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Numerical stability of discrete energy for a thermoelastic-Bresse system with second sound

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Abstract. Our contribution consists of studying numerical methods based on finite element space and finite difference schema in time of the linear one-dimensional thermoelastic Bresse system with second sound. We establish some a priori error estimates, and present some numerical analysis results of discrete energy under different decay rate profiles. Moreover, we study the behaviors of discrete energy with respect to the system parameters and the initial data. Some numerical simulations will be given in order to validate the theoretical results.

Keywords: Discrete energy, numerical approximation, finite element method, numerical stability, thermoelastic-Bresse system with second sound.
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1 Introduction

Generally, the Bresse system, also known as the curved beam model [6], is represented by the system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi+lw)_x - lk_0(w_x - l\varphi) = 0, & \text{in } (0,1)\times(0,\infty),\\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi+lw) = 0, & \text{in } (0,1)\times(0,\infty),\\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x+\psi+lw) = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{1}$$

where the functions φ,ψ, and w represent, respectively, the transverse displacement of a curved beam, the rotation angle of the filament, and the longitudinal displacement are represented. These physical quantities are influenced by the beam's material properties, represented by positive constants such as $k_0 = EH, k = GH, b = EI, l = 1/R$ and ρ_1,ρ_2,l,G,E,H,R. The model considers axial force, shear force, and

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bending moment indicated respectively, by the terms $k_0(w_x - l\varphi), k(\varphi_x + \psi + lw)$ and $b\psi_x$. Extensive research on the Bresse system focuses on its well-posedness and stability, examining the influence of feedback mechanisms and wave speed parameters $\alpha_1 = k/\rho_1$, $\alpha_2 = b/\rho_2$ and $\alpha_3 = k_0/\rho_1$, references to significant contributions in this field include a range of studies in [1–3, 8–11, 13, 14, 17].

Recently, the authors in [2, 3, 13] consider a linear Bresse system coupled with heat equation via Cattaneo's law known as thermoelastic-Bresse system with second sound as follow:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + \varepsilon_1\delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \varepsilon_2\delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + lw) + \varepsilon_3\delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta(\varepsilon_1\varphi_{xt} + \varepsilon_2\psi_{xt} + \varepsilon_3 w_{xt}) = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{2}$$

with the initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & \text{in } (0,1) \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & \text{in } (0,1) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & \text{in } (0,1) \\ \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x), & \text{in } (0,1) \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = q(0,t) = 0, & \text{in } (0,\infty) \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta(1,t) = 0, & \text{in } (0,\infty) \end{cases} \tag{3}$$

where $\varepsilon_i \in \{0,1\}$ and $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$ with ρ_3, τ, β and δ are positive constants, θ is the temperature deviations and q is the heat fluxe.

The authors in [13] analyzed the system represented by (2), with setting $\varepsilon_2 = 1$ and $\varepsilon_1 = \varepsilon_3 = 0$. They demonstrated that this case leads to exponential stability under the following conditions:

$$\alpha_1 = \alpha_3, \quad \left(1 - \frac{\rho_2 k}{\rho_1 b}\right)\left(\frac{\rho_1}{\tau k \rho_3} - 1\right) - \frac{\delta^2}{b\rho_2} = 0 \quad \text{and} \quad l \text{ is small},$$

and the system, as discussed in [13], is not exponentially stable if either of the following conditions is met

$$\alpha_1 \neq \alpha_3 \quad \text{or} \quad \left(1 - \frac{\rho_2 k}{\rho_1 b}\right)\left(\frac{\rho_1}{\tau k \rho_3} - 1\right) - \frac{\delta^2}{b\rho_2} \neq 0,$$

demonstrating that the polynomial stability of the system's responses exhibit a decay rate proportional to $t^{-\frac{1}{2}}$. Recently, in [2], the authors considered system (1) with $\varepsilon_3 = 1, \varepsilon_1 = \varepsilon_2 = 0$ under the restriction $l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}$. The authors showed that the solution lacks exponential stability if either condition

$$\left(1 - \frac{k}{k_0}\right)\left(\frac{\rho_3 \tau k}{\rho_1} - 1\right) - \frac{\delta^2 \tau k}{k_0 \rho_1} = 1 - \frac{k\rho_2}{b\rho_1} = 0, \tag{4}$$

or

$$l^2 \neq \left(1 + \frac{b\rho_1}{k_0\rho_2}\right)\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k}{\rho_2(k + k_0)}, \quad \forall m \in \mathbb{Z}. \tag{5}$$

is not satisfied.

