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Special approximation method for solving system of ordinary and fractional integro-differential equations

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Abstract. This paper concerns with some special approximate methods in order to solve the system of ordinary and the fractional integro-differential equations. The approach that we use begins by a method of converting the fractional integro-differential equations into an integral equation including both Volterra and the Fredholm parts. Then a specific successive approximation technique is applied to the Volterra part. Due to the presence of the factorial factor in the denominator of its kernel, the Volterra part tends to zero in the next iterations, leading us to discard the Volterra's sentence as an error of the method that we use. The analytical-approximate solution to the problem is then obtained by solving the resulting equation, as a Fredholm integral equation of the second kind. This method is applied to the boundary value problems in two distinct cases involving system of ordinary and fractional differential equations.

Keywords: System of ordinary differential equations, Boundary value problems, Successive approximations, fractional integro-differential equations.
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1 Introduction

The initial and boundary value problems are considered as important mathematical models for physical, engineering, biological, demographic, and the economic systems. Therefore, these problems are of significant interest and importance to people involving in many different disciplines, from which we can name, the mathematicians, physicists, engineers, and even the people whose speciality is economic. In mathematics, the Volterra integral equations are mostly used in order to solve the initial value problems, while the Fredholm integral equations are commonly used to solve the boundary value problems. After converting the initial value problems into the Volterra integral equations and the boundary value problems to the Fredholm integral equations, analytical and the approximate methods could be applied to solve the resulting equations studied in [2, 6–9, 11]. In some classical, advanced books and papers, the Galerkin

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method and the successive approximation methods are commonly used as classical methods to solve these problems [3,12,14–16,18,20]. In this paper, our aim is to develop a suitable convergent successive approximation method for solving the system of non-homogeneous first-order differential equations with variable coefficients and general boundary conditions. Our focus will be mostly providing a general approach to solve various applied problems using different methods.

This paper is organized as follows. Section 2 is devoted to the system of ordinary differential equations. In Section 3, fractional integro-differential equations will be investigated, and we will provide the interested reader with some interesting examples, presented in Section 4.

2 System of ordinary differential equations

2.1 Mathematical statement of the problem

For $t \in (a,b)$, and α, β real constant matrices of order n, and γ a column vector with constant values, consider the following system of first-order ordinary differential equations presented in the matrix form

$$\dot{X}(t) = A(t)X(t) + F(t), \tag{1}$$
$$\alpha X(a) + \beta X(b) = \gamma, \tag{2}$$

with general linear boundary conditions, where $X(t)$ is an unknown vector function, $A(t)$ is an $n \times n$ square matrix with continuous functions as its entries, and $F(t)$ a known continuous vector function on the interval $[a,b]$.

In addition, the general linear boundary conditions (2) will be considered to be linearly independent.

Remark 1. *Note that, system (1)-(2) includes variable coefficients. In [3], a method was discussed for such kind of systems, but the presented method for solving the problem is very complicated and lengthy.*

Remark 2. *The special sequential approximation method has been applied to the second-order ordinary differential equation $y'' + m^2 y = f(x)$. The solution to the problem has been presented as an analytical expressions. This method also has been applied in a more general case for the boundary value problems [11]*

$$y'' + p(x)y = f(x),$$

with a general boundary conditions

$$\begin{cases} \alpha_1 y(a) + \beta_1 y(b) = \gamma_1, \\ \alpha_2 y'(a) + \beta_2 y'(b) = \gamma_2. \end{cases}$$

2.2 Converting to integral equation

First, we convert system (1)-(2) into an integral equation by integrating over the interval $[a,t]$ as follows

$$X(t) = X(a) + \int_a^t A(\tau)X(\tau)\,d\tau + \int_a^t F(\tau)\,d\tau. \tag{3}$$

Then by substituting the value $t := b$ into the obtained integral equation (3), we get the value $X(b)$ as follows

$$X(b) = X(a) + \int_a^b A(\tau)X(\tau)\,\mathrm{d}\tau + \int_a^b F(\tau)\,\mathrm{d}\tau.$$

Substituting the result into the general linear boundary condition (2) and assuming $\det(\alpha+\beta) \neq 0$, we obtain

$$X(a) = (\alpha+\beta)^{-1}\gamma - (\alpha+\beta)^{-1}\beta \int_a^b A(\tau)X(\tau)\,\mathrm{d}\tau - (\alpha+\beta)^{-1}\beta \int_a^b F(\tau)\,\mathrm{d}\tau,$$

and

$$X(b) = (\alpha+\beta)^{-1}\gamma + \left[I - (\alpha+\beta)^{-1}\beta\right] \int_a^b A(\tau)X(\tau)\,\mathrm{d}\tau + \left[I - (\alpha+\beta)^{-1}\beta\right] \int_a^b F(\tau)\,\mathrm{d}\tau.$$

Again by substituting $X(a)$ in Eq. (3), we get

$$\begin{aligned} X(t) = {} & (\alpha+\beta)^{-1}\gamma - (\alpha+\beta)^{-1}\beta \int_a^b A(\tau)X(\tau)\,\mathrm{d}\tau - (\alpha+\beta)^{-1}\beta \int_a^b F(\tau)\,\mathrm{d}\tau \\ & + \int_a^t A(\tau)X(\tau)\,\mathrm{d}\tau + \int_a^t F(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{4}$$

And after performing some algebraic operations, for

$$f(t) = (\alpha+\beta)^{-1}\gamma - \int_a^b (\alpha+\beta)^{-1}\beta F(\tau)\,\mathrm{d}\tau + \int_a^t F(\tau)\,\mathrm{d}\tau,$$

we get the following integral equation

$$X(t) = \int_a^t A(\tau)X(\tau)\,\mathrm{d}\tau + f(t) + C, \tag{5}$$

where

$$C = \int_a^b B(\tau)X(\tau)\,\mathrm{d}\tau, \quad \text{and} \quad B(\tau) = -(\alpha+\beta)^{-1}\beta A(\tau).$$

2.3 Applying the method

Remark 3. ***Leibnitz rule for differentiation of integrals.*** *Let $f(x,t)$ and $\frac{\partial f}{\partial t}$ be continuous in a domain of the $x-t$ plane including the rectangle $a \le x \le b$, $t_0 \le t \le t_1$ and let*

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)\,\mathrm{d}t. \tag{6}$$

Then, the derivative of the integral in (6) exists, and is given by

$$F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}\,\mathrm{d}t. \tag{7}$$

