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Blow-up phenomena for a couple of parabolic equations with memory and source terms: Analytical and simulation

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Abstract. In this paper, the focus is on investigating the asymptotic behavior of the solution for a system of parabolic equations with memory terms acting in both equations. This system has many applications in various scientific fields, including heat conduction in materials with memory effects and the study of biological systems exhibiting memory phenomena. The system of parabolic equations with a memory term provides a powerful framework for understanding and predicting the behavior of such complex systems, with emphasis on the role of the memory term in capturing the system's history-dependent behavior. Firstly, we assume that the relaxation functions $\mu_2(t) \leq \mu_1(t)$, for all $t \geq 0$, and under certain conditions regarding the function p$(\cdot)$ we prove that the solution with positive initial energy blows up in finite time. Finally, we present the theoretical results as numerical findings in the form of figures that illustrate and confirm the results by studying examples in two dimensions.

Keywords: Semilinear parabolic equation, source term, memory term, variable-exponent, blow up.
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1 Introduction

We consider the following problem:

$$\begin{cases} \rho_t - \Delta\rho + \int_0^t \mu_1(t-s)\Delta\rho(s)\,ds = \varphi_1(\rho,\ \theta), & \text{in } \Omega_T, \\ \theta_t - \Delta\theta + \int_0^t \mu_2(t-s)\Delta\theta(s)\,ds = \varphi_2(\rho,\ \theta), & \text{in } \Omega_T, \\ \rho(x,0) = \rho_0(x),\ \theta(x,0) = \theta_0(x), & \text{in } \Omega, \\ \rho(x,t) = \theta(x,t) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a bounded domain in $\mathbb{R}^n$, $(n = 1,\ 2,\ 3)$, with smooth boundary $\partial\Omega$. The $\Omega_T = \Omega \times (0,\ T)$, and q is a given continuous function on Ω. Also, the functions $\varphi_1,\ \varphi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are given by

$$\varphi_1(\rho,\ \theta) = a\,|\rho+\theta|^{2q(x)+2}(\rho+\theta) + b\,|\rho|^{q(x)}\rho\,|\theta|^{q(x)+2},$$
$$\varphi_2(\rho,\ \theta) = a\,|\rho+\theta|^{2q(x)+2}(\rho+\theta) + b\,|\rho|^{q(x)+2}|\theta|^{q(x)}\theta,$$

where a and b are positive constants.

The function q satisfies

$$\begin{cases} 0 \le q_1 \le q(x) \le q_2, & \text{if } n = 1, 2, \\ 0 \le q(x), & \text{if } n = 3, \end{cases} \tag{2}$$

where

$$q_1 := ess \inf_{x\in\Omega} q(x) \le q(x) \le q_2 := ess \sup_{x\in\Omega} q(x),$$

and satisfying

$$|q(x_1) - q(x_2)| \le -\frac{R}{\log|x_1 - x_2|},\ \text{for all } x_1,\ x_2 \in \Omega,\ \text{with } |x_1 - x_2| < \kappa, \tag{3}$$

where $R > 0$ and $0 < \kappa < 1$.

Consider the following semilinear heat equation

$$\rho_t - \Delta\rho + \int_0^t \mu(t-s)\,\Delta\rho(s)\,ds = |\rho|^{q-2}\rho. \tag{4}$$

It arises in a range of mathematical models employed in the fields of engineering and the physical sciences, heat transfer, and ecology models. Messaoudi in **[8]** investigeted equation (4) under appropriate conditions on μ and q. He established a result indicating the occurrence of a explosion in certain solutions characterized by positive initial energy. The study of this type (4) has draw the attention of considerable esearchers, see **[1,5,10–12,19]**. Ouaoua and Maouni have investigated problem (4) in the absence of the memory term, leading to various outcomes such as exponential growth and non-global existence being documented in prior studies **[13]**. Wu in **[18]** considered the following boundary values problem

$$\begin{cases} \rho_t - \operatorname{div}\left(\sigma\left(|\nabla\rho|^2\right)\nabla\rho\right) = \varphi_1(\rho,\ \theta), & \text{in } \Omega\times[0,\ \infty), \\ \theta_t - \operatorname{div}\left(\sigma\left(|\nabla\theta|^2\right)\nabla\theta\right) = \varphi_2(\rho,\ \theta), & \text{in } \Omega\times[0,\ \infty), \\ \rho(x,0) = \rho_0(x),\ \theta(x,0) = \theta_0(x), & \text{in } \Omega, \\ \rho(x,t) = \theta(x,t) = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

He proved that, under suitable conditions concerning nonlinearity and certain initial data when Ω is a bounded domain in $\mathbb{R}^3$, the lower bound of the blow-up time is determined if a blow up occurs, and a condition is imposed to guarantee the occurrence of the blow up, while an upper bound of the blow-up time is also provided.

For a single equation, when the source term $f(u) = |u|^{p-2}u$, Payne et al. in **[14]**, achieved the occurrence of solution blow-up under specific conditions on the nonlinearities. In the case $\sigma \equiv 1$, Messaoudi in **[7],** under appropriate conditions on f, proved the occurrence of a rapid growth or explosion in solutions characterized by either zero or negative initial energy. Piskin and Ekinci in [16, 17] treated the following system

$$\begin{cases} \rho_t - \operatorname{div}\left(|\nabla\rho|^{p-2}\nabla\rho\right) + |\rho|^{q-2}\rho_t = \varphi_1(\rho,\ \theta), \\ \theta_t - \operatorname{div}\left(|\nabla\theta|^{p-2}\nabla\theta\right) + |\theta|^{q-2}\theta_t = \varphi_2(\rho,\ \theta). \end{cases} \tag{6}$$

The authors have demonstrated that, for the specified parameter values $(p > 2, q > 2)$ and $(p = 2, q > 2)$, there are nonexistence of global solutions and exponential growth of solution with negative initial energy. Pang and Qiao in [15] in the case $p = 2$, where $q > 2$, investigated the blow-up properties of system (6) for both negative initial energy and positive initial energy. In the absence of $|\rho|^{q-2}\rho_t$ and $|\theta|^{q-2}\theta_t$ terms in equation (6) when $p = 2$, many researchers [2–4] have focused on studying this type of equations, and various results have been obtained.

