frequencies λ_n do not depend on (x, y) .

Definition 2. We say that (3) is of the cooperative type [34], if for every $t \in \mathbb{R}$,

$$\frac{\partial u}{\partial y}(t, x, y) \geq 0, \quad \frac{\partial v}{\partial x}(t, x, y) \geq 0.$$

Moreover, $(\alpha(t), \beta(t))$ is sub-solution if

$$\xi'(t) \leq u(t, \xi(t), \eta(t)), \tag{4}$$

$$\eta'(t) \leq v(t, \xi(t), \eta(t)). \tag{5}$$

By reversing the inequalities, we obtain the notion of a super-solution $(\Xi(t), H(t))$. A pair $(\xi(t), \eta(t))$ and $(\Xi(t), H(t))$ is ordered if

$$\xi(t) \leq \Xi(t), \quad \eta(t) \leq H(t), \quad \forall t \in \mathbb{R}.$$

The following that will be useful in our development can be found in [9] as Theorem 2.

Theorem 1. Suppose that $(\xi(t), \eta(t))$ and $(\Xi(t), H(t))$ is a sub-super-solution ordered pair of the competitive ODE (3). Then, there exists an almost periodic solution satisfying $\xi(t) < x(t) < \Xi(t)$ and $\eta(t) < y(t) < H(t)$. The set of almost periodic solutions, having initial conditions in the rectangle $\xi(0) < x(0) < \Xi(0)$ and $\eta(0) < y(0) < H(0)$, is totally ordered, provided there is no equilibrium. If $(\check{x}(t), \check{y}(t)), (\hat{x}(t), \hat{y}(t))$, denote the minimal and maximal almost periodic solutions. Then

$$\check{x}(t) \leq x(t) \leq \hat{x}(t), \quad \check{y}(t) \leq y(t) \leq \hat{y}(t).$$

4 Results

Given an almost periodic function $v : \mathbb{R} \rightarrow \mathbb{R}$, we denote

$$v_* := \inf_{t \in \mathbb{R}} v(t) \quad \text{and} \quad v^* := \sup_{t \in \mathbb{R}} v(t).$$

Now we state our main result.

Theorem 2. For $i, j = 1, 2$ and $i \neq j$. Assume that $J_i(t), a_{ij}(t), b_{ij}(t) \geq 0$ and $c_i(t) \geq 0$ are continuous almost periodic functions and that at least one of them is not a constant function. Assume that with $c_{i*} > 0$ and that (2) does not admit equilibrium points with positive coordinates. Then, the following statements are valid.

- (i) There exists at least one almost periodic solution (u_1, u_2) of (2) whose components are positive.
- (ii) If $(a_{12}b_{12})^*(a_{21}b_{21})^* < c_{1*}c_{2*}$, there exists a unique almost periodic solution in $\mathbb{R}_{>0}^2$ and any other solution of (2) with positive initial conditions converges to this almost periodic solution, when $t \rightarrow \infty$.

Proof. For part (i). A direct calculation indicates that system (2) is cooperative. We construct sub- and super-solution pairs. For a sub-solution pair; we consider $(\xi(t), \eta(t)) = (0, 0)$, since $\tanh(0) = 0$, then $\xi(t)' = 0 \leq J_1(t)$ and $\eta(t)' = 0 \leq J_2(t)$. Thus, this pair fulfills the required condition (4).

For a super-solution pair, we consider

$$(\Xi(t), H(t)) = (R, R), \quad R > 0. \tag{6}$$

Now $\tanh(\cdot)$ is bounded and $J_i(t), a_{ij}(t)$ are almost-periodic and therefore bounded. Thus

$$J_1(t) + a_{12}(t) \tanh(b_{12}(t)R),$$

is dominated by $-c_{1*}R$ when R is large enough. Hence

$$0 \geq J_1(t) + a_{12}(t) \tanh(b_{12}(t)R) - c_{1*}R \geq J_1(t) + a_{12}(t) \tanh(b_{12}(t)R) - c_1(t)R.$$

Similarly

$$0 \geq J_2(t) + a_{21}(t) \tanh(b_{21}(t)R) - c_{2*}R \geq J_2(t) + a_{21}(t) \tanh(b_{21}(t)R) - c_2(t)R.$$

Consequently, (6) is a super-solution pair for R big enough R . Therefore, by Theorem 1, there exists at least one almost periodic solution for system (2).

This finishes the proof of the existence of an almost periodic solution.