In the integral equation (5), we apply the special method of successive approximations exclusively to the Volterra component. Therefore, we calculate $X(\tau)$ in the integral equation (5), as follows

$$X(\tau)=\int_a^\tau A(\tau_1)X(\tau_1)\,\mathrm{d}\tau_1+f(\tau)+C,$$

which by putting in the integral equation (5), we have

$$\begin{aligned}X(t)&=\int_a^t A(\tau)\Big[\int_a^\tau A(\tau_1)X(\tau_1)\,\mathrm{d}\tau_1+\big(f(\tau)+C\big)\Big]\,\mathrm{d}\tau_1+\big(f(t)+C\big)\\&=\int_a^t \mathrm{d}\tau_1\int_{\tau_1}^t A(\tau)A(\tau_1)X(\tau_1)\,\mathrm{d}\tau+\int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau+\big(f(t)+C\big)\\&=\int_a^t \mathrm{d}\tau_1\Big(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\Big)A(\tau_1)X(\tau_1)+\int_a^t A(\tau)\big(f(\tau)+C\big)+\big(f(t)+C\big).\end{aligned}$$

By using Remark 3, we get

$$X(t)=-\int_a^t d_{\tau_1}\frac{\left(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\right)^2}{2!}X(\tau_1)+\int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau+\big(f(t)+C\big). \tag{8}$$

For the second iteration, we calculate $X(\tau_1)$ in the integral equation (8) as following

$$X(\tau_1)=\int_a^{\tau_1} A(\tau_2)X(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_2+\big(f(\tau_1)+C\big).$$

Now, by substituting $X(\tau_1)$ into the integral equation (8), we obtain

$$\begin{aligned}X(t)\;&=\;-\int_a^t d_{\tau_1}\frac{\left(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\right)^2}{2!}\left[\int_a^{\tau_1} A(\tau_2)X(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_2+\big(f(\tau_1)+C\big)\right]+\int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau\\&\quad+\big(f(t)+C\big)\\&=\;-\int_a^t d_{\tau_2}\int_{\tau_2}^t d_{\tau_1}\frac{\left(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\right)^2}{2!}A(\tau_2)X(\tau_2)-\int_a^t d_{\tau_1}\frac{\left(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\right)^2}{2!}\big(f(\tau_1)+C\big)\\&\quad+\int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau+\big(f(t)+C\big),\\X(t)\;&=\;-\int_a^t d_{\tau_2}\frac{\left(\int_{\tau_2}^t A(\tau)\,\mathrm{d}\tau\right)^3}{3!}X(\tau_2)-\int_a^t \mathrm{d}_{\tau_1}\frac{\left(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\right)^2}{2!}\big(f(\tau_1)+C\big)\\&\quad+\int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau+\big(f(t)+C\big).\end{aligned} \tag{9}$$

For the third iteration, we calculate $X(\tau_2)$ in the integral equation (9):

$$X(\tau_2)=\int_a^{\tau_2} A(\tau_3)X(\tau_3)\,\mathrm{d}\tau_3+\big(f(\tau_2)+C\big).$$

Next, by substituting $X(\tau_2)$ into the integral equation (9), we obtain

$$X(t) = -\int_a^t d_{\tau_3} \frac{\left(\int_{\tau_3}^t A(\tau)\,\mathrm{d}\tau\right)^4}{4!} X(\tau_3) - \int_a^t d_{\tau_2} \frac{\left(\int_{\tau_2}^t A(\tau)\,\mathrm{d}\tau\right)^3}{3!} \big(f(\tau_2)+C\big)$$
$$- \int_a^t d_{\tau_1} \frac{\left(\int_{\tau_1}^t A(\tau)\,\mathrm{d}\tau\right)^2}{2!} \big(f(\tau_1)+C\big) + \int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau + \big(f(t)+C\big).$$

After the nth iteration, we will have

$$X(t) = -\int_a^t d_{\tau_{n-1}} \frac{\left(\int_{\tau_{n-1}}^t A(\tau)\,\mathrm{d}\tau\right)^n}{n!} X(\tau_{n-1}) - \int_a^t \sum_{k=1}^{n-1} d_{\tau_k} \frac{\left(\int_{\tau_k}^t A(\tau)\,\mathrm{d}\tau\right)^k}{k!} \big(f(\tau_k)+C\big)$$
$$+ \int_a^t A(\tau)\big(f(\tau)+C\big)\,\mathrm{d}\tau + \big(f(t)+C\big).$$

The first term in the above relation is a Volterra term, and due to the presence of the factorial factor $n!$, tending to zero as n approaches infinity. Therefore, we can disregard this term, and the remaining term provides the analytical approximate solution to the problem

$$X(t) \approx -\int_a^t \sum_{k=1}^{n-1} d_{\tau_k} \frac{\left(\int_{\tau_k}^t A(\tau)\,\mathrm{d}\tau\right)^k}{k!} \big(f(\tau_k)+C\big) + \big(f(t)+C\big). \tag{10}$$

To calculate the constant C using the relation (10), we first multiply both sides of the relation by $B(t)$, and then integrate over the interval $[a,b]$. In this case, we obtain

$$C = -\sum_{k=1}^{n-1} \int_a^b B(t)\,\mathrm{d}t \int_a^t \mathrm{d}_\tau \frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!} \big[f(\tau)+C\big] + \int_a^b B(t)\big[f(t)+C\big]\,\mathrm{d}t.$$

By substituting the above value in the relation (5) and categorizing the sentences relative to C, we have the following relation

$$\left[I + \sum_{k=1}^{n-1} \int_a^b B(t)\,\mathrm{d}t \int_a^t \mathrm{d}_\tau \frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!} - \int_a^b B(t)\,\mathrm{d}t\right] C$$
$$= -\sum_{k=1}^{n-1} \int_a^b B(t)\,\mathrm{d}t \int_a^t \mathrm{d}_\tau \frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!} f(\tau) + \int_a^b B(t) f(t)\,\mathrm{d}t,$$

provided that

$$\det\Big[I+\sum_{k=1}^{n-1}\int_a^b B(t)\,\mathrm{d}t\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}-\int_a^b B(t)\,\mathrm{d}t\Big]\neq 0.$$

