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# A neuro-fuzzy approach to compute the solution of a $Z$-numbers system with Trapezoidal fuzzy data

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**Abstract.** Linear systems of equations with $Z$-numbers have recently attracted some interest. Some approaches have been developed for solving these systems. Since, there are many ambiguities and uncertainties in such issues, there is no analytic solution for these kinds of systems. Therefore, numerical schemes are usually used to estimate the solution of them. In this research, a computational scheme for solving linear systems involving trapezoidal $Z$-numbers is presented. The proposed approach is designed in such a way that it is firstly converted the $Z$-numbers coefficients to the corresponding fuzzy numbers and then using a ranking function, the fuzzy coefficients are converted to real coefficients. In this trend, after two stages, firstly, the original $Z$-numbers system becomes a fuzzy linear system and the fuzzy system is converted to a real system. Then, the obtained crisp linear system is solved based on the artificial neural network algorithm. Finally, two sample trapezoidal $Z$-numbers systems are solved based on the given approach to illustrate the process of the proposed algorithm.

*Keywords*: $Z$-Numbers, Trapezoidal Fuzzy number, Weighted $Z$-number, Linear system of equations, Artificial Neural Networks, Ranking Function.
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## 1 Introduction

Fuzzy linear systems are one of the considerable mathematical models in science and engineering and presenting any efficient method to evaluate the solution of such systems is important. Fuzzy system of equations have been considered by many researchers to develop this subject [1, 5, 15, 21–23, 25, 38, 50]. Artificial neural networks (ANN) were applied in [38] to solve fuzzy linear systems of equations. In real world problems considering the reliability of uncertain information is essential. Thus, in order to

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remove this drawback in information analysis, Zadeh [48] introduced the concept of $Z$-number in 2011 to model uncertain information. $Z$-number is combined with two parts, the first part is restriction and the second part is reliability, which consists of thinking of humans. $Z$-number has an extreme ability to demonstrate human knowledge in comparison to the classical fuzzy number [2, 10, 11, 36, 43, 46]. One of the issues with the $Z$-number is the yielded results can not be represented in the form of $Z$-number. Many researchers applied $Z$-number in their works, such as arithmetic of discrete or continuous $Z$-numbers [3, 4], $Z$-evaluations [42], the $Z$-Advanced numbers process [6], solving system of equations [36], sensor data fusion [30], ranking $Z$-numbers [43], $Z$-numbers DEA [9], and other decision making models with $Z$-number [27, 31, 33, 39, 44–46]. Ranking $Z$-numbers is also an important topic in solving some issues with uncertainty. Many ranking approaches of $Z$-number suggested such as ranking using the concept of value and ambiguity at levels of decision-making [14], ranking by using tangent function [18], ranking by defining the sigmoid function of $Z$-number [19], ranking based on the developed golden rule representative value [13] and others.

In [28], the authors proposed an algorithm to solve a triangular $Z$-number system of equations by means of an aspect of the neural network. In this work, we develop this idea to propose an approach by means of another ranking function to approximate the solution of a $Z$-based linear system of equations (LSE) in the form $\widetilde{A_Z}Y = \widetilde{b_Z}$, where $Y$ is a crisp unknown vector and $\widetilde{A_Z}$ and $\widetilde{b_Z}$ are the trapezoidal $Z$-numbers coefficients matrix and right hand side vector, respectively. In order to compute the vector $Y$, at first by applying a special conversion method, the $Z$-numbers are converted to regular fuzzy numbers, then a ranking function is used to convert the obtained fuzzy numbers to the crisp real numbers. Finally, the system is solved by the ANN algorithm.

The novelty of this work can be summarized as follows:

- Considering a trapezoidal $Z$-number LSE with $Z$-number coefficients of matrix and also $Z$-number right hand side vector,
- Converting the required $Z$-number system by choosing a ranking function and a conversion method, to determine a crisp LSE,
- Applying the ANN technique with backward propagation in the proposed algorithm to solve the obtained crisp LSE.

In Section 2, the notions of fuzzy numbers and concept of $Z$-numbers are recalled. Section 3 establishes some methods for converting the weighted $Z$-number to the regular fuzzy number. In Section 4 the review of the ANN method and the proposed algorithm to solve the LSE with trapezoidal $Z$-number coefficients are presented. In Section 5, two sample trapezoidal $Z$-number systems are considered to solve based on the given algorithm and all steps of the presented algorithm are described in detail. Finally, concluding remarks are summarized.

## 2 Preliminaries

Here, some definitions of fuzzy sets theory ( [47]) such as fuzzy number and also the concept of $Z$-numbers with their properties are reviewed.