(ii) For uniqueness, we consider a maximal pair (\hat{u}_1, \hat{u}_2) and minimal pair $(\check{u}_1, \check{u}_2)$ of almost periodic solutions. We just need to prove that $\hat{u}_1(t) = \check{u}_1(t)$ and $\hat{u}_2(t) = \check{u}_2(t)$, to do this, we consider the following assertion that can be proved by using Parseval's equality.

Claim 1. *Let $\hat{\phi}, \check{\phi}$ be almost periodic functions such that*

$$\hat{\phi}(t) \geq \check{\phi}(t) \geq 0, \quad \mathcal{M}[\hat{\phi}] = \mathcal{M}[\check{\phi}].$$

Then, $\hat{\phi}(t) = \check{\phi}(t)$ for every $t \in \mathbb{R}$.

Note that by the fundamental calculus theorem the mean $\mathcal{M}[(\hat{u}_i)'] = \mathcal{M}[(\check{u}_i)'] = 0$. Then

$$\begin{aligned} \mathcal{M}[J_i(t)] &= \mathcal{M}[c_i(t)\hat{u}_i] - \mathcal{M}[a_{ij}(t)\tanh(b_{ij}(t)\hat{u}_j)] \\ &= \mathcal{M}[c_i(t)\check{u}_i] - \mathcal{M}[a_{ij}(t)\tanh(b_{ij}(t)\check{u}_j)], \quad i \neq j. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{M}[c_i(t)(\hat{u}_i - \check{u}_i)] &= \mathcal{M}[a_{ij}(t)(\tanh(b_{ij}(t)\hat{u}_j) - \tanh(b_{ij}(t)\check{u}_j))] \\ &\leq \mathcal{M}[a_{ij}(t)b_{ij}(t)(\hat{u}_j - \check{u}_j)], \quad i \neq j. \end{aligned}$$

The last inequality results from applying the mean value theorem to $\tanh(z)$ and the fact that the derivative of $\tanh(z)$ is bounded by 1. Hence

$$0 \leq c_{1*}\mathcal{M}[\hat{u}_1 - \check{u}_1] \leq (a_{12}b_{12})^* \mathcal{M}[\hat{u}_2 - \check{u}_2] \leq \frac{(a_{12}b_{12})^*(a_{21}b_{21})^*}{c_{2*}} \mathcal{M}[\hat{u}_1 - \check{u}_1].$$

If $\mathcal{M}[\hat{u}_1 - \check{u}_1] > 0$ this leads to a contradiction with the condition in (ii). Therefore $\mathcal{M}[\hat{u}_1] = \mathcal{M}[\check{u}_1]$, whence $\hat{u}_1 = \check{u}_1$ by the Claim above. From the above inequality we have $\hat{u}_2 = \check{u}_2$.

Finally, we can conclude the proof of Theorem 2. We can build arbitrarily large super-solutions. Thus, we have a single almost periodic orbit which is an attractor in an arbitrarily large compact set inside $\mathbb{R}_{>0}^2$. Then, the almost periodic orbit is an attractor at the entire set $\mathbb{R}_{>0}^2$. This concludes (ii) and therefore ends the proof of Theorem 2. \square

5 Numerical simulations of the solutions of the model

In this section, examples of the behaviour of the solutions of model (2) are shown, when the functions $J_i(t), a_{ij}(t), c_i(t)$ and $b_{ij}(t)$, for $i, j = 1, 2$ and $i \neq j$, are almost periodic functions that satisfy $J_i(t), a_{ij}(t), b_{ij}(t) \geq 0$, $c_i(t) \geq 0$ and $c_{i*} > 0$. Additionally, for comparison, scenarios in which these functions are periodic are considered. To model rates to have almost periodic dynamics. The following

functions will be used

$$\begin{aligned}
J_1(t) &= j_{11} (j_{12} + j_{15} \sin(j_{13}t) + j_{16} \cos(j_{14}t)), \\
J_2(t) &= j_{21} (j_{22} + j_{25} \sin(j_{23}t) + j_{26} \cos(j_{24}t)), \\
a_{12}(t) &= a'_{11} \left(a'_{12} + a'_{15} \sin(a'_{13}t) + a'_{16} \cos(a'_{14}t) \right), \\
a_{21}(t) &= a'_{21} \left(a'_{22} + a'_{25} \sin(a'_{23}t) + a'_{26} \cos(a'_{24}t) \right), \\
b_{12}(t) &= b'_{11} \left(b'_{12} + b'_{15} \sin(b'_{13}t) + b'_{16} \cos(b'_{14}t) \right), \\
b_{21}(t) &= b'_{21} \left(b'_{22} + b'_{25} \sin(b'_{23}t) + b'_{26} \cos(b'_{24}t) \right), \\
c_1(t) &= c_{11} (c_{12} + c_{15} \sin(c_{13}t) + c_{16} \cos(c_{14}t)), \\
c_2(t) &= c_{21} (c_{22} + c_{25} \sin(c_{23}t) + c_{26} \cos(c_{24}t)).
\end{aligned} \tag{7}$$