Additionally, the authors established that the solution achieves exponential stability when both (4) and (5) are met. Furthermore, for the system represented by (2) with sitting $\varepsilon_3 = 1$, $\varepsilon_1 = \varepsilon_2 = 0$, the study in [2] demonstrated polynomial stability with a decay rate of $t^{-\frac{1}{8}}$, provided that (5) is satisfied while (4) is not.

In [3], the authors focused on the problem defined by (2) – (3), particularly when $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = 0$. Utilizing semi-group theory, they investigated the existence, uniqueness, and smoothness of the solution. Additionally, they provided and substantiated a result concerning the lack of exponential stability of the system's solutions, which is contingent upon the following specific constants defined as:

$$\begin{cases} \mu_0 = 1 - \frac{k_0\rho_2}{b\rho_1}, \\ \mu_1 = \frac{\tau\delta^2}{\rho_1} - \left(1 - \frac{k\rho_2}{b\rho_1}\right)\left(\frac{\tau b\rho_3}{\rho_2} - 1\right), \\ \mu_2 = \frac{\tau\delta^2}{\rho_1} - \left(1 - \frac{k}{k_0}\right)\left(1 - \frac{\tau\rho_3 k_0}{\rho_1}\right), \end{cases}$$

and further restrictions on l and μ_0, μ_1, μ_2 that follows:

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}. \tag{6}$$

$$l^2 \neq \left(1 - \frac{b\rho_1}{k_0\rho_2}\right)\left(\frac{\pi}{2} + m\pi\right)^2 - \frac{k\rho_1}{\rho_2(k+k_0)}, \quad \forall m \in \mathbb{Z}. \tag{7}$$

$$\mu_0 \neq 0 \quad \text{and} \quad \mu_1 = \mu_2 = 0. \tag{8}$$

The Bresse system achieves indirect stabilization through its second equation when $\varepsilon_2 = 1$ and $\varepsilon_1 = \varepsilon_3 = 0$, or through its third equation when $\varepsilon_3 = 1$ and $\varepsilon_1 = \varepsilon_2 = 0$. Conversely, the system's first hyperbolic equation, defined by $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = 0$, experiences indirect damping due to its interaction with the latter two equations, which collectively describe the heat conduction process within Cattaneo's law. At present, there are some theoretical and numerical results on the asymptotic behavior of thermoelastic problems [15, 16].

In this paper, we consider the following model

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l w)_x - lk_0(w_x - l\varphi) + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l w) = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + l w) = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\varphi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{9}$$

with the initial and boundary conditions

$$\begin{cases} cc\varphi(x,0) = \varphi_0(x),\, \varphi_t(x,0) = \varphi_1(x),\, \theta(x,0) = \theta_0(x), & \text{for } x \in (0,1), \\ \psi(x,0) = \psi_0(x),\, \psi_t(x,0) = \psi_1(x),\, q(x,0) = q_0(x), & \text{for } x \in (0,1), \\ w(x,0) = w_0(x),\, w_t(x,0) = w_1(x), & \text{for } x \in (0,1), \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = q(0,t) = 0, & \forall t \geq 0, \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta(1,t) = 0, & \forall t \geq 0. \end{cases} \tag{10}$$

where ρ_1, ρ_2, ρ_3, b, k, k_0, τ, β, δ and l are positive constants, the initial data φ_0, φ_1, ψ_0, ψ_1, w_0, w_1, θ_0 and q_0 belong to a suitable Hilbert space, and the unknowns of (9)-(10) are the following variables:

$$(\varphi,\psi,w,\theta,q):(0,1)\times(0,\infty)\to\mathbb{R}^5. \tag{11}$$

In [3], the authors stated the well-posedness and stability of the system (9)-(10) and proved the nonexponential and exponential decay under new conditions on the parameters of the system. The present paper is mainly concerned with the solution of (9)-(10). Our paper is organized as follows. In Section 2, we establish the stability property of the discrete energy and get some a priori error estimates. In Section 3, we validate the theoretical analysis with a numerical simulator and deduce the energy decay of the discrete energy. Finally, a conclusion will be given.