The goal of this paper is to investigate the following. In Section 2 , we present some definitions and important theories. In Section 3, we prove the asymptotic behavior of the solution, which demonstrates a blow-up in finite time. In the final Section, we attempt to provide three tests to validate the previous results.

2 Preliminaries

In this section, we present some important results that are used in the paper.

Lemma 1 ([6])**.** *Suppose that* $r \in C(\Omega)$ *such that*

$$\begin{cases} 1 \le r_1 \le r(x) \le r_2 < 2\frac{n}{n-2}, & \text{if } n \ge 3, \\ 1 \le r(x) < +\infty, & \text{if } n = 1,\ 2. \end{cases}$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is continuous and compact.*

Lemma 2 ([6])**.** *If* $r \in C(\Omega)$ *such that*

$$1 \le r_1 := ess \inf_{x\in\Omega} r(x) \le r(x) \le r_2 := ess \sup_{x\in\Omega} r(x) < \infty,$$

then we have

$$\min\left\{\|\rho\|_{r(\cdot)}^{r_1},\ \|\rho\|_{r(\cdot)}^{r_2}\right\} \le \int_\Omega |\rho|^{r(x)}\,dx \le \max\left\{\|\rho\|_{r(\cdot)}^{r_1},\ \|\rho\|_{r(\cdot)}^{r_2}\right\},$$

for any $\rho \in L^{r(\cdot)}(\Omega)$.

We define the energy functional

$$\begin{aligned} \mathscr{E}(t) = {} & \frac{1}{2}\left(1 - \int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2 + \frac{1}{2}\left(1 - \int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2 + \frac{1}{2}(\mu_1 \circ \nabla\rho)(t) \\ & + \frac{1}{2}(\mu_2 \circ \nabla\theta)(t) - \int_\Omega \Phi(\rho,\ \theta)\,dx, \end{aligned} \tag{7}$$

where

$$(\zeta \circ w)(t) = \int_0^t \zeta(t-\tau) \|w(t) - w(\tau)\|_2^2 d\tau.$$

For μ_1 and μ_2, we suppose that

$$\begin{cases} \mu_1(t) \geq 0, \mu_1'(t) \leq 0, \ 1 - \int_0^\infty \mu_1(s)\, ds = l > 0, \\ \mu_2(t) \geq 0, \mu_2'(t) \leq 0, \ 1 - \int_0^\infty \mu_2(s)\, ds = k > 0, \end{cases} \tag{8}$$

and

$$\mu_2(t) \leq \mu_1(t), \text{ for all } t \geq 0. \tag{9}$$

By recalling the definition of $\varphi_1(\rho,\ \theta)$ and $\varphi_2(\rho,\ \theta)$, one can easily verify that

$$\rho\varphi_1(\rho,\ \theta) + \theta\varphi_2(\rho,\ \theta) = 2(q(x)+2)\ \Phi(\rho,\ \theta),\ \ \forall(\rho,\ \theta) \in \mathbb{R}^2, \tag{10}$$

where

$$\Phi(\rho,\ \theta) = \frac{1}{2q(x)+4}\left[a|\rho+\theta|^{2q(x)+4} + 2b|\rho\theta|^{q(x)+2}\right].$$

We have the fallowing results, which represents a bound for $\int_\Omega \Phi(\rho,\ \theta)\, dx$ and is used in Section 3.

Lemma 3 ([9]). *There exist c_0 and c_1 such that*

$$\frac{c_0}{2q(x)+4}\left(|\rho|^{2q(x)+4} + |\theta|^{2q(x)+4}\right) \leq \Phi(\rho,\ \theta) \leq \frac{c_1}{2q(x)+4}\left(|\rho|^{2q(x)+4} + |\theta|^{2q(x)+4}\right). \tag{11}$$

Corollary 1. *There exist a_0 and a_1 such that*

$$a_0 \int_\Omega \left(|\rho|^{2q(x)+4} + |\theta|^{2q(x)+4}\right) dx \leq \int_\Omega \Phi(\rho,\ \theta)\, dx \leq a_1 \int_\Omega \left(|\rho|^{2q(x)+4} + |\theta|^{2q(x)+4}\right) dx, \tag{12}$$

where $a_0 = \frac{c_0}{2(q_2+2)}$, $a_1 = \frac{c_1}{2(q_1+2)}$.

Definition 1. *A pair $(\rho,\ \theta)$ solution of (1) is said to be strong solution, if*

$$\rho,\ \theta \in C\left((0,\ T),\ H_0^1(\Omega)\right) \cap C^1\left((0,\ T),\ L^2(\Omega)\right),$$

satisfying $\frac{d}{dt}\mathscr{E}(t) \leq 0$, and

$$\int_0^t \int_\Omega \left(\rho_t\phi + \nabla\rho\nabla\phi - \int_0^s \mu_1(s-\tau)\nabla\rho(\tau)\nabla\phi(s)\, d\tau - \varphi_1(\rho,\ \theta)\phi\right) dxds = 0,$$

$$\int_0^t \int_\Omega \left(\theta_t\psi + \nabla\theta\nabla\psi - \int_0^s \mu_2(s-\tau)\nabla\theta(\tau)\nabla\psi(s)\, d\tau - \varphi_2(\rho,\ \theta)\psi\right) dxds = 0,$$

for all t in $(0,\ T)$ and for all test functions $\phi,\ \psi \in C\left((0,\ T),\ H_0^1(\Omega)\right)$.