In the following, we show that solutions of the model converge to a global attractor when the conditions in Theorem 2 case (ii) are satisfied. To do this, in a first scenario, we use the following values of the parameters: $j_{11} = 0.5, j_{12} = 1.25, j_{13} = \sqrt{3}, j_{14} = \sqrt{2}, j_{15} = 0.5, j_{16} = 0.3, a'_{11} = 0.8, a'_{12} = 1.25, a'_{13} = \sqrt{3}, a'_{14} = \sqrt{2}, a'_{15} = 0.01, a'_{16} = 0.02, b'_{11} = 0.9, b'_{12} = 1.25, b'_{13} = \sqrt{3}, b'_{14} = \sqrt{2}, b'_{15} = 0.2, b'_{16} = 0.3, c_{11} = 2, c_{12} = 1.15, c_{13} = \sqrt{3}, c_{14} = \sqrt{2}, c_{15} = 0.005, c_{16} = 0.01, j_{21} = 0.9, j_{22} = 1.25, j_{23} = \frac{1}{\sqrt{2}}, j_{24} = \frac{1}{\sqrt{3}}, j_{25} = 0.1, j_{26} = 0.9, a'_{21} = 0.1, a'_{22} = 1.25, a'_{23} = \frac{1}{\sqrt{2}}, a'_{24} = \frac{1}{\sqrt{3}}, a'_{25} = 0.3, a'_{26} = 0.5, b'_{21} = 0.02, b'_{22} = 1.25, b'_{23} = \frac{1}{\sqrt{2}}, b'_{24} = \frac{1}{\sqrt{3}}, b'_{25} = 0.1, b'_{26} = 0.5, c_{21} = 1.5, c_{22} = 1.25, c_{23} = \frac{1}{\sqrt{2}}, c_{24} = \frac{1}{\sqrt{3}}, c_{25} = 0.02, c_{26} = 0.01$. For these values of the parameters, the rates given in (7) are almost periodic functions. Notice that, in this case, the conditions of Theorem 2 are satisfied since $(a_{12}b_{12})^* = (2.9229)$, $(a_{21}b_{21})^* = 0.25329$, $c_{1*} = 2.27$ and $c_{2*} = 1.83$. Therefore, $(a_{12}b_{12})^*(a_{21}b_{21})^* < c_{1*}c_{2*}$ is satisfied since $0.740341341 < 4.1541$. In Figure 2, we show the geometrical behaviour of the almost periodic rates, given in (7), for these values of the parameters. In Figure 3 cases (a) and (b), the numerical solutions of the model are shown when all the rates are almost periodic functions. In a second scenario, in Figure 3, cases (c) and (d), the solutions of the model are shown when the decay rates are constant functions. To do this, we use the following values of the parameters: $c_{15} = c_{16} = c_{25} = c_{26} = 0$. All other values of the parameters are the same as in cases (a) and (b). In this case, $c_{1*} = 2.30$ and $c_{2*} = 1.875$. Then, $c_{1*}c_{2*} = 4.3125$. Therefore, the conditions in Theorem 2 are satisfied.

For comparison purposes, in Figure 5, we show the behavior of the solutions of the model in two cases: when the rates are almost periodic functions and when they are periodic functions. To show the dynamics of the solutions in the almost periodic scenario, we use the same values as those used in Figure 3 cases (a) and (b). To show the dynamics of the solutions in the periodic scenario, we use $j_{13} = a'_{13} = b'_{13} = c_{13} = 1.7, j_{14} = a'_{14} = b'_{14} = c_{14} = 1.4, j_{23} = a'_{23} = b'_{23} = c_{23} = \frac{1}{1.4}, j_{24} = a'_{24} = b'_{24} = c_{24} = \frac{1}{1.7}$. All other values of the parameters are taken as in the almost periodic scenario. In Figure 4, we show the geometrical behavior of the rates when they are almost periodic or periodic functions.

In Figure 6, we show the behaviour of the solutions of model (2) when all the functions $J_i(t), a_{ij}(t), c_i(t), b_{ij}(t)$ are constant. In this scenario, all the values of the parameters associated with the trigonometric functions in (7) are zero. For the numerical simulations of the solutions of the model, we use $j_{11} = 0.5, j_{12} = 1.25, a_{11} = 0.8, a_{12} = 1.25, b_{11} = 0.9, b_{12} = 1.25, c_{11} = 5, c_{12} = 1.25, j_{21} = 0.9, j_{22} = 1.25, a_{21} = 0.1, a_{22} = 1.25, b_{21} = 0.02, b_{22} = 1.25, c_{21} = 10, c_{22} = 1.25$. All other parameters of the model are equal to zero. In this scenario, the solutions converge to a nontrivial equilibrium.