**Definition 1.** *[17] A function $\tilde{F} : \mathbb{R} \to [0, 1]$ is said to be a fuzzy number when the following conditions are satisfied:*

*(1) $\tilde{F}$ is normal, that is $\exists a \in \mathbb{R}$ such that $\tilde{F}(a) = 1$,*

*(2)* $\tilde{F}$ *is fuzzy convex set*

$$(i.e.\ \tilde{F}(\theta x_1+(1-\theta)x_2) \geq \min\left\{\tilde{F}(x_1),\tilde{F}(x_2)\right\} \quad for\ all\ x_1,x_2 \in \mathbb{R}, \theta \in [0,1]),$$

*(3)* $\tilde{F}$ *is upper semi-continuous on* $\mathbb{R}$*,*

*(4) the closure of* $supp(\tilde{F})$*, i.e. the* $\overline{\left\{ x\in\mathbb{R} : \tilde{F}(x)>0\right\}}$ *is compact set.*

Let $\mathscr{E}$ denote the set of all fuzzy numbers. The $r-$cut set, $0<r\leq 1$, of a fuzzy number is defined as a closed interval $[\tilde{F}]_r=\{x\in\mathbb{R} : \tilde{F}(x)\geq r\}=[\underline{\tilde{F}}(r),\overline{\tilde{F}}(r)]$, for all $r\in[0,1]$ (see [8]), where $\underline{\tilde{F}},\overline{\tilde{F}} : [0,1]\to\mathbb{R}$, as the end-points of $r$-cuts of $\tilde{F}$ are increasing and decreasing functions, respectively.

**Definition 2.** *[49] Let* $F_1,F_2\in\mathscr{E}$ *and* $[\widetilde{F_1}]_r=[\underline{\widetilde{F_1}}(r),\overline{\widetilde{F_1}}(r)]$, $[\widetilde{F_2}]_r=[\underline{\widetilde{F_2}}(r),\overline{\widetilde{F_2}}(r)]$ *and* $r\in[0,1]$. *Then*

$$\begin{aligned}
[\widetilde{F_1}+\widetilde{F_2}]_r &= [\underline{\widetilde{F_1}}(r)+\underline{\widetilde{F_2}}(r),\overline{\widetilde{F_1}}(r)+\overline{\widetilde{F_2}}(r)],\\
[-\widetilde{F_1}]_r &= [-\overline{\widetilde{F_1}}(r),-\underline{\widetilde{F_1}}(r)],\\
[\widetilde{F_1}-\widetilde{F_2}]_r &= [\underline{\widetilde{F_1}}(r)-\overline{\widetilde{F_2}}(r),\overline{\widetilde{F_1}}(r)-\underline{\widetilde{F_2}}(r)],\\
[\beta\widetilde{F_1}]_r &= [\beta\underline{\widetilde{F_1}}(r),\beta\overline{\widetilde{F_1}}(r)], \qquad (\beta\geq 0),\\
[\beta\widetilde{F_1}]_r &= [\beta\overline{\widetilde{F_1}}(r),\beta\underline{\widetilde{F_1}}(r)], \qquad (\beta< 0).
\end{aligned}$$

**Definition 3.** *[16] A triangular fuzzy number* $\widetilde{T}$ *can be defined by a triple* $(t_1,t_2,t_3)$ *in which the values* $t_1$*,* $t_2$ *and* $t_3$ *are real numbers with following membership function:*

$$\widetilde{T}(x)=\begin{cases}0, & -\infty\leqslant x\leqslant t_1,\\ \frac{x-t_1}{t_2-t_1}, & t_1\leqslant x\leqslant t_2,\\ \frac{t_3-x}{t_3-t_2}, & t_2\leqslant x\leqslant t_3,\\ 0, & t_3\leqslant x\leqslant +\infty.\end{cases} \tag{1}$$

**Definition 4.** *[40] A generalized trapezoidal fuzzy number can be expressed as* $\widetilde{T}=(t_1,t_2,t_3,t_4;\omega)$ *where the values* $t_1$*,* $t_2$*,* $t_3$ *and* $t_4$ *are real numbers,* $\omega\in(0,1]$ *is a constant, and the membership function of* $\widetilde{T}$ *is defined as:*

$$\widetilde{T}(x)=\begin{cases}0, & -\infty\leqslant x\leqslant t_1,\\ \frac{\omega(x-t_1)}{t_2-t_1}, & t_1\leqslant x\leqslant t_2,\\ \omega, & x\in[t_2,t_3],\\ \frac{\omega(t_4-x)}{t_4-t_3}, & t_3\leqslant x\leqslant t_4,\\ 0, & t_4\leqslant x\leqslant +\infty.\end{cases} \tag{2}$$

In the special case of $\omega=1$, $\widetilde{T}$ is called the normal trapezoidal fuzzy number, which is denoted by $\widetilde{T}=(t_1,t_2,t_3,t_4)$.