3 Blow up for positive initial energy

In this section, we establish the concept of blow-up in solutions exhibiting positive energy. Let $B=\max\{c_*/l,\ c_*/k\}$ where c_* is the embedding constant of $H_0^1(\Omega)\hookrightarrow L^{2(q_2+2)}(\Omega)$, and set

$$\alpha_1=\left(\frac{1}{2c_1}\right)^{\frac{1}{2(q_2+1)}}\left(\frac{q_1+2}{q_2+2}\right)^{\frac{1}{2(q_2+1)}}B^{-\frac{q_2+2}{q_2+1}};\ \mathscr{E}_1=\frac{1}{2}\left(1-\frac{1}{q_2+2}\right)\alpha_1^2-\frac{c_1}{q_1+2}|\Omega|, \tag{13}$$

$$\mathscr{F}(t)=\mathscr{E}_1-\mathscr{E}(t). \tag{14}$$

Lemma 4. *Let $(\rho_0,\ \theta_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ and (2) holds, then we have*

$$\begin{aligned}\mathscr{E}'(t)=&-(\frac{1}{2}\mu_1(t)\|\nabla\theta\|_2^2+\frac{1}{2}\mu_2(t)\|\nabla\theta\|_2^2+\|\rho_t\|_2^2+\|\theta_t\|_2^2\\&-\frac{1}{2}\left(\mu_1'\circ\nabla\rho(t)\right)-\frac{1}{2}\left(\mu_2'\circ\nabla\theta(t)\right))\le 0,\end{aligned} \tag{15}$$

and

$$\mathscr{E}(t)\le\mathscr{E}(0). \tag{16}$$

Proof. We multiply the first equation of (1) by ρ_t, the second by θ_t and integrate over Ω,We obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\left(1-\int_0^t\mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\frac{1}{2}\left(1-\int_0^t\mu_2(s)\,ds\right)\|\nabla\theta\|_2^2\right.\\&\quad\left.+\frac{1}{2}(\mu_1\circ\nabla\rho)(t)+\frac{1}{2}(\mu_2\circ\nabla\theta)(t)-\int_\Omega\Phi(\rho,\ \theta)\,dx\right\}\\&=-(\frac{1}{2}\mu_1(t)\|\nabla\rho\|_2^2+\frac{1}{2}\mu_2(t)\|\nabla\theta\|_2^2+\|\rho_t\|_2^2+\|\theta_t\|_2^2-\frac{1}{2}\left(\mu_1'\circ\nabla\rho(t)\right)-\frac{1}{2}\left(\mu_2'\circ\nabla\theta(t)\right)).\end{aligned}$$

Then

$$\begin{aligned}\mathscr{E}'(t)=&-(\frac{1}{2}\mu_1(t)\|\nabla\rho\|_2^2+\frac{1}{2}\mu_2(t)\|\nabla\theta\|_2^2+\|\rho_t\|_2^2+\|\theta_t\|_2^2\\&-\frac{1}{2}\left(\mu_1'\circ\nabla\rho(t)\right)-\frac{1}{2}\left(\mu_2'\circ\nabla\theta(t)\right))\le 0,\end{aligned} \tag{17}$$

integrating (17) over $(0,\ t)$, we get

$$\mathscr{E}(t)\le\mathscr{E}(0).$$ □

Now, we present two important lemmas that are the key to this new work. Through them, we obtain the main result, which is to find the relation between $\|\rho\|_{2(q_1+2)}^{2(q_1+2)}+\|\theta\|_{2(q_1+2)}^{2(q_1+2)}$ and $\int_\Omega|\rho|^{2(q(x)+2)}\,dx+\int_\Omega|\theta|^{2(q(x)+2)}\,dx$.

Lemma 5 ([9]). *Suppose that (2) holds. Hence the solution of (1) satisfies, for $c>0$,*

$$\int_\Omega|\rho|^{2(q(x)+2)}\,dx+\int_\Omega|\theta|^{2(q(x)+2)}\,dx\ge c\left(\|\rho\|_{2(q_1+2)}^{2(q_1+2)}+\|\theta\|_{2(q_1+2)}^{2(q_1+2)}\right). \tag{18}$$

Lemma 6. *Let $(\rho,\ \theta)$ be a solution of problem (1). Assume condition (2) and (3) hold. Then*

$$\int_{\Omega}|\rho|^{2q(x)+4}dx+\int_{\Omega}|\theta|^{2q(x)+4}dx\leq 2|\Omega|+\|\rho\|_{2(q_2+2)}^{2(q_2+2)}+\|\theta\|_{2(q_2+2)}^{2(q_2+2)},$$

where $|\Omega|$ is Lebesgue's measure of Ω.

Proof. Let

$$\Omega_1^-=\{x\in\Omega,\ |\rho(x,t)|\leq 1\}\text{ and }\Omega_1^+=\{x\in\Omega,\ |\rho(x,t)|>1\},$$
$$\Omega_2^-=\{x\in\Omega,\ |\theta(x,t)|\leq 1\}\text{ and }\Omega_2^+=\{x\in\Omega,\ |\theta(x,t)|>1\}.$$

We have

$$\int_{\Omega}|\rho|^{2(q(x)+2)}dx\leq\int_{\Omega_1^-\cap\Omega}|\rho|^{2(q_2+2)}dx+\int_{\Omega_1^+\cap\Omega}|\rho|^{2(q_2+2)}dx.$$

So, we get

$$\int_{\Omega}|\rho|^{2(q(x)+2)}dx\leq|\Omega|+\int_{\Omega_1^+\cap\Omega}|\rho|^{2(q_2+2)}dx.$$

This implies that

$$\int_{\Omega}|\rho|^{2(q(x)+2)}dx\leq|\Omega|+\|\rho\|_{2(q_2+2)}^{2(q_2+2)}. \tag{19}$$

Similarly, we obtain

$$\int_{\Omega}|\theta|^{2(q(x)+2)}dx\leq|\Omega|+\|\theta\|_{2(q_2+2)}^{2(q_2+2)}. \tag{20}$$

Combining (19) and (20), we get

$$\int_{\Omega}|\rho|^{2q(x)+4}dx+\int_{\Omega}|\theta|^{2q(x)+4}dx\leq 2|\Omega|+\|\rho\|_{2(q_2+2)}^{2(q_2+2)}+\|\theta\|_{2(q_2+2)}^{2(q_2+2)}.$$ □

Before proving the main theorem, the following theorem must be proven.