Then, we have the final analytical-approximate solution of the integral equation as follows

$$\begin{aligned}X(t)\approx&\left[I-\sum_{k=1}^{n-1}\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}\right]\left[I+\sum_{k=1}^{n-1}\int_a^b B(t)\,\mathrm{d}t\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}-\int_a^b B(t)\,\mathrm{d}t\right]^{-1}\\&\times\left[-\sum_{k=1}^{n-1}\int_a^b B(t)\,\mathrm{d}t\int_a^\tau \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}f(\tau)+\int_a^b B(t)f(t)\,\mathrm{d}t\right]-\sum_{k=1}^{n-1}\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}f(\tau)\\&+f(t).\end{aligned}$$

Then by letting

$$A_k=\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}f(\tau),\quad\text{and}\quad B_k=\int_a^b B(t)\,\mathrm{d}t\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!},$$

the final solution is summarized as follows

$$\begin{aligned}X(t)\approx&\Big[I-\sum_{k=1}^{n-1}\int_a^t \mathrm{d}_\tau\frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}\Big]\\&\times\Big[I+\sum_{k=1}^{n-1}B_k-\int_a^b B(t)\,\mathrm{d}t\Big]^{-1}\Big[-\sum_{k=1}^{n-1}\int_a^b B(t)\,\mathrm{d}tA_k+\int_a^b B(t)f(t)\,\mathrm{d}t\Big]-\sum_{k=1}^{n-1}A_k+f(t).\end{aligned}$$

2.4 Analyzing the error of the method

We consider the non-homogeneous integral equation including both Volterra and the Fredholm terms, as follows

$$y(x)=\lambda\int_a^x k_0(x,\xi)y(\xi)\,\mathrm{d}\xi+\mu\int_a^b k_1(x,\xi)y(\xi)\,\mathrm{d}\xi+f(x). \tag{11}$$

Note that λ is the parameter for the Volterra term, and μ is the parameter for the Fredholm term. To apply the special method of successive approximations to the Volterra term, we express $y(\xi)$ as follows

$$y(\xi)=\lambda\int_a^\xi k_0(\xi,\eta)y(\eta)\,\mathrm{d}\eta+\mu\int_a^b k_1(\xi,\eta)y(\eta)\,\mathrm{d}\eta+f(\xi).$$

By substituting the above value in the Volterra term of the integral equation (11), we obtain

$$y(x)=\lambda\int_a^x k_0(x,\xi)\Big[\lambda\int_a^{\xi} k_0(\xi,\eta)y(\eta)\,\mathrm{d}\eta+\mu\int_a^b k_1(\xi,\eta)y(\eta)\,\mathrm{d}\eta+f(\xi)\Big]\,\mathrm{d}\xi$$
$$+\mu\int_a^b k_1(x,\xi)y(\xi)\,\mathrm{d}\xi+f(x)=\lambda^2\int_a^x y(\eta)\mathrm{d}\eta\int_a^{\xi} k_0(x,\xi)k_0(\xi,\eta)\mathrm{d}\xi$$
$$+\lambda\mu\int_a^b y(\eta)\mathrm{d}\eta\int_a^x k_0(x,\xi)k_1(\xi,\eta)\mathrm{d}\xi+\lambda\int_a^x k_0(x,\xi)f(\xi)\mathrm{d}\xi$$
$$+\mu\int_a^b k_1(x,\eta)y(\eta)\mathrm{d}\eta+f(x),$$

and therefore, we have

$$y(x)=\lambda^2\int_a^x k_{02}(x,\eta)y(\eta)\mathrm{d}\eta+\mu\int_a^b\Big[k_1(x,\eta)+\lambda\int_a^x k_0(x,\xi)k_1(\xi,\eta)\xi\Big]y(\eta)\mathrm{d}\eta$$
$$+\Big[f(x)+\lambda\int_a^x k_0(x,\xi)f(\xi)\mathrm{d}\xi\Big]. \tag{12}$$

Hence, once again, we use the method of special sequential approximations

$$y(\eta)=\lambda\int_a^t k_0(\eta,t)y(t)\mathrm{d}t+\mu\int_a^b k_1(\eta,t)y(t)\mathrm{d}t+f(\eta).$$

Now by substituting this value in the equation (12), we obtain

$$y(x)=\lambda^2\int_a^x k_{02}(x,\eta)\Big[\lambda\int_a^{\eta} k_0(\eta,t)y(t)\mathrm{d}t+\mu\int_a^b k_1(\eta,t)y(t)\,\mathrm{d}t+f(\eta)\Big]\mathrm{d}\eta$$
$$+\mu\int_a^b\Big[k_1(x,\eta)+\lambda\int_a^x k_0(x,\xi)k_1(\xi,\eta)d\xi\Big]y(\eta)\mathrm{d}\eta+\Big[f(x)+\lambda\int_a^x k_0(x,\xi)f(\xi)\mathrm{d}\xi\Big].$$

And after applying the special sequential approximation method twice, we get

$$y(x)=\lambda^3\int_a^x k_{03}(x,t)y(t)\,\mathrm{d}t$$
$$+\mu\int_a^b\Big[k_1(x,\eta)+\lambda\int_a^x k_0(x,\xi)k_1(\xi,\eta)\,\mathrm{d}\xi+\lambda^2\int_a^x k_{02}(x,\eta)k_1(\eta,t)\,\mathrm{d}\eta\Big]y(t)\,\mathrm{d}t \tag{13}$$
$$+\Big[f(x)+\lambda\int_a^x k_0(x,\xi)f(\xi)\,\mathrm{d}\xi+\lambda^2\int_a^x k_{02}(x,\eta)f(\eta)\,\mathrm{d}\eta\Big].$$

Now, if we use this method for the third time, we get

$$y(t)=\lambda\int_a^t k_0(t,\tau)y(\tau)\,\mathrm{d}\tau+\mu\int_a^b k_1(t,\tau)y(\tau)\,\mathrm{d}\tau+f(t). \tag{14}$$

By substituting $y(t)$ in (13) and doing some calculations and rearranging the integrals, we get the following relation

$$y(x)=\lambda^4\int_a^x k_{04}(x,\tau)y(\tau)\,\mathrm{d}\tau$$
$$+\mu\int_a^b\Big[k_1(x,\eta)+\lambda\int_a^x k_0(x,\xi)k_1(\xi,\eta)\,\mathrm{d}\xi+\lambda^2\int_a^x k_{02}(x,\eta)k_1(\eta,t)\,\mathrm{d}\eta+\lambda^3\int_a^x k_{03}(x,t)k_1(t,\tau)\,\mathrm{d}t\Big]y(\tau)\,\mathrm{d}\tau$$
$$+\Big[f(x)+\lambda\int_a^x k_0(x,\xi)f(\xi)\,\mathrm{d}\xi+\lambda^2\int_a^x k_{02}(x,\eta)f(\eta)\,\mathrm{d}\eta+\lambda^3\int_a^x k_{03}(x,t)f(t)\,\mathrm{d}t\Big].$$