Yager in [41] proposed a ranking function like $\mathscr{R}$ for a given fuzzy number $\widetilde{F}$ with $[\widetilde{F}]_r = [\underline{\widetilde{F}}(r), \overline{\widetilde{F}}(r)]$ as follows:

$$\mathscr{R}(\widetilde{F}) = \frac{1}{2}\int_0^1 r\left(\overline{\widetilde{F}}(r) + \underline{\widetilde{F}}(r)\right)dr, \tag{3}$$

which is reduced for the trapezoidal fuzzy number $\widetilde{T} = (t_1, t_2, t_3, t_4)$ as:

$$\mathscr{R}(\widetilde{T}) = \frac{t_1 + t_2 + t_3 + t_4}{4}. \tag{4}$$

The following ranking property is satisfied for $\widetilde{F_1}$, $\widetilde{F_2} \in \mathscr{E}$ and any $a \in \mathbb{R}$:

$$\mathscr{R}(a\widetilde{F_1} + \widetilde{F_2}) = a\mathscr{R}(\widetilde{F_1}) + \mathscr{R}(\widetilde{F_2}).$$

**Definition 5.** *Let $\widetilde{T}$ and $\widetilde{T}'$ be two normal trapezoidal fuzzy numbers, then*

1. $\widetilde{T} \prec \widetilde{T}' \Leftrightarrow \mathscr{R}(\widetilde{T}) < \mathscr{R}(\widetilde{T}')$.
2. $\widetilde{T} \succ \widetilde{T}' \Leftrightarrow \mathscr{R}(\widetilde{T}) > \mathscr{R}(\widetilde{T}')$.
3. $\widetilde{T} \approx \widetilde{T}' \Leftrightarrow \mathscr{R}(\widetilde{T}) = \mathscr{R}(\widetilde{T}')$.

**Definition 6.** *[7, 32] The ordered pair $(\widetilde{F}, \widetilde{R})$, where $\widetilde{F}$ is the restriction of an uncertain variable on real values and $\widetilde{R}$ is a measure of reliability of $\widetilde{F}$, is called a Z-number, denoted by $\widetilde{Z} = (\widetilde{F}, \widetilde{R})$.*

The $\widetilde{R}$, as the reliability part of $\widetilde{Z}$, can be converted to a classical number. To this end, let $\widetilde{R} = \{(t, \widetilde{R}(t)) : t \in [0,1]\}$, where $\widetilde{R}(t)$ is the membership function. Then, the corresponding crisp number can be obtained by means of the following center of gravity method:

$$\alpha = \frac{\int_S t\widetilde{R}(t)dt}{\int_S \widetilde{R}(t)dt}, \tag{5}$$

where $S = supp(\tilde{R}) = [0,1]$. For triangular fuzzy number $\widetilde{R} = (t_1, t_2, t_3)$, Eq. (5) may be described as:

$$\alpha = \frac{t_1 + t_2 + t_3}{3}. \tag{6}$$

**Definition 7.** *[32] The fuzzy expectation of the fuzzy set $\widetilde{F} = \{(t, \widetilde{F}(t)); t \in U\}$ is defined as:*

$$E_{\widetilde{F}}(t) = \int_U t\widetilde{F}(t)dt. \tag{7}$$

Let $\widetilde{Z^\alpha}$ denote as the weighted $Z$-number ($WZ$-number), defined by

$$\widetilde{Z^\alpha} = \{(t, \widetilde{A^\alpha}(t)) : \widetilde{Z^\alpha}(t) = \alpha\widetilde{F}(t), t \in U\}, \tag{8}$$

then, from (5), (7) and (8), we have

$$\begin{aligned} &E_{\widetilde{A^\alpha}} = \alpha E_{\widetilde{F}}, \qquad t \in U, \quad \alpha \in (0,1], \\ &\text{Subject to} \quad \widetilde{A^\alpha}(t) = \alpha\widetilde{F}(t), \qquad t \in U. \end{aligned}$$

# 3 Methods of conversion

In the sequel, two conversion methods to convert a $WZ$-number to a normal trapezoidal fuzzy number are presented.