Theorem 1. *Let $(\rho_0,\ \theta_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ satisfy*

$$\alpha_1<\left(\|\nabla\rho_0\|_2^2+\|\nabla\theta_0\|_2^2\right)^{\frac{1}{2}},\ \ \mathscr{E}(0)<\mathscr{E}_1,$$

then there exists a constant $\alpha_2>\alpha_1$ such that

$$\left(\left(1-\int_0^t\mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\left(1-\int_0^t\mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+(\mu_1\circ\nabla\rho)(t)+(\mu_1\circ\nabla\theta)(t)\right)^{\frac{1}{2}}\geq\ \alpha_2. \tag{21}$$

Proof. By Corollary 1, we have

$$\begin{aligned}\mathscr{E}(t)&=\frac{1}{2}\left(1-\int_0^t\mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\frac{1}{2}\left(1-\int_0^t\mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+\frac{1}{2}(\mu_1\circ\nabla\rho)(t)\\&+\frac{1}{2}(\mu_2\circ\nabla\theta)(t)-\int_{\Omega}\Phi(\rho,\ \theta)\,dx\\&\geq\frac{1}{2}\left(1-\int_0^t\mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\frac{1}{2}\left(1-\int_0^t\mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+\frac{1}{2}(\mu_1\circ\nabla\rho)(t)\\&+\frac{1}{2}(\mu_2\circ\nabla\theta)(t)-\frac{c_1}{2(q_1+2)}\int_{\Omega}\left(|\rho|^{2q(x)+4}+|\theta|^{2q(x)+4}\right)dx.\end{aligned}$$

Thanks to Lemma 6 and Poicaré's inequality, we obtain

$$\begin{aligned}
\mathscr{E}(t) \geq & \frac{1}{2}\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\frac{1}{2}\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+\frac{1}{2}(\mu_1\circ\nabla\rho)(t) \\
& +\frac{1}{2}(\mu_2\circ\nabla\theta)(t)-\frac{c_1}{2(q_1+2)}\left(2|\Omega|+\|\rho\|_{2(q_2+2)}^{2q_2+4}+\|\theta\|_{2(q_2+2)}^{2q_2+4}\right) \\
\geq & \frac{1}{2}\left(\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t)\right) \\
& -\frac{c_1}{2(q_1+2)}\times\left(B^{2(q_2+2)}\left(l^{2(q_2+2)}\left(\|\nabla\rho\|_2^2\right)^{q_2+2}+k^{2(q_2+2)}\left(\|\nabla\theta\|_2^2\right)^{q_2+2}\right)+2|\Omega|\right) \\
\geq & \frac{1}{2}\left(\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t)\right) \\
& -\frac{c_1}{2(q_1+2)}\times B^{2(q_2+2)}\left(\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2\right. \\
& \left.+(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t)\right)^{q_2+2}-\frac{c_1}{(q_1+2)}|\Omega| \\
\geq & \frac{1}{2}\xi^2-\frac{c_1}{q_1+2}B^{2(q_2+2)}\xi^{2(q_2+2)}-\frac{c_1}{q_1+2}|\Omega|,
\end{aligned}$$

where $\xi=[\gamma(t)]^{\frac{1}{2}}$ and

$$\gamma(t)=\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2+\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2+(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t).$$

Let

$$f(\xi)=\frac{1}{2}\xi^2-\frac{c_1}{q_1+2}B^{2(q_2+2)}\xi^{2(q_2+2)}-\frac{c_1}{q_1+2}|\Omega|. \tag{22}$$

It is clear that f is increasing for $0<\xi<\alpha_1$ and decreasing for $\xi>\alpha_1$; $f(\xi)\to-\infty$ and $f(\alpha_1)=\mathscr{E}_1$ where α_1 and $\mathscr{E}_1$ are constants defined in (13). Since $\mathscr{E}(0)<\mathscr{E}_1$, $\left(\|\nabla\rho_0\|_2^2+\|\nabla\rho_0\|_2^2\right)^{\frac{1}{2}}>\alpha_1$, we can deduce the existence of a constant $\alpha_2>\alpha_1$ such that $\mathscr{E}(0)=f(\alpha_2)$. Then by (22), we have

$$f\left[\left(\|\nabla\rho_0\|_2^2+\|\nabla\rho_0\|_2^2\right)^{\frac{1}{2}}\right]<\mathscr{E}(0)=f(\alpha_2),$$

which implies that $\left(\|\nabla\rho_0\|_2^2+\|\nabla\rho_0\|_2^2\right)^{\frac{1}{2}}>\alpha_2$. To establish (21), we assume that there exists a t_0 such that

$$[\gamma(t_0)]^{\frac{1}{2}}<\alpha_2,$$

Given a positive value $t_0>0$ and utilizing the continuity of γ. we have the flexibility to select t_0 in such a way that

$$[\gamma(t_0)]^{\frac{1}{2}}>\alpha_1.$$

We use again (22), and obtain

$$\mathscr{E}(t_0)\geq f(\gamma(t_0))>f(\alpha_2)=\mathscr{E}(0),$$

which is a contraction since $\mathscr{E}(t)\leq\mathscr{E}(0)$, for all $t\in[0,\ T]$. Then (21) is confirmed. □

Lemma 7. *Let $(\rho_0,\ \theta_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and (2) be fulfilled, then we have*

$$0 < \mathscr{F}(0) \leq \mathscr{F}(t) \leq \int_\Omega \Phi(\rho,\ \theta)\,dx, \tag{23}$$

and

$$\frac{1}{2(q_2+2)}\alpha_2^2 + \frac{c_1}{q_1+2}|\Omega| \leq \int_\Omega \Phi(\rho,\ \theta)\,dx. \tag{24}$$