A particular scenario is shown in Figure 7. In this scenario, it is shown that when the functions $J_i(t), a_{ij}(t), b_{ij}(t)$ are constant and $c_1(t)$ or $c_2(t)$ is an almost periodic function and the other is periodic,

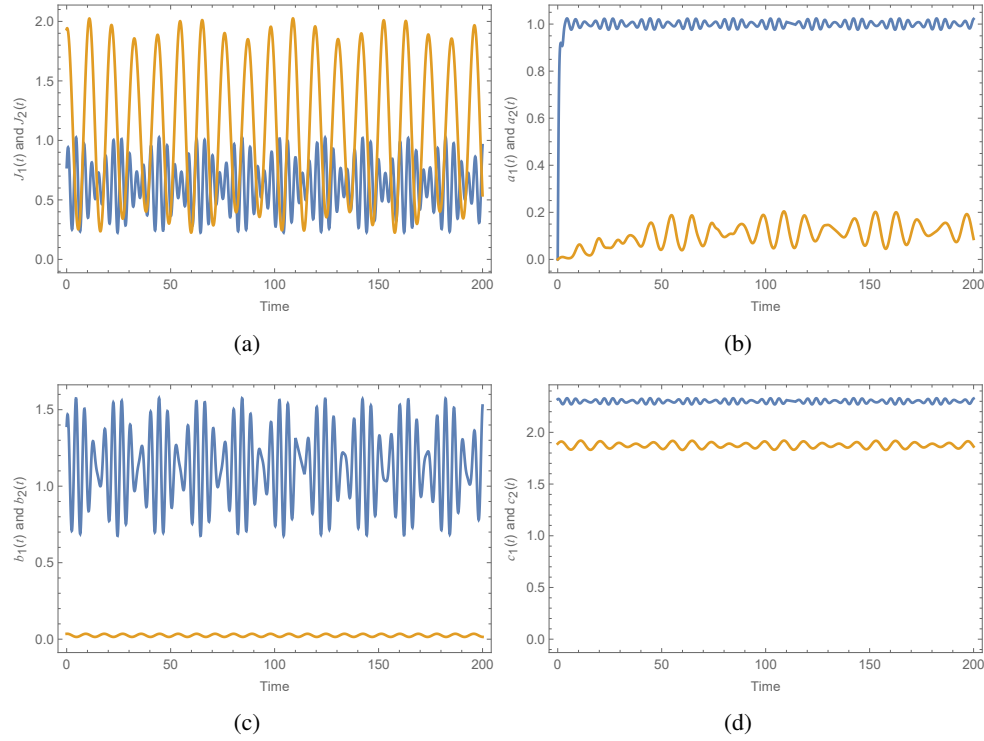


Figure 2: Almost periodic functions used in Figure 3 cases (a) and (b).

the neuronal dynamics has a stable almost periodic solution. For the numerical simulations shown in Figure 7 cases (a)-(b), $c_2(t) = 10(1.25 + 0.2\sin(0.4\sqrt{3}t) + 0.7\cos(0.3\sqrt{7}t))$ and all other values of the parameters are the same as those used in Figure 6. For the numerical simulations shown in Figure 7 cases (c)-(d), $c_1(t) = 2(1.25 + 0.2\sin(\sqrt{3}t) + 0.7\cos(\sqrt{7}t))$, $a_{11} = 10.8$, $a_{12} = 10.25$, $b_{11} = 1.25$ and other values of the parameters are equal to case shown in Figure 6. This example is relevant because the results of the existence and stability of an almost periodic solution of Hopfield models or some of its generalizations assume that the coefficients $c_i(t)$ are just continuous and the almost periodicity is not considered in the assumptions of these models; see [3, 26–28, 38, 39]. We show that almost periodic solutions of the proposed model occur when the decay rates are constant functions.