## 3.1 First conversion method [32]

In this method, the $WZ$-number in Eq. (8) is converted to the following normal fuzzy number:

$$\widetilde{Z'} = \{(t, \widetilde{Z'}(t)) : \widetilde{Z'}(t) = \alpha \widetilde{F}(\frac{t}{\sqrt{\alpha}}), t \in U\}. \tag{9}$$

By applying Eqs. (5), (7) and (9), we have

$$\begin{aligned} &E_{\widetilde{Z'}}(t) = \alpha E_{\widetilde{F}}(t), \qquad (t \in \sqrt{\alpha}U), 0 < \alpha \leq 1, \\ &\text{Subject to} \quad \widetilde{Z'}(t) = \widetilde{F}(\frac{t}{\sqrt{\alpha}}), \qquad (t \in \sqrt{\alpha}U). \end{aligned}$$

The following example illustrates the mentioned scheme. Let the $Z$-number as:

$$\widetilde{Z} = (\widetilde{F}, \widetilde{R}) = \big[(0.6, 0.7, 0.8, 0.9; 1), (0.1, 0.25, 0.4; 1)\big].$$

From Eq. (5), the reliability is converted to the following crisp number:

$$\alpha = 0.25, \quad \widetilde{Z^{\alpha}} = \{0.6, 0.7, 0.8, 0.9; 0.25\},$$

where, $\widetilde{Z^{\alpha}}$ is the $WZ$-number that is converted to the following regular fuzzy number:

$$\begin{aligned} \widetilde{Z'} &= \{\sqrt{0.25} \times 0.6, \sqrt{0.25} \times 0.7, \sqrt{0.25} \times 0.8, \sqrt{0.25} \times 0.9; 1\} \\ &= \{0.3, 0.35, 0.4, 0.45; 1\}. \end{aligned}$$

## 3.2 Second conversion method (The proposed scheme)

In [9], first the triangular fuzzy number of the first part of $Z$-number is converted to the corresponding $WZ$-number. Then, the $WZ$-number related to first part of $Z$-number, is converted into a regular fuzzy number. In this work, the mentioned idea is applied for the $Z$-number with the first part as a trapezoidal fuzzy number and a triangular fuzzy number as the reliability part of it.

It is noted that the method of converting $WZ$-number, based on its reliability, into a regular fuzzy number, increases the domain of the ambiguity of the $Z$-number in case of normality. Since the membership function of the $WZ$-number is trapezoidal, the corresponding regular fuzzy number is trapezoidal too. Furthermore, the slope of the sides of the normal trapezoidal fuzzy number is equal to the sides of the $WZ$-number and there is a relationship between the slope of sides and the value of $\alpha$. Since the $WZ$-number has a trapezoidal membership function with $\widetilde{Z^{\alpha}} = (t_1, t_2, t_3, t_4)$, then its corresponding regular fuzzy number has trapezoidal membership function as $\widetilde{Z'} = (t'_1, t'_2, t'_3, t'_4)$. Let $t_2 = t'_2$, $t_3 = t'_3$, with equal slope of sides. In order to evaluate the value of $t'_1$, the left side slope of $\widetilde{Z^{\alpha}}$ must be considered, which is equals to $\frac{\alpha}{t_2 - t_1}$ and hence the left side linear equation of the fuzzy number is

$$\widetilde{Z'}(x) = \frac{\alpha}{t_2 - t_1} x + h, \quad x \leq t_2.$$

In order to find the value of $h$, we put the point of $(t_2,1)$ in this equation, so we have $h = 1-\frac{\alpha t_2}{t_2-t_1}$. Thus, the left side membership function of normal fuzzy number is obtained as follows:

$$\widetilde{Z}'(x) = \frac{\alpha}{t_2-t_1}x + 1 - \frac{\alpha t_2}{t_2-t_1}, \quad x \leq t_2, \tag{10}$$

where $\alpha$ is defined in Eq. (5). To identify the value of $t_1'$, we solve the equation $\widetilde{Z}'(t_1') = 0$ in terms of $t_1'$. Thus

$$t_1' = \frac{\alpha t_2 - t_2 + t_1}{\alpha}. \tag{11}$$

The same steps are carried out to find the value of $t_4'$. The right side slope of the $WZ$-number and its corresponding normal fuzzy number is $\frac{\alpha}{t_3-t_4}$. Then, the rule of membership function of the right hand side of a normal fuzzy number is

$$\widetilde{Z}'(x) = \frac{-\alpha}{t_4-t_3}x + h, \quad x \geq t_3.$$

To find the value of $h$, we put the point of $(t_3,1)$ in this equation, so we have $h = 1+\frac{\alpha t_3}{t_4-t_3}$. Therefore, the right side membership function of normal fuzzy number is obtained as follows:

$$\widetilde{Z}'(x) = \frac{-\alpha}{t_4-t_3}x + 1 + \frac{\alpha t_3}{t_4-t_3}, \quad x \geq t_3. \tag{12}$$