Proof. Using (7), (16) and (14), we obtain

$$\begin{aligned}
0 < \mathscr{E}_1 - \mathscr{E}(0) &= \mathscr{F}(0) \leq \mathscr{F}(t) \\
&\leq \mathscr{E}_1 - \frac{1}{2}\left(1 - \int_0^t \mu_1(s)\,ds\right)\|\nabla\rho\|_2^2 - \frac{1}{2}\left(1 - \int_0^t \mu_2(s)\,ds\right)\|\nabla\theta\|_2^2 \\
&\quad - \frac{1}{2}(\mu_1 \circ \nabla\rho)(t) - \frac{1}{2}(\mu_2 \circ \nabla\theta)(t) + \int_\Omega \Phi(\rho,\ \theta)\,dx.
\end{aligned}$$

Since $\mathscr{E}_1 = f(\alpha_1)$ and $\alpha \geq \alpha_2 > \alpha_1$, we have

$$\begin{aligned}
\mathscr{F}(t) &\leq \frac{1}{2}\left(1 - \frac{1}{q_2+2}\right)\alpha_1^2 - \frac{c_1}{q_1+2}|\Omega| - \frac{1}{2}\alpha_1^2 + \int_\Omega \Phi(\rho,\ \theta)\,dx \\
&= -\frac{1}{2(q_2+2)}\alpha_1^2 - \frac{c_1}{q_1+2}|\Omega| + \int_\Omega \Phi(\rho,\ \theta)\,dx \\
&\leq \int_\Omega \Phi(\rho,\ \theta)\,dx.
\end{aligned}$$

To prove the second estimate, we use (7) and the decreasing property of $\mathscr{E}$ to get

$$\mathscr{E}(0) \geq \mathscr{E}(t) \geq \frac{1}{2}\alpha^2 - \int_\Omega \Phi(\rho,\ \theta)\,dx.$$

Consequently,

$$\int_\Omega \Phi(\rho,\ \theta)\,dx \geq \frac{1}{2}\alpha^2 - \mathscr{E}(0).$$

Since $\mathscr{E}(0) = f(\alpha_2)$ and $\alpha \geq \alpha_2$, we get

$$\int_\Omega \Phi(\rho,\ \theta)\,dx \geq \frac{1}{2}\alpha_2^2 - f(\alpha_2) = \frac{1}{2(q_2+2)}\alpha_1^2 + \frac{c_1}{q_1+2}|\Omega|. \qquad \square$$

Our main result of this section is the following theorem. We prove that the solution blows up in a finite time by using a Lyapunov function and some suitable conditions on the initial energy and initial conditions.

Theorem 2. *Suppose that* (2) *and* (8) *are fulfilled and* $(\rho_0,\ \theta_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfying*

$$\alpha_1 < \left(\|\nabla\rho_0\|_2^2 + \|\nabla\theta_0\|_2^2\right)^{\frac{1}{2}},\ \ \mathscr{E}(0) < \mathscr{E}_1.$$

If

$$\frac{1-\frac{3}{4}\int_0^\infty \mu_1(s)\,ds}{1-\int_0^\infty \mu_2(s)\,ds} < \frac{q_1+2}{\eta}, \tag{25}$$

where η *we specify it later, then* $(\rho,\ \theta)$ *blows-up in finite time.*

Proof. Let the Lyaponov function

$$L(t) = \frac{1}{2}\int_\Omega \left(\rho^2(x,\ t) + \theta^2(x,\ t)\right)dx, \tag{26}$$

and take the derivative to obtain

$$\begin{aligned}
L'(t) &= \int_\Omega \rho\rho_t(x,t)\,dx + \int_\Omega \theta\theta_t(x,t)\,dx \\
&= -\|\nabla\rho\|_2^2 - \|\nabla\theta\|_2^2 + \int_\Omega\int_0^t \mu_1(t-s)\nabla\rho(x,t).\nabla\rho(x,s)\,dsdx \\
&\quad + \int_\Omega\int_0^t \mu_2(t-s)\nabla\theta(x,t).\nabla\theta(x,s)\,dsdx + \int_\Omega(\rho\varphi_1(\rho,\ \theta) + \theta\varphi_2(\rho,\ \theta))\,dx \\
&\geq -\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho(t)\|_2^2 - \int_0^t \mu_1(t-s)\int_\Omega |\nabla\rho(t).[\nabla\rho(s)-\nabla\rho(t)]|\,dxds \\
&\quad - \left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta(t)\|_2^2 - \int_0^t \mu_2(t-s)\int_\Omega |\nabla\theta(t).[\nabla\theta(s)-\nabla\theta(t)]|\,dxds \\
&\quad + \int_\Omega(\rho\varphi_1(\rho,\ \theta) + \theta\varphi_2(\rho,\ \theta))\,dx.
\end{aligned} \tag{27}$$

By using Schwartz inequality, (27) becomes

$$\begin{aligned}
L'(t) &\geq -\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho(t)\|_2^2 - \int_0^t \mu_1(t-s)\|\nabla\rho(t)\|_2\|\nabla\rho(s)-\nabla\rho(t)\|_2\,ds \\
&\quad - \left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla\theta(t)\|_2^2 - \int_0^t \mu_2(t-s)\|\nabla\theta(t)\|_2\|\nabla\theta(s)-\nabla\theta(t)\|_2\,ds \\
&\quad + \int_\Omega(\rho\varphi_1(\rho,\ \theta) + \theta\varphi_2(\rho,\ \theta))\,dx.
\end{aligned} \tag{28}$$

Using Young's inequality on the two terms from (28), we deduce

$$\begin{aligned}
L'(t) \geq &-\left(1-\frac{3}{4}\int_0^t \mu_1(s)\,ds\right)\|\nabla\rho(t)\|_2^2 - (\mu_1\circ\nabla\rho)(t) \\
&-\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)\|\nabla\rho(t)\|_2^2 - (\mu_2\circ\nabla\theta)(t) \\
&+\int_\Omega(\rho\varphi_1(\rho,\ \theta) + \theta\varphi_2(\rho,\ \theta))\,dx.
\end{aligned} \tag{29}$$