So after applying the iterative method, n times, we will have

$$\begin{aligned} y(x) &= \lambda^n \int_a^x k_{0n}(x,\tau) y(\tau)\,\mathrm{d}\tau \\ &+ \mu \int_a^b \Big[k_1(x,\tau) + \lambda \int_a^x k_0(x,\xi) k_1(\xi,\tau)\,\mathrm{d}\xi + \int_a^x \sum_{j=2}^n \lambda^j k_{0j}(x,\xi) k_1(\xi,\tau)\,\mathrm{d}\xi\Big] y(\tau)\,\mathrm{d}\tau \\ &+ \Big[f(x) + \lambda \int_a^x k_0(x,\xi) f(\xi)\,\mathrm{d}\xi + \int_a^x \sum_{j=2}^n \lambda^j k_{0j}(x,\xi) f(\xi)\,\mathrm{d}\xi\Big]. \end{aligned}$$

Then by assuming the following equations

$$\begin{aligned} K(x,\lambda,n) &= \mu \int_a^b \Big[k_1(x,\tau) + \lambda \int_a^x k_0(x,\xi) k_1(\xi,\tau)\,\mathrm{d}\xi + \int_a^x \sum_{j=2}^n \lambda^j k_{0j}(x,\xi) k_1(\xi,\tau)\,\mathrm{d}\xi\Big] y(\tau)\,\mathrm{d}\tau, \\ F(x,\lambda,n) &= f(x) + \lambda \int_a^x k_0(x,\xi) f(\xi)\,\mathrm{d}\xi + \int_a^x \sum_{j=2}^n \lambda^j k_{0j}(x,\xi) f(\xi)\,\mathrm{d}\xi, \end{aligned}$$

we obtain

$$y(x) = \lambda^n \int_a^x k_{0n}(x,\tau) y(\tau)\,\mathrm{d}\tau + \mu \int_a^b K(x,\lambda,n) y(\tau)\,\mathrm{d}\tau + F(x,\lambda,n).$$

And if we assume

$$\Big|k_0(x,\xi)\Big| \le k,$$

then we get

$$\begin{aligned} \Big|k_{02}(x,\eta)\Big| &= \Big|\int_\eta^x k_0(x,\xi) k_0(\xi,\eta)\,\mathrm{d}\xi\Big| \le k^2 \frac{(x-\eta)}{1!}, \\ \Big|k_{03}(x,t)\Big| &= \Big|\int_t^x k_{02}(x,\eta) k_0(\eta,t)\,\mathrm{d}\eta\Big| \le k^3 \int_t^x (x-\eta)\,\mathrm{d}\eta = k^3 \frac{(x-t)^2}{2!}, \end{aligned}$$

and finally, we have the nth sentence

$$\Big|k_{0n}(x,t)\Big| \le k^n \frac{(x-t)^{n-1}}{(n-1)!}, \qquad n \in \mathbb{N}.$$

If we consider the maximum value of the error with the desired $\varepsilon \ge 0$, then we will have

$$k^n \frac{(b-a)^{n-1}}{(n-1)!} |\lambda|^n < \varepsilon \Rightarrow |\lambda| < \frac{1}{k} \sqrt[n]{\frac{\varepsilon(n-1)!}{(b-a)^{n-1}}}.$$

For this range of parameter λ in Volterra term, the method will always be convergent.

3 Fractional integro-differential equations

In the last two decades, fractional calculus has increasingly influenced many fields, such as engineering, physics, and the economics. This is due to the new possibilities that the fractional calculus offers in modeling various problems. A significant advantage of the fractional derivatives over the classical (integer-order) derivatives is their ability to account for effects that are generally ignored in the traditional approaches [1,4,5,10,13,17,19,21].

Definition 1. *The Rieman-Liouville fractional integral of order* $\alpha > 0$ *is defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad (I^0 f)(t) = f(t).$$

Definition 2. *Fractional derivative of order* $\alpha > 0$ *is written as follows*

$$D^{\alpha} f(t) = (\frac{d}{dt})^n (I^{n-\alpha} f)(t), \qquad (n-1 < \alpha \le n),$$

where n is an integer number.

Now, we use a special sequential method to solve the following fractional integro-differential equation

$$D^{\alpha} y(x) + \int_a^x k_0(x,t) y(t)\,\mathrm{d}t + \int_a^b k_1(x,t) y(t)\,\mathrm{d}t = f(x), \quad 0 < a < b, \quad \alpha \in (0,1), \tag{15}$$

with the following boundary condition

$$a_1 y(a) + b_1 y(b) = c_1,$$

including both the Volterra and the Fredholm terms. We do this by taking α-order fractional integral of both sides of Eq. (15), in order to have

$$I^{\alpha} D^{\alpha} y(x) = I^{\alpha} \int_a^x k_0(x,t) y(t)\,\mathrm{d}t + I^{\alpha} \int_a^b k_1(x,t) y(t)\,\mathrm{d}t = I^{\alpha} f(x),$$

and therefore we have

$$y(x) = C\frac{x^{\alpha-1}}{(\alpha-1)!} + \int_a^x \frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!}\,\mathrm{d}\xi \int_a^b k_0(\xi,t) y(t)\,\mathrm{d}t + \int_a^x \frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!}\,\mathrm{d}\xi \int_a^b k_1(\xi,t) y(t)\,\mathrm{d}t$$
$$+ \int_a^x \frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!} f(\xi)\,\mathrm{d}\xi.$$

Notice that C is an arbitrary constant and $C\frac{x^{\alpha-1}}{(\alpha-1)!}$ is the solution of the homogeneous part of (15). By using

$$\int_a^b \mathrm{d}x \int_a^x f(x,t)\,\mathrm{d}t = \int_a^b \mathrm{d}t \int_t^b f(x,t)\,\mathrm{d}x,$$

and doing some calculations and rearranging the integrals, we get the following relation

$$y(x) = -\int_a^x H_{0,\alpha}(x,t) y(t)\,\mathrm{d}t - \int_a^b H_{1,\alpha}(x,t) y(t)\,\mathrm{d}t + \frac{x^{\alpha-1}}{(\alpha-1)!} C + f_1(x), \tag{16}$$

where

$$H_{0,\alpha}(x,t) = \int_t^x \frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!} k_0(\xi,t)\,\mathrm{d}\xi, \quad H_{1,\alpha}(x,t) = \int_a^x \frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!} k_1(\xi,t)\,\mathrm{d}\xi, \quad f_1(x) = \int_a^x \frac{(x-\xi)^{\alpha-1}}{(\alpha-1)!} f(\xi)\,\mathrm{d}\xi.$$