2 Discrete energy behavior

In this section, we introduce a fully discrete approximation of system (9) with initial and boundary conditions (10) by using the finite elements for the spatial approximation and the implicit Euler scheme to discretize the time derivatives. Then we establish the stability property, from which we deduce the energy decay of the discrete energy.

2.1 Variational formulation

We introduce the energy functional $\mathscr{E}(t)$ associated to $(9)-(10)$ by

$$\mathscr{E}(t)=\frac{1}{2}\int_0^1\left[k\left(\varphi_x+\psi+lw\right)^2+k_0\left(w_x-l\varphi\right)^2+b\left(\psi_x\right)^2\;+\rho_1u^2+\rho_2v^2+\rho_1z^2+\rho_3\theta^2+\tau q^2\right]dx, \tag{12}$$

where $u=\varphi_t, v=\psi_t$ and $z=w_t$.

The following energy decay property was proved in [3].

Theorem 1. *We assume that* (6)*,* (7) *and* (8) *hold. Then the energy $\mathscr{E}(t)$ decays exponentially, ie, there exist positive constant a_1 et a_2 such that the energy function satisfies $\mathscr{E}(t)\le a_1\mathscr{E}(0)e^{-a_2t}$, $\forall t\ge 0$.*

To obtain the variational formulation of Problem (9), we consider the following spaces

$$H^1_*=\{f\in H^1(0,1): f(0)=0\},\qquad \tilde{H}^1_*=\{f\in H^1(0,1): f(1)=0\},$$

and the following functions $\widetilde{\varphi}=\varphi_t$ $\widetilde{\psi}=\psi_t$ $\widetilde{w}=w_t$ and denote by $(\cdot)$ the scalar product in the space $L^2(0,1)$, with corresponding norm $\|\cdot\|$. We rewrite system (9) as follows

$$\begin{cases}\rho_1\widetilde{\varphi}_t-k(\varphi_x+\psi+lw)_x-lk_0(w_x-l\varphi)+\delta\theta_x=0,\\ \rho_2\widetilde{\psi}_t-b\psi_{xx}+k(\varphi_x+\psi+lw)=0,\\ \rho_1\widetilde{w}_t-k_0(w_x-l\varphi)_x+lk(\varphi_x+\varphi+lw)=0,\\ \rho_3\theta_t+q_x+\delta\widetilde{\varphi}_x=0,\\ \tau q_t+\beta q+\theta_x=0.\end{cases} \tag{13}$$

To get the weak form associated to system (13), we multiply the equations by test functions $\zeta,\ \alpha\in H^1_*(0,1)$ *and* $\chi,\ \xi,\ \eta\in\tilde{H}^1_*(0,1)$ and integrating by parts that follows:

$$\begin{cases} \rho_1(\widetilde{\varphi}_t,\zeta)+k(\varphi_x+\psi+lw,\zeta_x)-k_0 l(w_x-l\varphi,\zeta)-\delta(\theta_x,\chi_x)=0,\\ \rho_2(\widetilde{\psi}_t,\ \chi)+b(\psi_x,\ \chi_x)+k(\varphi_x+\psi+lw,\chi)=0,\\ \rho_1(\widetilde{w}_t,\xi)+k_0(w_x-l\varphi,\xi_x)+lk(\varphi_x+\psi+lw,\xi)=0,\\ \rho_3(\theta_t,\eta)+(q_x,\eta)+\delta(\widetilde{\varphi}_x,\eta)=0,\\ \tau(q_t,\alpha)+\beta(q,\alpha)+(\theta_x,\alpha)=0. \end{cases} \tag{14}$$