Now, the value of $t_4'$ is identified by solving the equation $\widetilde{Z}'(t_4') = 0$, so, we get:

$$t_4' = \frac{\alpha t_3 - t_3 + t_4}{\alpha}. \tag{13}$$

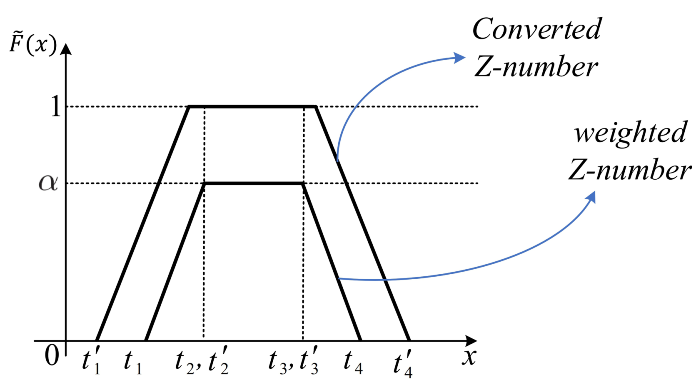


Figure 1: A $WZ$-number and its conversion to the regular fuzzy number.

Fig. 1 shows the weighted $Z$-number with height $\alpha$ and presents the conversion into the regular fuzzy number as schematically. The following example shows the mentioned method. Let

$$\widetilde{Z} = (\widetilde{F}, \widetilde{R}) = \big[(0.6, 0.7, 0.8, 0.9; 1), (0.1, 0.25, 0.4; 1)\big].$$

From (5), the part of reliability should be converted to a crisp real numbers as follows:

$$\alpha = 0.25, \quad \widetilde{Z^{\alpha}} = \{0.6, 0.7, 0.8, 0.9; 0.25\}.$$

Now, we find the regular trapezoidal fuzzy number using Eqs. (11) and (13) as follows:

$$\begin{aligned}
t_1' &= \frac{\alpha t_2 - t_2 + t_1}{\alpha} = \frac{0.25 \times 0.7 - 0.7 + 0.6}{0.25}, \\
t_2' &= t_2 = 0.7, \\
t_3' &= t_3 = 0.8, \\
t_4' &= \frac{\alpha t_3 - t_3 + t_4}{\alpha} = \frac{0.25 \times 0.8 - 0.8 + 0.9}{0.25}.
\end{aligned}$$

Hence, we obtain

$$\widetilde{Z'} = (0.3, 0.7, 0.8, 1.2; 1).$$

# 4 The proposed algorithm

In this section, at first a brief review of the ANN which is applied in the given algorithm is presented. Then, the proposed algorithm to approximate the solution of the $Z$-number system is presented.

The ANN in backward propagation case, with the following two-stage learning algorithm, is one of the efficient computational schemes in engineering sciences (See [12, 20]):
(1) forward computation of working signal,
(2) backward propagation of training error.

In order to solve a crisp LSE by applying the ANN method, a two-layer backward propagation neural network is designated. Fig. 2 shows the architecture of this ANN where, $w_i$'s show the weights of the ANN related to the $y_i$'s (the components of the unknown vector $Y$), $a_{ij}$'s, and $t_i$, $i, j = 1, 2, \ldots, n$, are the inputs and output of the ANN method, respectively to evaluate the crisp linear system $Ay = b$, with following objective function:

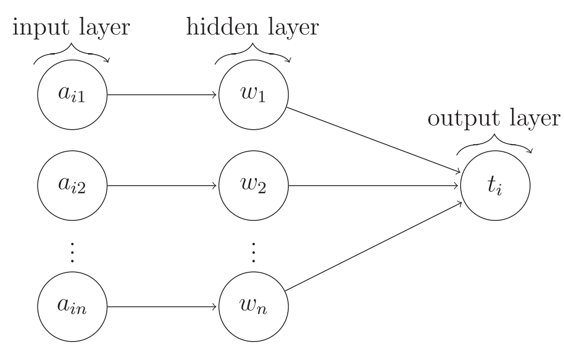


Figure 2: The architecture of an ANN to solve linear system $AY = b$.

$$\min \sum_{j=1}^{n} (e_i)^2 \ , \quad e_i = t_i - y_i.$$

Here, a numerical algorithm is proposed in order to approximate the solution of such systems. To this end, the ANN with a two-layer network is considered, where the inputs are in the first layer and the weights of the neural network are the second layer as outputs. Using optimization techniques the training of ANN is based on minimizing the sum of squared errors [20, 38]. It should be noted that one of the important criteria in training ANN is the repetitions that the network performs during the training. The higher the number of repetitions in the network training, the lower the network simulation error; but when the number of iterations exceeds a value, the error of the method will also increase.