3.1 Application of the special sequential approximation method

In this subsection, we apply the special method of successive approximations to the equation (16). We do this by calculating $y(t)$ by substituting $x := t$ into the left hand side of Eq. (16)

$$y(t) = -\int_a^t H_{0,\alpha}(t,\tau) y(\tau)\,\mathrm{d}\tau - \int_a^b H_{1,\alpha}(t,\tau) y(\tau)\,\mathrm{d}\tau + \frac{t^{\alpha-1}}{(\alpha-1)!} C + f_1(t).$$

Now we substitute $y(t)$ into the Volterra integral (16) and we do not change anything related to the Fredholm part, and we get

$$\begin{aligned} y(x) = &-\int_a^x H_{0,\alpha}(x,t)\Big[-\int_a^t H_{0,\alpha}(t,\tau)y(\tau)\,\mathrm{d}\tau - \int_a^b H_{1,\alpha}(t,\tau)y(\tau)\,\mathrm{d}\tau + \frac{t^{\alpha-1}}{(\alpha-1)!}C + f_1(t)\Big]\,\mathrm{d}t \\ &-\int_a^b H_{1,\alpha}(x,t)y(t)\,\mathrm{d}t + \frac{x^{\alpha-1}}{(\alpha-1)!}C + f_1(x). \end{aligned} \tag{17}$$

By changing the order of the integration in Eq. (17) we obtain

$$\begin{aligned} y(x) = &\int_a^x y(\tau)\,\mathrm{d}\tau \int_\tau^x H_{0,\alpha}(x,t)H_{0,\alpha}(t,\tau)\,\mathrm{d}t + \int_a^b y(\tau)\,\mathrm{d}\tau\Big[-\int_a^x H_{0,\alpha}(x,t)H_{1,\alpha}(t,\tau)\,\mathrm{d}t - H_{1,\alpha}(x,\tau)\Big] \\ &+\Big[\frac{x^{\alpha-1}}{(\alpha-1)!} - \int_a^x H_{0,\alpha}(x,t)\frac{t^{\alpha-1}}{(\alpha-1)!}\,\mathrm{d}t\Big]C + \Big[f_1(x) - \int_a^x H_{0,\alpha}(x,t)f_1(t)\,\mathrm{d}t\Big]. \end{aligned}$$

But, in order to have a little bit simpler calculations, we consider

$$\begin{aligned} H_{0^2,\alpha}(x,t) &:= \int_\tau^x H_{0,\alpha}(x,t)H_{0,\alpha}(t,\tau)\,\mathrm{d}t, \\ H_{1,0,\alpha} &:= \Big[\int_a^x H_{0,\alpha}(x,t)H_{1,\alpha}(t,\tau)\,\mathrm{d}t - H_{1,\alpha}(x,\tau)\Big], \\ H_{0,\alpha}(x) &:= \Big[\frac{x^{\alpha-1}}{(\alpha-1)!} - \int_a^x H_{0,\alpha}(x,t)\,\mathrm{d}t\frac{t^{\alpha-1}}{(\alpha-1)!}\Big], \\ f_2(x) &:= \Big[f_1(x) - \int_a^x H_{0,\alpha}(x,t)f_1(t)\,\mathrm{d}t\Big]. \end{aligned}$$

Thus, we have

$$y(x) = \int_a^x H_{0^2,\alpha}(x,\tau)y(\tau)\,\mathrm{d}\tau + \int_a^b H_{1,0,\alpha}(x,t)y(t)\,\mathrm{d}t + H_{0,\alpha}(x)C + f_2(x). \tag{18}$$

Now if we do the same procedure n times $\big($for $y(t)$, from (18)$\big)$ we get

$$y(x) = \int_a^x H_{0^n,\alpha}(x,t)y(t)\,\mathrm{d}t + \int_a^b H_{1,0^{n-1},\alpha}(x,t)y(t)\,\mathrm{d}t + H_{0^{n-1},\alpha}(x)C + f_{n-1}(x).$$

Finally, if we do the process with a large enough n, we obtain

$$\forall\varepsilon > 0, \exists N_0 > 0, \forall n > N_0 \Longrightarrow \Big|H_{0^n,\alpha}(x,t)\Big| < \varepsilon.$$

When $n = N_0 + 1$, we will get the value $(n+1)!$ in the denominator. So, by disregarding the Volterra term as the error and considering its final zero value, we will have

$$y(x) = \int_a^b H_n(x,t)y(t)\,\mathrm{d}t + F_n(x) + H_n(x)C, \tag{19}$$

where

$$H_n(x,t) = H_{1,0^{n-1},\alpha}(x,t), \quad F_n(x) = f_{n-1}(x), \quad H_n(x) = H_{0^{n-1}\alpha}(x).$$

In this stage, the boundary condition is applied to calculate the value of C and reach an ordinary integral equation (in relation (19) we use the boundary condition):

$$a_1\Big[\int_a^b H_n(a,t)y(t)\,\mathrm{d}t + F_n(a) + H_n(a)C\Big] + b_1\Big[\int_a^b H_n(b,t)y(t)\,\mathrm{d}t + F_n(b) + H_n(b)C\Big] = C_1.$$

Finally, the constant C will be calculated as

$$C = \frac{C_1 - a_1 \int_a^b H_n(a,t)y(t)\,\mathrm{d}t - a_1 F_n(a) - b_1 \int_a^b H_n(b,t)y(t)\,\mathrm{d}t - b_1 F_n(b)}{a_1 H_n(a) + b_1 H_n(b)}. \tag{20}$$

Assuming

$$\beta = \frac{1}{a_1 H_n(a) + b_1 H_n(b)},$$

and substituting C in the equation (20), we get

$$\begin{aligned} y(x) = &\int_a^b H_n(x,t)y(t)\,\mathrm{d}t + F_n(x) - \beta a_1 H_n(x) \int_a^b H_n(a,t)y(t)\,\mathrm{d}t \\ &- \beta b_1 H_n(x) \int_a^b H_n(b,t)y(t)\,\mathrm{d}t - \beta a_1 H_n(x)F_n(a) - \beta b_1 H_n(x)F_n(b), \end{aligned}$$

where

$$\begin{aligned} H(x,t) &= H_n(x,t) - \beta a_1 H_n(x)H_n(a,t) - \beta b_1 H_n(x)H_n(b,t), \\ F(x) &= F_n(x) - \beta a_1 H_n(x)F_n(a) - \beta b_1 H_n(x)F_n(b). \end{aligned}$$