6 Discussion

From the Hippocratic era, humanity has tried to understand the mechanism inducing biological oscillators. In the Corpus Hippocraticum, associations between infections diseases and social and environmental drivers appear [33]. Although more than two thousand years have passed since these discoveries until today. It was in 1829 that Fecher experimentally found that an electrochemical cell showed oscillations [10]. In this direction, it was only almost a century ago that the first advances were made in the construction of mathematical models that described biological oscillators. The first advances in this theory are associated to Lotka with his work on simultaneous reactions through kinetic models [29–31]

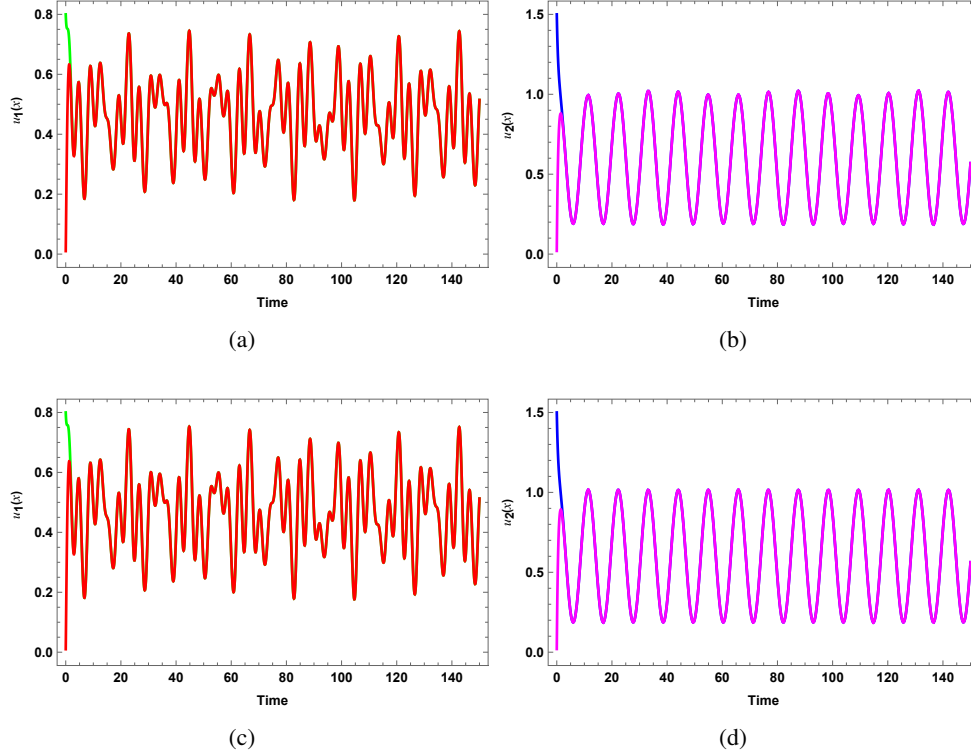


Figure 3: The neuronal state variables show an almost periodic behavior due to its excitatory and inhibitory inputs. In cases (a) and (b), the decay rates $c_1(t)$ and $c_2(t)$ are almost periodic functions. In cases (c) and (d), the decay rates are constant. In both scenarios, the solutions converge to a global almost periodic attractor for the initial conditions $(0.2, 0.15)$, $(0.01, 0.02)$.

and the second advances are associated to Volterra with his works on mathematical ecology [37]. From a neurobiological perspective, it was in 1952 that Hodgkin and Huxley proposed a mathematical model to describe the action potential in excitable cells [18]. Although there are important advances on theory of biological oscillators and rhythmic biological phenomena that are ubiquitous in nature, we are far from understanding the phenomenology behind the mechanisms that produce oscillatory behavior in biological systems.

Among oscillatory biological systems, neuronal networks stand out for their complexity. However, for a better understanding of these complex systems, it was necessary to analyze each neuron individually because the complexity associated to a neuronal networks is because of the number of connections among neurons. From the Hodgkin-Huxley model passing through the FitzHugh-Nagumo model [11, 12, 32] to computational models currently used that emulate the complex dynamics of the brain, there have been advances in understanding how the brain works [3, 38].

In this work, it is analyzed a two-neuron model when all parameters of the model are almost periodic functions. The model proposed generalizes, in a sense, the results obtained in [3, 26–28, 38, 39] since, in the modeling process, we consider that all the rates are almost periodic functions. The proposed model describes a discrete entity of the neuronal network in which an upstream system $u_i(t)$ is input-output