In this work, the MATLAB Neuro Solution package is applied and the final solution is evaluated with specific standard criteria [24, 26, 29, 34, 37, 49].

The following algorithm is introduced in order to compute the solution of a LSE with trapezoidal $Z$-number coefficients as:

$$\widetilde{A_Z} Y = \widetilde{b_Z}, \tag{14}$$

where, the coefficients of the matrix ($\widetilde{A_Z}$) and the components of the vector ($\widetilde{b_Z}$) are $Z$-numbers containing of the trapezoidal fuzzy numbers, and the crisp unknown vector is $Y = (y_1, y_2, y_3, \ldots, y_n)^T$ where $y_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots, n$. The MATLAB package is applied to implement this algorithm.

**Algorithm 1.**

**1.** Transform the coefficients of $\widetilde{A_Z}$ and $\widetilde{b_Z}$ to $WZ$-number by means of Eq. (5).

**2.** Rewrite the obtained $WZ$-number coefficients to normal fuzzy number coefficients to achieve the fuzzy system as $\widetilde{A'}Y = \widetilde{b'}$.

**3.** Transform the obtained fuzzy coefficients, apply Eq. (4) to the classical corresponding coefficients to get the crisp LSE as $A''Y = b''$, where $A'' = [a''_{ij}], i, j = 1, 2, \ldots, n$ and $b'' = (b''_1, b''_2, \ldots, b''_n)^T$.

**4.** Apply the following ANN steps to compute the obtained crisp LSE:

**4.1.** Input data "$\varepsilon$" (tolerance), $\gamma$ (learning parameter), random (initial) vector weight $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, \ldots, w_n^{(0)})^T$, and let $k = 0$ and $e^{(0)} = 0$.

**4.2.** Calculate $t_i^{(k)} = \sum_{j=1}^{n} a''_{ij} w_j^{(k)}$, $e_i^{(k)} = b''_i - t_i^{(k)}$.

**4.3.** Evaluate the residual error as $E^{(k)} = \frac{1}{2} \|b'' - A'' w^{(k)}\|^2 = \frac{1}{2} \sum_{i=1}^{n} \left(e_i^k\right)^2$ and update the weights by the following formula:

$$w_j^{(k+1)} = w_j^{(k)} + \Delta w_j^{(k)}, \quad j = 1, 2, \ldots, n \quad \text{where} \quad \Delta w_j^{(k)} = \gamma \sum_{i=1}^{n} e_i^{(k)} a''_{ij}.$$

**4.4.** If $E^{(k)} \geq \varepsilon$ then $k = k + 1$ and return to step 4.2, else go to step 4.5.

**4.5** Write $w_j, \quad j = 1, 2, \ldots, n$.

# 5 Illustrative examples

In the sequel, Algorithm 1 is applied on two sample trapezoidal $Z$-number linear systems. The learning parameter is evaluated as $\gamma = \frac{1}{2tr(A''^T A'')}$ [35], the initial weight vector is considered as $w^{(0)} = 0$, and $\varepsilon = 5 \times 10^{-7}$ in both examples.

**Example 1.** Consider the $Z$-number system $\widetilde{A_Z}Y = \widetilde{b_Z}$, where

$$\widetilde{A_Z} = \begin{bmatrix} [(5,7,8,10),(0.4,0.5,0.6)] & [(2,2,4,4),(0.7,0.8,0.9)] \\ [(5,7,8,10),(0.4,0.5,0.6)] & [(-4,-4,-2,-2),(0.7,0.8,0.9)] \end{bmatrix},$$
$$\widetilde{b_Z}^T = \begin{bmatrix} [(7,9,12,16),(0.28,0.4,0.54)] & [(1,3,6,8),(0.28,0.4,0.54)] \end{bmatrix},$$

with the crisp exact solution $y_1 = y_2 = 1$.

**Step 1.** The $Z$-number inputs are transformed to the following weighted fuzzy number entries by means of Eq. (6):

$$\widetilde{A^\alpha} = \begin{bmatrix} (5,7,8,10\,;0.5) & (2,2,4,4\,;0.8) \\ (5,7,8,10\,;0.5) & (-4,-4,-2,-2\,;0.8) \end{bmatrix},$$
$$\widetilde{b^\alpha}^T = \begin{bmatrix} (7,9,12,16\,,0.4067) & (1,3,6,8\,;0.4067) \end{bmatrix}.$$