We obtain the following second kind Fredholm integral equation, which can be solved by using the ordinary sequential approximation method

$$y(x) = \int_a^b H(x,t)y(t)\,\mathrm{d}t + F(x).$$

4 Examples

Example 1. Consider the following boundary value problem

$$\begin{aligned} &y'' = x + y, \quad x \in (0,1), \\ &y(0) = y(1) = 0. \end{aligned}$$

The exact solution to this problem is

$$y(x) = \frac{e^x}{e - e^{-1}} - \frac{e^{-x}}{e - e^{-1}} - x.$$

We now solve Example 1 by using the method of special successive approximations. The general solution to the homogeneous part of the equation is $y(x) = ax + b$. By applying the method of variable change (Lagrange method), we obtain a particular solution for the non-homogeneous equation, and additionally, by applying the boundary conditions, we can express the solution to the problem in the form of the following integral equation

$$y(x) = \int_0^1 x(t-1)(t + y(t))\,\mathrm{d}t + \int_0^x (x-t)(t + y(t))\,\mathrm{d}t.$$

So we have

$$y(x) = \frac{x(x^2-1)}{6} + \int_0^1 x(t-1)y(t)\,\mathrm{d}t + \int_0^x (x-t)y(t)\,\mathrm{d}t.$$

It can be seen that the solution includes the Fredholm and the Volterra parts. Now we use the method of special sequential approximations. For this we substitute the previous value in the Volterra part

$$\begin{aligned} y(x) &= \frac{x(x^2-1)}{6} + \int_0^1 x(t-1)y(t)\,\mathrm{d}t + \int_0^x (x-t)\left[\frac{t(t^2-1)}{6} + \int_0^1 t(\eta-1)y(\eta)\,\mathrm{d}\eta + \int_0^1 (t-\eta)y(\eta)\,\mathrm{d}\eta\right]\mathrm{d}t \\ &= \frac{x(x^2-1)}{6} + \frac{1}{6}\left[\frac{1}{20}x^5 - \frac{1}{6}x^3\right] + \int_0^1 x(\eta-1)y(\eta)\,\mathrm{d}\eta + \int_0^1 y(\eta)\int_0^x \left[t(x-t)(\eta-1)\,\mathrm{d}t\right]\mathrm{d}\eta \\ &\quad + \int_0^x y(\eta)\left[\int_\eta^x (x-t)(t-\eta)\,\mathrm{d}t\right]\mathrm{d}\eta. \end{aligned}$$

If we denote the outside parts of the integral by

$$\phi(x) = \frac{x(x^2-1)}{6} + \frac{1}{6}\left[\frac{1}{20}x^5 - \frac{1}{6}x^3\right],$$

then we have

$$\begin{aligned} y(x) = \phi(x) &+ \int_0^1 y(\eta)\left[x(\eta-1) + (\eta-1)\int_0^1 t(x-t)\,\mathrm{d}t\right]\mathrm{d}\eta \\ &+ \int_0^x y(\eta)\left[x\frac{x^2-\eta^2}{2} - x\eta(x-\eta) - \frac{x^2-\eta^2}{2} - \eta\frac{x^2-\eta^2}{2}\right]\mathrm{d}\eta. \end{aligned}$$

By calculating and rearranging the sentences, we get

$$y(x) = \phi(x) + \int_0^1 \left(x + \frac{x^3}{3!}\right)[t-1]y(t)\,\mathrm{d}t - \int_0^x \frac{(x-t)^2}{3!}y(t)\,\mathrm{d}t.$$

As we see, the second sentence is a Fredholm integral, and the third sentence is a Volterra integral. At this point, if we omit the Volterra part as an error, we get to the following Fredholm integral equation

$$y(x) = \phi(x) + \left[x + \frac{x^3}{3!}\right]\int_0^1 (t-1)y(t)\,\mathrm{d}t.$$

Now in order to solve this Fredholm integral equation and to obtain the corresponding algebraic system, we multiply both sides of the above equation by $(x-1)$ and take the integrals over the interval $[0,1]$

$$\int_0^1 (x-1)y(x)\,\mathrm{d}x = \int_0^1 (x-1)\phi(x)\,\mathrm{d}x + \int_0^1 (x-1)\left(x + \frac{x^3}{3!}\right)\mathrm{d}x \int_0^1 (t-1)y(t)\,\mathrm{d}t.$$

Now, letting

$$a = \int_0^1 (x-1)\left(x + \frac{x^3}{3!}\right)\mathrm{d}x, \quad b = \int_0^1 (x-1)\phi(x)\,\mathrm{d}x, \quad c = \int_0^1 (x-1)y(x)\,\mathrm{d}x,$$

we get the following algebraic equation

$$c = ac + b \quad \text{if} \quad a \neq 1 \Longrightarrow c(1-a) = b \quad \text{if} \quad a \neq 1 \Longrightarrow c = \frac{b}{1-a}.$$

Finally we will have the solution to the integral equation with the following relation

$$\tilde{y(x)} \approx \phi(x) + \left(x + \frac{x^3}{3!}\right)\frac{\int_0^1 (t-1)\phi(t)\,\mathrm{d}t}{1 - \int_0^1 (t-1)\left(t + \frac{t^3}{3!}\right)\mathrm{d}t}.$$

If we apply the method of successive approximations, for the second time, then the solution will be more accurate, but we will need to engage more calculations. We now compare the exact solution of the problem with the

Table 1: Comparison of the exact and approximate solutions.

x	$y(x)$	$\overline{y(x)}$	$Error(x)$
0	0	0	0
0.25	-0.03504	-0.03505	1×10^{-5}
0.5	-0.05659	-0.05656	3×10^{-5}
0.5846	-0.058331	-0.058265	6.6×10^{-5}
0.5846	-0.05833	-0.05826	7×10^{-5}
0.75	-0.05027	-0.05085	5.8×10^{-4}
1	0	0.00104	1.04×10^{-3}

approximate solution obtained from the method of successive approximations, shown in Figure 1. Additionally, we compare the exact solution with the approximate solution obtained using the Picard iteration method, shown in Figure 2. In Table 1, we have calculated the values of $y(x)$ (exact solution) and $\overline{y(x)}$ (approximate solution) and the amount of $Error(x)$.