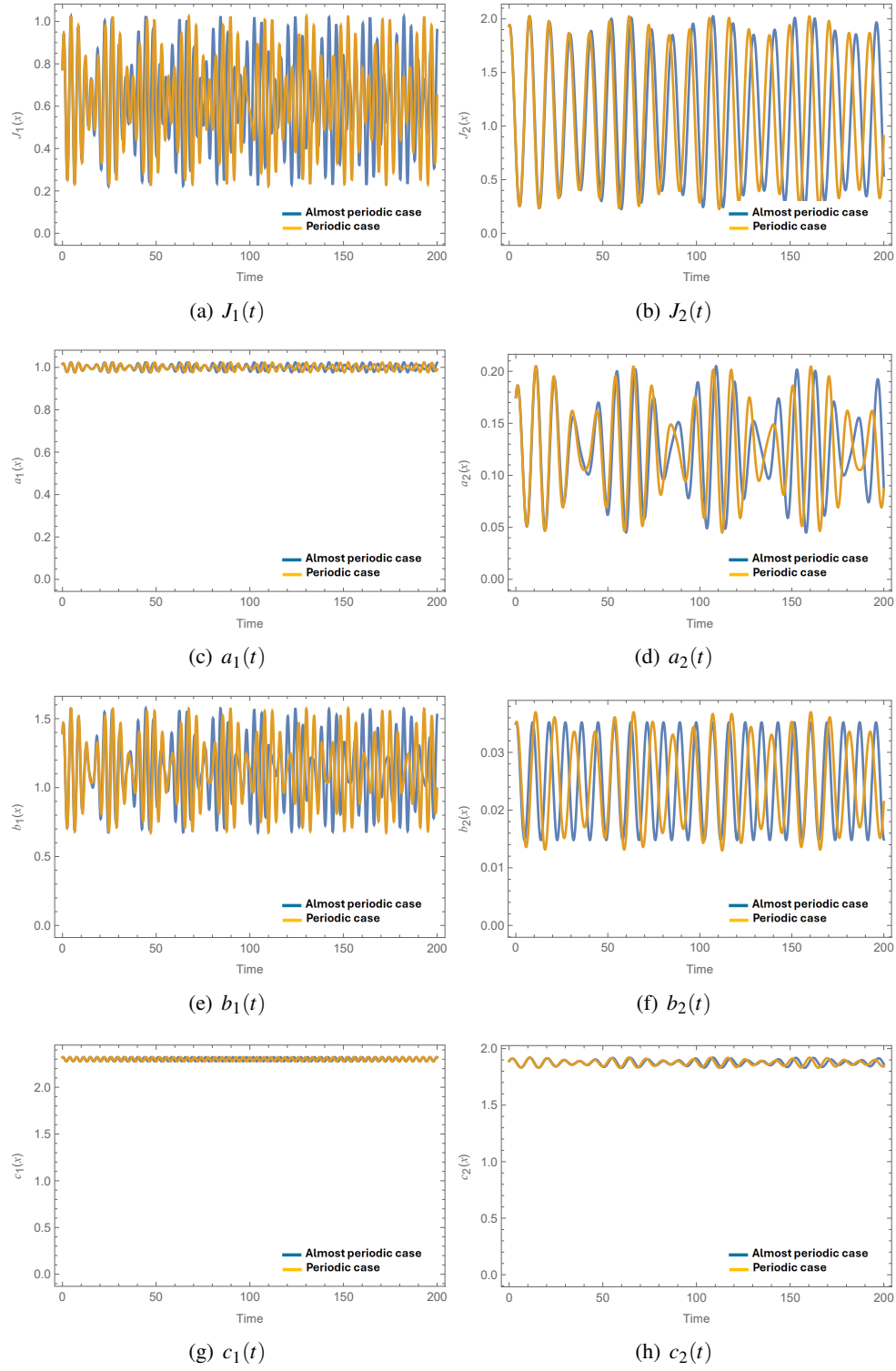


Figure 4: Almost periodic and periodic functions used in the model for comparison purposes.

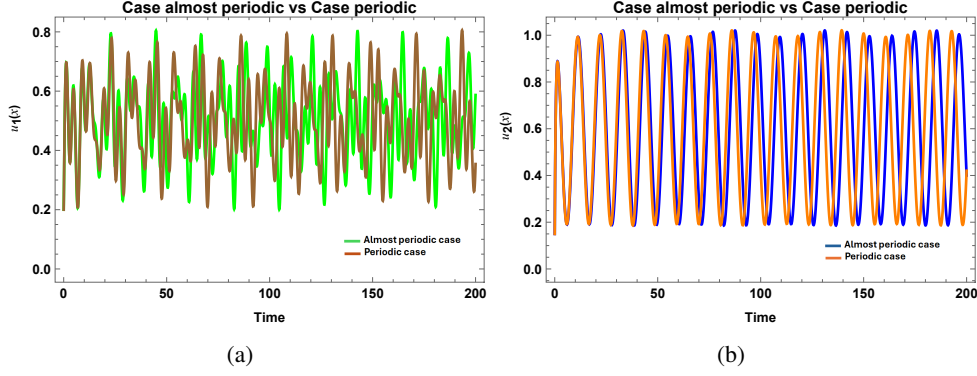


Figure 5: Although the solutions are very similar at the beginning of time in the almost and periodic cases, when time increases, the solutions are separated. For the numerical simulation, the initial condition is $(0.2, 0.15)$.