**Step 2.** The obtained weighted fuzzy numbers are converted to the following normal fuzzy numbers based on the first conversion method ( [32]):

$$\widetilde{A'} = \begin{bmatrix} (3.53554.94975.65697.0711) & (1.78891.78893.57773.5777) \\ (3.53554.94975.65697.0711) & (-3.5777-3.5777-1.7889-1.7889) \end{bmatrix},$$
$$\widetilde{b'}^T = \begin{bmatrix} (4.46415.73967.652810.2037) & (0.63771.91323.82645.1018) \end{bmatrix}.$$

**Step 3.** The entries of the obtained fuzzy numbers system are converted to the following classical LSE by applying the ranking method mentioned in Eq. (4):

$$A'' = \begin{bmatrix} 5.3033 & 2.6833 \\ 5.3033 & -2.6833 \end{bmatrix},$$
$$b''^T = \begin{bmatrix} 7.0151 & 2.8698 \end{bmatrix}.$$

**Step 4.** Solve the above crisp LSE based on the ANN method to find the outputs in Table 1.

Table 1: Numerical results of Example 1, with $\varepsilon = 0.5 \times 10^{-6}$ using method of [32].

| $y_1$ | $y_2$ |
|---|---|
| 0.9320 | 0.7724 |

If in Step 2, each element of matrix $\widetilde{A^{\alpha}}$ and vector $\widetilde{b^{\alpha}}$ is converted to normal fuzzy numbers, by applying the proposed conversion method, we get

$$\widetilde{A'} = \begin{bmatrix} (3,7,8,12) & (2,2,4,4) \\ (3,7,8,12) & (-4,-4,-2,-2) \end{bmatrix},$$
$$\widetilde{b'}^{T} = \begin{bmatrix} (4.0824,9,12,21.8353) & (-1.9176,3,6,10.9176) \end{bmatrix}.$$

Then, after converting the coefficients of the above fuzzy linear system to the corresponding crisp system by using the ranking function mentioned in Eq. (5), we get

$$A'' = \begin{bmatrix} 7.5 & 3 \\ 7.5 & -3 \end{bmatrix},$$
$$b''^{T} = \begin{bmatrix} 11.7294 & 4.5 \end{bmatrix}.$$

By solving the crisp LSE $A''Y = b''$ based on the ANN method, the results of Table 2 are obtained, which shows that the proposed conversion method is more accurate than the other existing conversion method results in Table 1.

Table 2: Numerical results of Example 1, with $\varepsilon = 0.5 \times 10^{-6}$ using the proposed conversion method.

| $y_1$ | $y_2$ |
|---|---|
| 1.0820 | 1.2049 |

**Example 2.** Let

$$\widetilde{A_Z} = \begin{bmatrix} [(6,8,9,10),Sure] & [\widetilde{0},Certainly] & [(-2,-1,0,0),Sure] & [\widetilde{0},Certainly] \\ [(4,6,7,8),Likely] & [(-4,-2,-1,0),Likely] & [\widetilde{0},Certainly] & [(1,3,4,5),Usually] \\ [\widetilde{0},Certainly] & [(3,5,6,7),Usually] & [(1,2,3,5),Sure] & [\widetilde{0},Certainly] \\ [(0,1,2,4),Sure] & [\widetilde{0},Certainly] & [(-4,-2,-1,0),Usually] & [(1,2,3,5),Likely] \end{bmatrix},$$

$$\widetilde{b_Z}^{T} = \begin{bmatrix} [(7,9,10,11),Likely] & [(8,10,12,14),Sure] & [(3,5,6,7),Usually] & [(1,3,5,7),Likely] \end{bmatrix},$$

where, $\widetilde{0} = (0,0,0)$. Table 3 shows the classification of the reliability values into four different levels (certainty, sure, usually and likely) [9].

The steps of Algorithm 1 are determined as follows:

**Step 1.** Find the weighted fuzzy number entries:

$$\widetilde{A^{\alpha}} = \begin{bmatrix} (6,8,9,10;0.9333) & (\widetilde{0};1) & (-2,-1,0,0;0.9333) & (\widetilde{0};1) \\ (4,6,7,8;0.6) & (-4,-2,-1,0;0.6) & (\widetilde{0};1) & (1,3,4,5;0.75) \\ (\widetilde{0};1) & (3,5,6,7;0.75) & ((1,2,3,5;0.9333) & (\widetilde{0};1) \\ (0,1,2,40.9333) & (\widetilde{0};1) & ((-4,-2,-1,0;0.75) & (1,2,3,5;0.6) \end{bmatrix},$$
$$\widetilde{b^{\alpha}}^{T} = \begin{bmatrix} (7,9,10,11;0.6) & (8,10,12,14;0.9333) & (3,5,6,7;0.75) & (1,3,5,7;0.6) \end{bmatrix}$$