Example 2. Consider the following fractional boundary value problem

$$D^{\alpha}y(x)=y(x),\qquad 1<x<2,\qquad \alpha\in(1,2), \tag{21}$$

$$y(2)=2y(1). \tag{22}$$

By taking the α-order fractional integral, we have

$$y(x)=\int_1^x\frac{(x-t)^{\alpha-1}}{(\alpha-1)!}y(t)\,\mathrm{d}t+C\frac{x^{\alpha-1}}{(\alpha-1)!}. \tag{23}$$

Note that the term

$$C\frac{x^{\alpha-1}}{(\alpha-1)!},$$

is the solution of the homogeneous fractional equation (21). By applying boundary condition (22) we get

$$\int_1^2\frac{(2-t)^{\alpha-1}}{(\alpha-1)!}y(t)\,\mathrm{d}t+C\frac{2^{\alpha-1}}{(\alpha-1)!}=2C\frac{1}{(\alpha-1)!}\quad\Longrightarrow\quad C=\frac{\int_1^2(2-t)^{\alpha-1}y(t)\,\mathrm{d}t}{2-2^{(\alpha-1)}}.$$

Now by substituting the constant C in Eq. (23) we obtain

$$y(x)=\int_1^x\frac{(x-t)^{\alpha-1}}{(\alpha-1)!}y(t)\,\mathrm{d}t+\frac{x^{\alpha-1}}{(\alpha-1)!(2-2^{\alpha-1})}\int_1^2(2-t)^{\alpha-1}y(t)\,\mathrm{d}t.$$

And by applying special sequential approximation for the first time in the Volterra term, we get

$$y(x)=\int_1^x\frac{(x-t)^{\alpha-1}}{(\alpha-1)!}\,\mathrm{d}t\int_1^t\frac{(t-\xi)^{\alpha-1}}{(\alpha-1)!}y(\tau)\,\mathrm{d}\tau+F_1(x). \tag{24}$$

Note that the residual of Eq. (24) is shown by $F_1(x)$, and by the variable changes $t-\tau=\xi(x-\tau)$, and changing the order of integration, we obtain

$$y(x)=\frac{1}{[(\alpha-1)!]^2}\int_1^x y(\tau)\,\mathrm{d}\tau\int_\tau^x(x-t)^{\alpha-1}(t-\xi)^{\alpha-1}\,\mathrm{d}t+F_1(x).$$

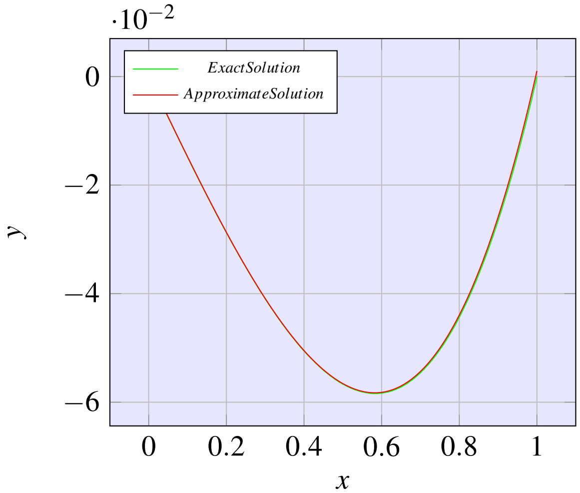

Figure 1: Comparison of the exact and approximate solutions.

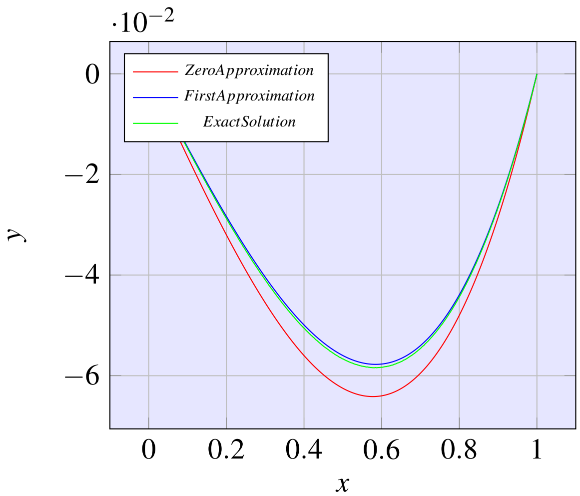

Figure 2: Comparison of the exact and the Picard iteration solutions.

And the Kernel of the Voltrra term will be equal to

$$\int_0^1 (x-\xi)^{2\alpha-1}\xi^{\alpha-1}(1-\xi^{\alpha-1})\,\mathrm{d}\xi = (x-\tau)^{2\alpha-1}\int_0^1 \xi^{\alpha-1}(1-\xi)^{\alpha-1}\,\mathrm{d}\xi.$$

Now by letting

$$A_1 = \int_0^1 \xi^{\alpha-1}(1-\xi)^{\alpha-1}\,\mathrm{d}\xi,$$

and by applying the special sequential approximation method for the second time to the Volterra term we will have

$$y(x) = \frac{A_1}{[(\alpha-1)!]^2}\int_0^x (x-\tau)^{2\alpha-1}y(\tau)\,\mathrm{d}\tau + F_2(x).$$

Applying the method for the third time will give

$$y(x) = \frac{A_1}{[(\alpha-1)!]^3} \int_1^x y(t)\mathrm{d}t \int_t^x (x-\tau)^{2\alpha-1}(\tau-t)^{\alpha-1}\,\mathrm{d}\tau + F_2(x).$$

By the change of variabl $\tau = (x-t)\eta + t$, and by calculating the second integral, we deduce

$$\int_t^x (x-\tau)^{2\alpha-1}(\tau-t)^{\alpha-1}\,\mathrm{d}\tau = (x-t)^{3\alpha-1}\int_0^1 \eta^{\alpha-1}(1-\tau)^{2\alpha-1}\,\mathrm{d}\eta.$$

and by assuming

$$A_2 = \int_0^1 \eta^{\alpha-1}(1-\eta)^{2\alpha-1}\,\mathrm{d}\eta,$$

we get

$$y(x) = \frac{A_1 A_2}{[(\alpha-1)!]^3}\int_1^x (x-t)^{3\alpha-1} y(t)\,\mathrm{d}t + F_3(x).$$

In the nth iteration of the special sequential approximation in the kernel of the Volterra part, we have

$$y(x) = \frac{\Pi_{i=1}^n A_i}{[(\alpha-1)!]^n}\int_1^x (x-t)^{n\alpha-1} y(t)\,\mathrm{d}t + F_n(x),$$

and clearly the kernel of the Volterra part, when $n \longrightarrow \infty$, tends to zero, and therefore we can remove the Volterra term and solve the Fredholm term as the final approximate solution.