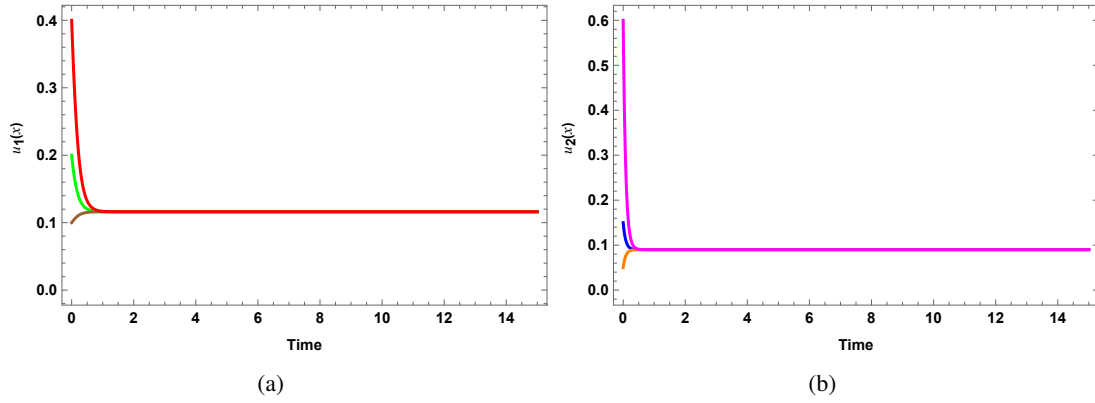


Figure 6: Behavior of u_1 and u_2 when all parameters of the model are constant. Notice that, the solutions asymptotically converge to a unique equilibrium point for different initial conditions, which are given by $(0.2, 0.15)$, $(0.1, 0.05)$, $(0.4, 0.6)$.

connected to a downstream one $u_j(t)$, for $i \neq j$. Results obtained from the analysis of the model show that a unique almost periodic attractor exists when the conditions in theorem 2 are satisfied. In particular when the decay rates satisfies that $(a_{12}b_{12})^*(a_{21}b_{21})^* < c_1^*c_2^*$. Therefore, for every initial condition, the solutions of model (2) tend to a unique almost periodic solution. This result is relevant from the neuronal networks models perspective since the existence of this attractor, that depends on the external inputs, is related to the recall of memories. In such a scenario, the brain returns from a current state to another state by the trigger provided for the system remembers the memories from outside as external inputs.

Numerical simulations show that solutions with different initial conditions converge rapidly to the almost periodic attractor even though the decay rates, $c_1(t)$ and $c_2(t)$ are constant or almost periodic functions; see Figure 3. For this case, the exogenous and endogenous stimuli are shown in Figure 2. Notice that, amplitude of the oscillations in the decay rates, $c_1(t)$ and $c_2(t)$, is not too large. Also, in Figure 5, we show that modeling the excitatory and inhibitory inputs using a periodic function when they actually are almost periodic functions might lead to underestimate or overestimate the value of the