Thanks to (9), then (29) takes the form

$$L'(t) \geq -\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)\left(\|\nabla\rho(t)\|_2^2+\|\nabla\theta(t)\|_2^2\right) - (\mu_1\circ\nabla\rho)(t) - (\mu_2\circ\nabla\theta)(t) + \int_\Omega (\rho\varphi_1(\rho,\ \theta)+\theta\varphi_2(\rho,\ \theta))\,dx. \tag{30}$$

We proceed to replace $\|\nabla\rho(t)\|_2^2+\|\nabla\theta(t)\|_2^2$ from (14), (7) and (9), hence (30) becomes

$$\begin{aligned} L'(t) \geq{}& 2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\mathscr{F}(t) - 2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\mathscr{E}_1 \\ &+\left(\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}-1\right)[(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t)] \\ &-2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\int_\Omega \Phi(\rho,\ \theta)\,dx + \int_\Omega 2(q(x)+2)\Phi(\rho,\ \theta)\,dx. \end{aligned}\tag{31}$$

By using (24), the estimate (31) takes the form

$$\begin{aligned} L'(t) \geq{}& 2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\mathscr{F}(t) + \left(\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}-1\right)[(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t)] \\ &-2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\left(\frac{1}{2(q_2+2)}\alpha_1^2+\frac{c_1}{q_1+2}|\Omega|\right)^{-1}\mathscr{E}_1\int_\Omega \Phi(\rho,\ \theta)\,dx \\ &-2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\int_\Omega \Phi(\rho,\ \theta)\,dx + \int_\Omega 2(q(x)+2)\Phi(\rho,\ \theta)\,dx. \end{aligned}\tag{32}$$

Then, the estimate (32) takes the form

$$\begin{aligned} L'(t) \geq{}& 2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\mathscr{F}(t) + \left(\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}-1\right)[(\mu_1\circ\nabla\rho)(t)+(\mu_2\circ\nabla\theta)(t)] \\ &+\left(2(q_1+2)-2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\eta\right)\int_\Omega \Phi(\rho,\ \theta)\,dx, \end{aligned}$$

where

$$\eta = 1+\left(\frac{1}{2(q_2+2)}\alpha_1^2+\frac{c_1}{q_1+2}|\Omega|\right)^{-1}\mathscr{E}_1.$$

So, we get

$$L'(t) \geq \gamma\int_\Omega \Phi(\rho,\ \theta)\,dx, \tag{33}$$

where

$$\gamma = 2(q_1+2)-2\frac{\left(1-\frac{3}{4}\int_0^t \mu_2(s)\,ds\right)}{\left(1-\int_0^t \mu_1(s)\,ds\right)}\eta > 0.$$

By using (33), (12) and Lemma 5, we obtain

$$L'(t) \geq \Gamma\left(\|\rho\|_{2(q_1+2)}^{2(q_1+2)} + \|\theta\|_{2(q_1+2)}^{2(q_1+2)}\right), \tag{34}$$

where $\Gamma = \gamma a_0 c$ is a positive constant and a_0, c are defined respectively in Corollary 1. Next, by the embedding theorem of $L^{2(q_1+2)}$, we get

$$\begin{aligned} L^{q_1+2}(t) &\leq \left(\frac{1}{2}\right)^{q_1+2}\left(\|\rho\|_2^2 + \|\theta\|_2^2\right)^{q_1+2} \\ &\leq \left(\|\rho\|_2^{2(q_1+2)} + \|\theta\|_2^{2(q_1+2)}\right) \\ &\leq c\left(\|\rho\|_{2(q_1+2)}^{2(q_1+2)} + \|\theta\|_{2(q_1+2)}^{2(q_1+2)}\right). \end{aligned} \tag{35}$$

Combining (34) and (35), we obtain

$$L'(t) \geq \lambda L^{q_1+2}(t). \tag{36}$$

After integrating (36) directly, we arrive at

$$L^{1+q_1}(t) \geq \frac{1}{L^{-1-q_1}(0) - \lambda t}.$$

Consequently, L blows up in a time $t^* \leq \frac{1}{\lambda L^{q_1+1}(0)}$. □

4 Numerical tests

This section provides details about a numerical application, to show the blow-up outcome from Theorem 1. To achieve this, we solve problem (1) numerically for $n = 2$. First, we propose a suitable numerical method to discretize (1) using finite differences for both time domain and spatial variable $x = (x_1, x_2) \in \Omega$. Then, we partition $[0,\ T]$ into N intervals $[t_{n-1},\ t_n]$, $\ t_n = n\ \delta t$, $\ n = 1, 2, \ldots, N+1$, where δt is the time step.

Let $\rho^n\ (x_1, x_2) = \rho(x_1, x_2,\ t_n)$ and $\theta^n\ (x_1, x_2) = \theta(x_1, x_2,\ t_n)$. Using the finite difference formulas, the time discrete problem of (1) reads: Given $(\rho_0,\ \theta_0)$, find $\left\{(\rho^1,\ \theta^1), \ldots, (\rho^{n+1},\ \theta^{n+1})\right\}$ such that

$$\begin{cases} \frac{\rho^{n+1}}{\delta t} - \Delta\rho^{n+1} = \frac{1}{\delta t}\rho^n - \int_0^{t_{n+1}} \mu_1(t_{n+1}-s)\Delta\rho^n(s)\,ds & \\ \quad + a|\rho^n + \theta^n|^{2q(x)+2}(\rho^n + \theta^n) + b|\rho^n|^{q(x)}\rho^n|\theta^n|^{q(x)+2}, & \text{in } \Omega_h, \\ \frac{\theta^{n+1}}{\delta t} - \Delta\theta^{n+1} = \frac{1}{\delta t}\theta^n - \int_0^{t_{n+1}} \mu_2(t_{n+1}-s)\Delta\theta^n(s)\,ds & \\ \quad + a|\rho^n + \theta^n|^{2q(x)+2}(\rho^n + \theta^{nn}) + b|\theta^n|^{p(x)}\theta^n|\rho^n|^{q(x)+2}, & \text{in } \Omega_h, \\ \rho^{n+1} = \theta^{n+1} = 0 & \text{on } \partial\Omega_h, \\ \rho^0 = \rho_0(x_1, x_2),\ \ \theta^0 = \theta_0(x_1, x_2), & \text{in } \Omega_h. \end{cases} \tag{37}$$

We offer and analyze the results obtained using the numerical method (37). The numerical results were obtained through the execution of Matlab codes. The parameters chosen for the numerical experiments are as follows:

- The relaxation functions are: $\mu_1(t) = \zeta e^{-t}$, $\mu_2(t) = \xi e^{-t}$, where $0 < \xi \leq \zeta < 1$.
- The variable exponet $q(x_1, x_2) = 2.1\left(1 + 3x_1^2 + 2x_2^2\right)$.