Example 3. Let us now consider the following integral equation including the Volterra and the Fredholm parts

$$u(x) = 1 + \int_0^x u(t)\,\mathrm{d}t + \int_0^1 (t^2+t-1)u(t)\,\mathrm{d}t, \tag{25}$$

with the analytical solution $u(t) = e^x$.

Now, we solve the integral equation (25) by using the method of successive special approximations. We have

$$u(t) = 1 + \int_0^t u(\xi)\,\mathrm{d}\xi + \int_0^1 (\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi. \tag{26}$$

Now, we substitute the relation (26) in the Volterra part of the relation (25):

$$u(x) = 1 + \int_0^1 (t^2+t-1)u(t)\,\mathrm{d}t + \int_0^x \left[1 + \int_0^t u(\xi)\,\mathrm{d}\xi + \int_0^1 (\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi\right]\mathrm{d}t,$$

$$u(x) = 1 + x\int_0^1 (t^2+t-1)u(t)\,\mathrm{d}t + \int_0^x\int_0^t u(\xi)\,\mathrm{d}\xi\,\mathrm{d}t + \int_0^x\int_0^1 (\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi\,\mathrm{d}t.$$

By solving the above integrals, we have

$$u(x) = (1+x) + (1+x)\int_0^1 (\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi + \int_0^x (x-\xi)u(\xi)\,\mathrm{d}\xi. \tag{27}$$

After applying the iteration twice by using $u(\xi)$ and substituting it in (26), we obtain

$$u(\xi) = 1 + \int_0^\xi u(\eta)\,\mathrm{d}\eta + \int_0^1 (\eta^2+\eta-1)u(\eta)\,\mathrm{d}\eta. \tag{28}$$

Here, we substitute the relation (28) in the Volterra part of the relation (27), and thus as a result, we get

$$u(x)=(1+x)+(1+x)\int_0^1(\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi+\int_0^x(x-\xi)\left[1+\int_0^\xi u(\eta)\,\mathrm{d}\eta+\int_0^1(\eta^2+\eta-1)u(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi.$$

And as a result, we obtain

$$u(x)=(1+x+\frac{1}{2}x^2)+(1+x+\frac{1}{2}x^2)\int_0^1(\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi+\int_0^x(\frac{1}{2}x^2+\frac{1}{2}\xi^2-x\xi)u(\xi)\,\mathrm{d}\xi.$$

Note that in the nth repetition we get

$$u(x)=\sum_{k=0}^n\frac{x^k}{k!}+\sum_{k=0}^n\frac{x^k}{k!}\int_0^1(\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi+\int_0^x\frac{(x-\xi)^n}{n!}u(\xi)\,\mathrm{d}\xi.$$

When $n\to\infty$, and by removing Volterra part we have

$$u(x)=e^x+e^x\int_0^1(\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi, \tag{29}$$

which is a Fredholm integral equation. Now, in order to solve Eq. (29), let

$$\alpha=\int_0^1(\xi^2+\xi-1)u(\xi)\,\mathrm{d}\xi.$$

Then, we have $u(x)=e^x+e^x\alpha$. The last integral gives $\alpha=0$, and $u(x)=e^x$, which is the same analytic solution.

Table 2: Comparison of the exact and approximate solutions.

x	$y(x)=e^x$	$u_3(x)$	$y_3(x)$	$Error u_3(x)$	$Error y_3(x)$
0	1	0.994475	0.97037	0.005525	0.02963
0.25	1.28403	1.27676	1.23513	0.00727	0.0489
0.5	1.64872	1.63674	1.56759	0.01198	0.08113
0.75	2.117	2.08995	1.98339	0.02705	0.13361
1	2.71828	2.65193	2.49815	0.06635	0.22013

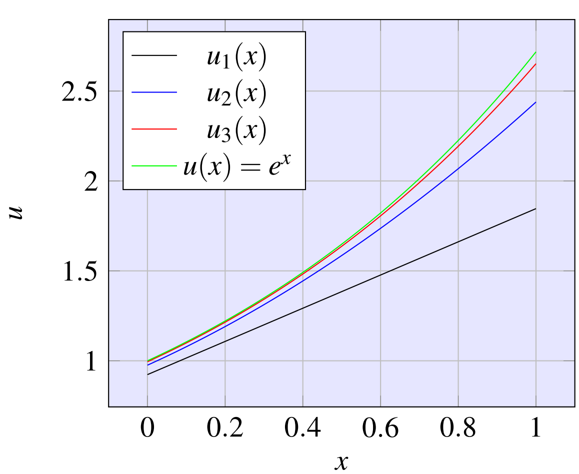

Figure 3: Comparison of the exact and approximate solutions.

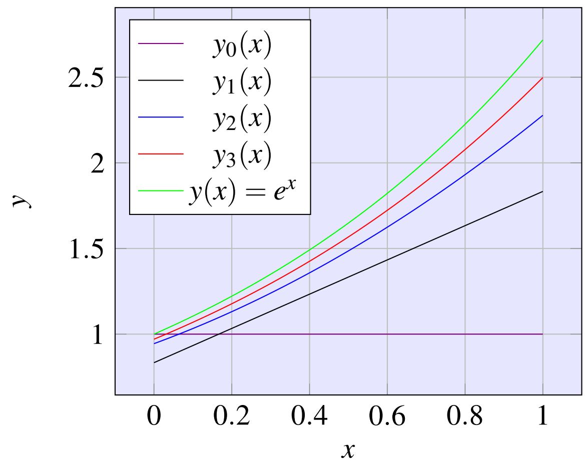

Figure 4: Comparison of the exact and the Picard iteration solutions.

5 Conclusion

In this paper, we converted the system of ordinary differential equations with variable coefficients into an integral equation including both the Fredholm and the Volterra components. We then applied the method of special successive approximations to solve both the system of ordinary and the fractional integro-differential equations. After a few finite iterations of the successive approximation method, it has been demonstrated that the Volterra kernel tends to zero due to the presence of the factorial factor $(n-1)!$ in its denominator, and as a result, the overlapping of the third iteration involved in this method, with the exact solution, has been illustrated in the diagram.

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