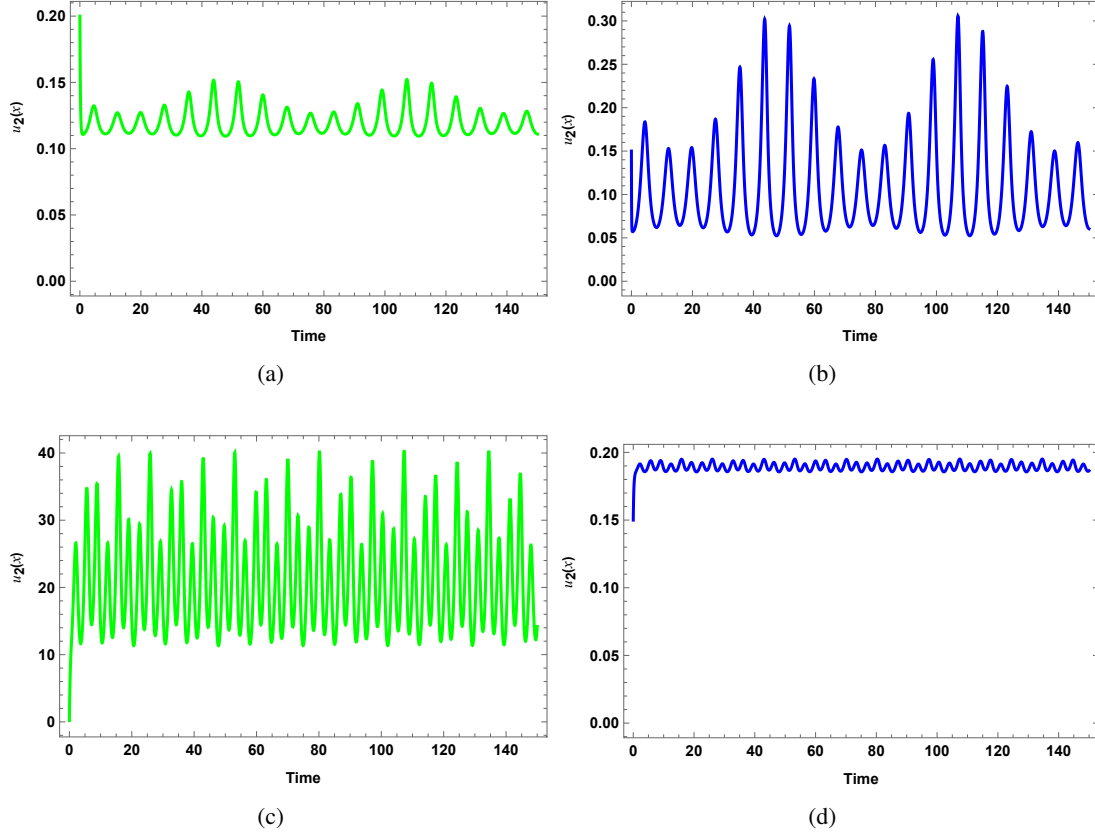


Figure 7: In cases (a)-(b), $c_1(t)$ is constant and $c_2(t)$ is an almost periodic function. In contrast, in cases (c)-(d), $c_2(t)$ is constant and $c_1(t)$ is an almost periodic function. Solutions of the almost periodic model converge to an almost periodic global attractor when one of the decay rates, $c_1(t)$ or $c_2(t)$, is an almost periodic function and all other are constant functions. For the numerical simulations, the initial condition is given by $(0.2, 0.15)$.

neuronal state variables. Notice that, in such a scenario, at the beginning of the process, both solutions are very similar, however, in the long term, there are intervals of time in which the solutions are separated. Numerical simulations in Figure 5 show that estimation errors can occur depending on, in the modeling process, whether the rates are periodic or almost periodic functions. Such estimation errors can lead to failure in forecasting the time in which neurons must coordinate, which can be catastrophic for the neuronal network. To simulate this scenario, we use the functions shown in Figure 4 to show both the exogenous and endogenous stimuli. For the numerical simulations, we consider that decay rates show a small amplitude.

Figure 6 shows a scenario in which all the parameters of the model are constant. This result agrees with the results of the Hopfield model and its generalization analyzed in [15], which states that the solutions converge to a unique equilibrium point. According to Gopalsamy and He, the existence of a globally stable solution is associated to the recall of perfect memories since they do not need hints and guesses to return from a current memory to another state in which it has been before.

Finally, Figure 7 shows the existence of a unique almost periodic solution of the model when the

rates $J_i(t), a_{ij}(t), b_{ij}(t)$ are constant and the functions $c_1(t)$ and $c_2(t)$, that measure the decay rate, are almost periodic functions. This result is relevant because other works on this subject assume continuity of the functions $c_1(t)$ and $c_2(t)$ to prove the existence and stability of almost periodic solutions.

7 Conclusion

The principal result obtained from the analysis of the model is the existence of a unique globally stable almost periodic solution which is associated with almost periodic endogenous and exogenous stimulus. This unique almost periodic attractor exists when some conditions over the rates of the model are satisfied. Particularly, a condition involving the decay rates which is given by $(a_{12}b_{12})^*(a_{21}b_{21})^* < c_{1*}c_{2*}$. Therefore, the neuronal state variable of both neurons converge to an equilibrium given by an almost periodic solution for all initial condition. This equilibrium, from a neuronal networks models perspective, is related to the recall of memories since the brain returns from different states to one in which it has been before.

Modeling neuronal networks using almost periodic models offers advantages because modeling the functioning of biological neuronal networks with periodic or constant rates is very restrictive. Numerical simulations of the solutions of the model show that if a periodic neuronal network model is used to model the neuronal state variables, when this phenomenon actually must be modeled by an almost periodic neuronal network, an overestimation or underestimation misleading is presented. Therefore, the prediction of the time in which the neurons must be synchronized can fail.

In ongoing work, we analyze Hopfield-type systems in higher dimensions to prove the existence and stability of almost periodic functions when the decay rates are almost periodic functions.

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