Test 1. The domain is taken to be square $\Omega = [-1,\ 1]^2$. We chosen

$$\rho_0(x_1, x_2) = 12(x_1 - 1)(x_1 + 1)(x_2 - 1)(x_2 + 1)$$

and

$$\theta_0(x_1, x_2) = 10(x_1 - 1)(x_1 + 1)(x_2 - 1)(x_2 + 1).$$

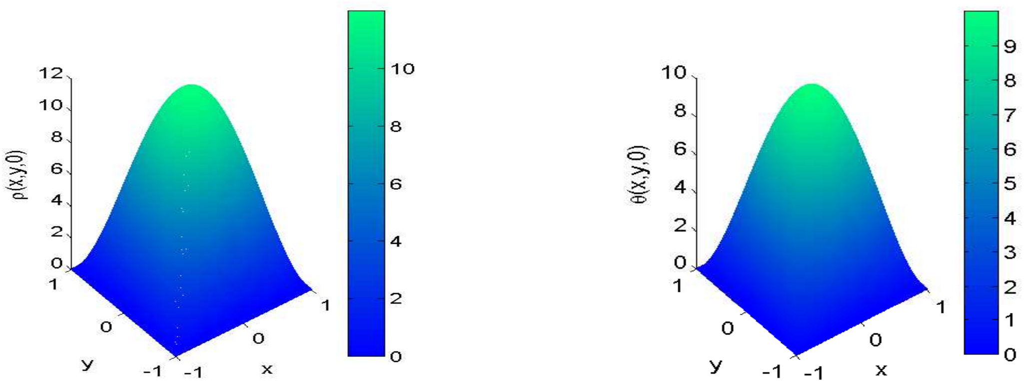

Figure 1: Solution (ρ, θ) at $t = 0.0$.

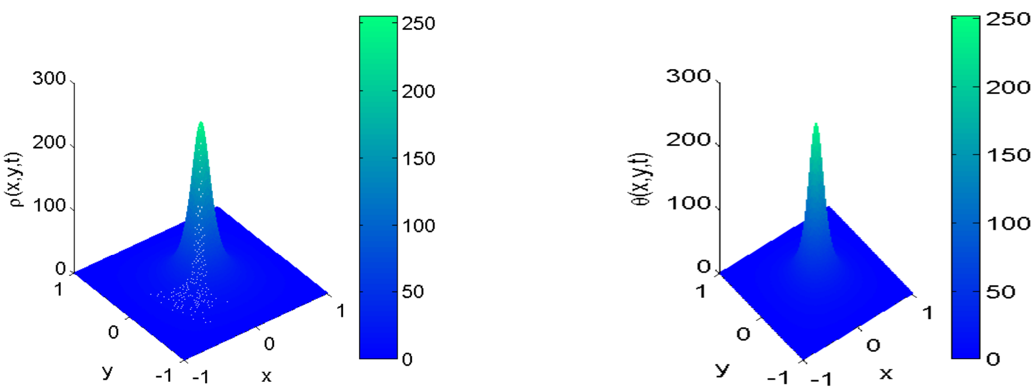

Figure 2: Solution (ρ, θ) at $t = 0.015$.

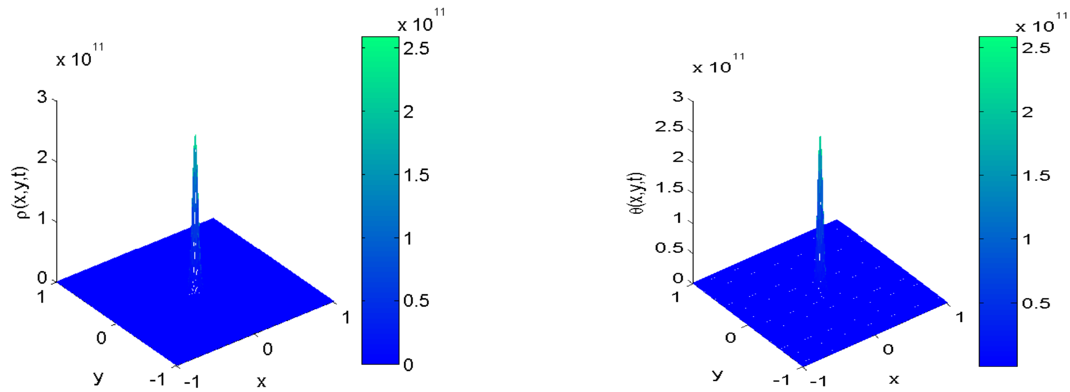

Figure 3: Solution $(\rho,\ \theta)$ at $t = 0.017$.

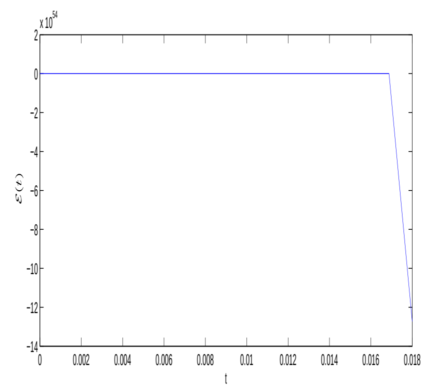

Figure 4: Blow up of $\mathscr{E}$(t) in finite time.