Table 3: Classifying the reliability values and related fuzzy numbers.

| Reliability levels | Corresponding fuzzy numbers |
|---|---|
| Certainly | (1, 1, 1), |
| Sure | (0.8, 1, 1), |
| Usually | (0.65, 0.75, 0.85), |
| Likely | (0.5, 0.6, 0.7). |

**Step 2.** Convert the above entries to the following regular fuzzy numbers based on the method in [32]:

$$\widetilde{A'} = \begin{bmatrix} (5.80,7.73,8.69,9.66) & \tilde{0} & (-1.93,-0.97,0,0) & \tilde{0} \\ (3.10,4.65,5.42,6.20) & (-3.10,-1.55,-0.77,0) & \tilde{0} & (0.87,2.60,3.46,4.33) \\ \tilde{0} & (2.60,4.33,5.20,6.06) & (0.97,1.93,2.90,4.83) & \tilde{0} \\ (0,0.87,1.73,3.46) & \tilde{0} & (-3.46,-1.73,-0.87,0) & (0.77,1.55,2.32,3.87) \end{bmatrix},$$

$$\widetilde{b'}^T = \begin{bmatrix} (5.42,6.97,7.75,8.52) & (7.73,9.66,11.59,13.53) & (2.60,4.33,5.20,6.06) & 0.772.323.875.42 \end{bmatrix}.$$

**Step 3.** Convert the above fuzzy components to achieve the following crisp system using the ranking function (Eq. (4)):

$$A'' = \begin{bmatrix} 7.9701 & 0 & -0.7246 & 0 \\ 4.8412 & -1.3556 & 0 & 2.8146 \\ 0 & 4.5467 & 2.6567 & 0 \\ 1.5156 & 0 & -1.5156 & 2.1302 \end{bmatrix},$$

$$b''^T = \begin{bmatrix} 7.1651 & 10.6268 & 4.5467 & 3.0984 \end{bmatrix}.$$

**Step 4.** Solve the yielded crisp LSE by the ANN technique, the final computed results are reported in Table 4.

Table 4: Approximate solution of Example 2, by applying conversion method in [32].

| $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|
| 1.0579 | -0.0215 | 1.7581 | 1.9456 |

If in Step 2, each element of matrix $\widetilde{A^\alpha}$ and vector $\widetilde{b^\alpha}$ is converted to the normal fuzzy numbers, by applying the proposed scheme, the following results are found:

$$\widetilde{A'} = \begin{bmatrix} (5.8571,8,9,10.0715) & \tilde{0} & (-2.0715,-1,0,0) & \tilde{0} \\ (2.6667,6,7,8.6667) & (-5.3333,3,-2,-1,0.6667) & \tilde{0} & (0.3333,3,4,5.3333) \\ \tilde{0} & (2.3333,5,6,7.3333) & (0.9285,2,3,5.1429) & \tilde{0} \\ (-0.0715,1,2,4.1429) & \tilde{0} & (-4.6667,-2,-1,0.333) & (0.3333,2,3,6.333) \end{bmatrix},$$

$$\widetilde{b'}^T = \begin{bmatrix} (5.6667,9,10,11.6667) & (7.8571,10,12,14.1429) & (2.3333,5,6,7.3333) & (-0.3333,3,5,8.3333) \end{bmatrix}.$$

Then, using ranking function (Eq. (5)), the following results are obtained:

$$A'' = \begin{bmatrix} 8.2322 & 0 & -0.7679 & 0 \\ 6.0834 & -0.9333 & 0 & 3.1666 \\ 0 & 5.1666 & 2.7679 & 0 \\ 1.7679 & 0 & -1.8333 & 2.9166 \end{bmatrix},$$

$$b''^T = \begin{bmatrix} 9.0833 & 11.0000 & 5.1666 & 4.0000 \end{bmatrix}.$$

At last, by solving the above crisp LSE based on the ANN scheme, the results in Table 5 are obtained.

Table 5: Approximate solution of Example 2, by applying the suggested conversion method.

| $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|
| 1.2005 | 0.4425 | 1.0407 | 1.2979 |

As we observe in Tables 4 and 5, since the proposed conversion method contains the whole support of each fuzzy number (first part) into the $Z$-number, the final results of the proposed scheme are more accurate than the method in [32].

## 6 Conclusion

In this work, solving a trapezoidal $Z$-number LSE based on the suggested algorithm was discussed. In this case, the given system was converted to a crisp LSE. Then, the ANN method was applied to get the final results, which are more accurate than the method in [32]. The suggested algorithm may be performed on other $Z$-numbers complicated systems.

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