Figures 1-3 represent the approximate solution $(\rho,\ \theta)$ at times $t = 0$, $t = 0.015$, and $t = 0.017$, where it is observed that the blow-up occurs at time $t = 0.017$.
Figure 4 represents the energy function $\mathscr{F}\,(t)$, where we can see that the blow-up occurs at time $t = 0.017$.

Test 2. The domain is taken to be rectangle $\Omega = [-1,\ 1] \times [0,\ 1]$. We chosen

$$\rho_0\,(x_1,x_2) = 28\,(x_1-1)\,(x_1+1)\,(x_2-1)\,x_2,$$

and

$$\theta_0\,(x_1,x_2) = 30\,(x_1-1)\,(x_1+1)\,(x_2-1)\,x_2.$$

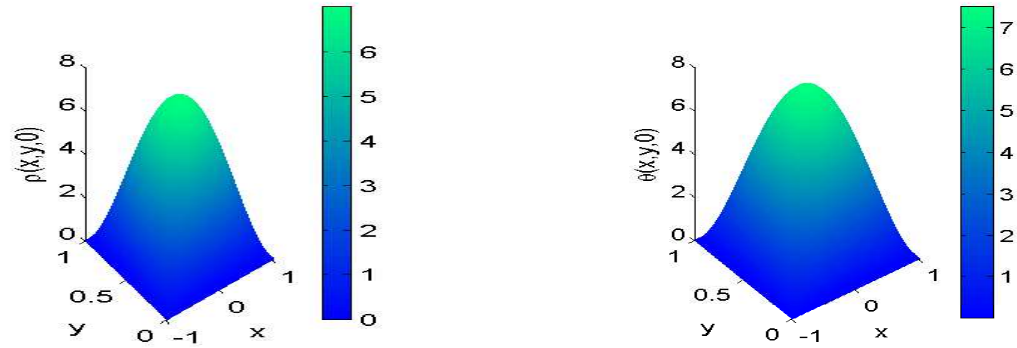

Figure 5: Solution $(\rho,\, \theta)$ at $t = 0.0$.

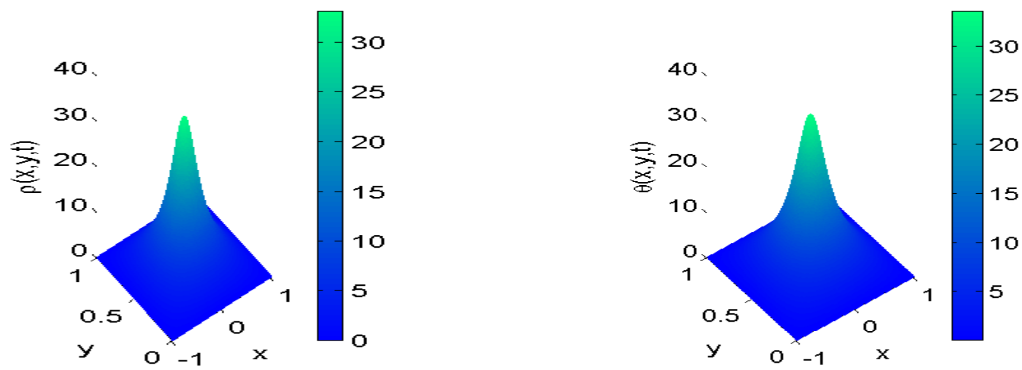

Figure 6: Solution $(\rho,\, \theta)$ at $t = 0.030$.

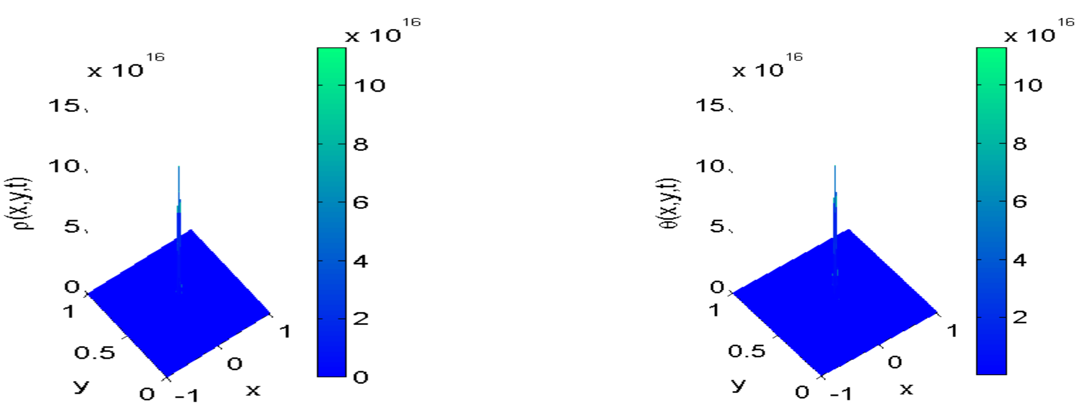

Figure 7: Solution $(\rho,\, \theta)$ at $t = 0.031$.

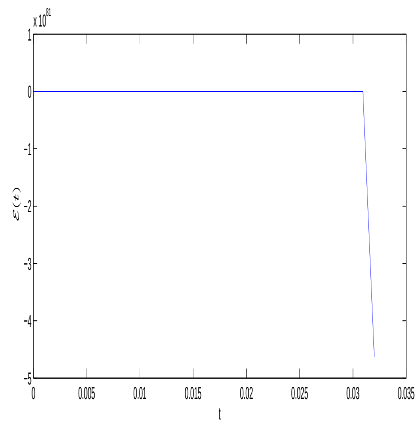

Figure 8: Blow up of $\mathscr{E}$(t) in finite time.

Figures 5-7 represent the approximate solution $(\rho,\ \theta)$ at times $t=0$, $t=0.030$, and $t=0.031$, where it is observed that the blow-up occurs at time $t=0.031$.

Figure 8 represents the energy function $\mathscr{E}(t)$, where we can see that the blow-up occurs at time $t=0.031$.

Theorem 1 blow up results are verified and agreed upon by the previously mentioned numerical application.

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