For our purposes, we considered J a nonnegative integer and $h=\frac{1}{J}$ a subdivision of the interval $[0,1]$ given by $0=x_0<x_1<\cdots<x_{J-1}<x_J=1$, such that $x_j=jh, \forall j=0,\dots,J$ and

$$S_0^h=\left\{u\in H_*^1(0,1)\,\middle|\, u\in C([0,1]),\ u\Big|_{(x_j,x_{j+1})} \text{ is a linear polynomial}\right\}, \tag{15}$$

$$S_1^h=\left\{u\in \tilde{H}_*^1(0,1)\,\middle|\, u\in C([0,1]),\ u\Big|_{(x_j,x_{j+1})} \text{ is a linear polynomial}\right\}. \tag{16}$$

For a given final time T and a positive integer N, let $\Delta t=\frac{T}{N}$ be the time step and $t_n=n\triangle t,\ n=0,\dots,N$.

The finite element method for (14) with Dirichlet homogeneous boundary conditions using the backward Euler scheme is to find $\widetilde{\varphi}_h^n, q_h^n\in S_0^h$ and $\widetilde{\psi}_h^n,\ \widetilde{w}_h^n$ and $\theta_h^n\in S_1^h\subset H^1(0,1)$ such that, for $n=1,\dots,N$ and for all $\zeta_h,\ \eta_h\in S_0^h,\ \chi_h,\ \xi_h,\ \alpha_h\in S_1^h$:

$$\begin{cases} \frac{\rho_1}{\triangle t}\left(\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1},\zeta_h\right)+k\left(\varphi_{hx}^n+\psi_h^n+lw_h^n,\zeta_{hx}\right)-lk_0\left(w_{hx}^n-l\varphi_h^n,\zeta_h\right)-\delta\left(\theta_h^n,\chi_{hx}\right)=0,\\ \frac{\rho_2}{\triangle t}\left(\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1},\chi_h\right)+b\left(\psi_{hx}^n,\chi_{hx}\right)+k\left(\varphi_{hx}^n+\psi_h^n+lw_h^n,\chi_h\right)=0,\\ \frac{\rho_1}{\triangle t}\left(\widetilde{w}_h^n-\widetilde{w}_h^{n-1},\ \xi_h\right)+k_0\left(w_{hx}^n-l\varphi_h^n,\xi_{hx}\right)+lk\left(\varphi_{hx}^n+\psi_h^n+lw_h^n,\ \xi_h\right)=0,\\ \frac{\rho_3}{\triangle t}\left(\theta_h^n-\theta_h^{n-1},\eta_h\right)+\left(q_{hx}^n,\eta_h\right)+\delta\left(\widetilde{\varphi}_x^n,\eta_h\right)=0,\\ \frac{\tau}{\triangle t}\left(q_h^n-q_h^{n-1},\alpha_h\right)+\beta\left(q_h^n,\alpha_h\right)+\left(\theta_{hx}^n,\alpha_h\right)=0, \end{cases} \tag{17}$$

where

$$\widetilde{\varphi}_h^n=\frac{\varphi_h^n-\varphi_h^{n-1}}{\triangle t},\ \ \widetilde{\psi}_h^n=\frac{\psi_h^n-\psi_h^{n-1}}{\triangle t},\ \ \widetilde{w}_h^n=\frac{w_h^n-w_h^{n-1}}{\triangle t}, \tag{18}$$

are approximations to $\varphi_h(t_n),\ \psi_h(t_n)$ and $w_h(t_n)$, respectively. Here, $\varphi_h^0, \widetilde{\varphi}_h^0,\ \psi_h^0,, \widetilde{\psi}_h^0,\ w_h^0,\ \widetilde{w}_h^0,\ \theta_h^0$ and q_h^0 are given approximations to the initial conditions $\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1, w_0, w_1,\ \theta_0, q_0$ respectively.

The following inequality will often be used

$$(a-b,a)=\frac{1}{2}\left(\|a-b\|^2+\|a\|^2-\|b\|^2\right). \tag{19}$$

The next result is a discrete version of the energy decay property satisfied by the solution of system (9).

Theorem 2. *Let the following discrete energy:*

$$\begin{aligned} \mathscr{E}_h^n=&\frac{1}{2}\left(\rho_1\left(\|\widetilde{\varphi}_h^n\|^2+\|\widetilde{w}_h^n\|^2\right)+\rho_2\|\widetilde{\psi}_h^n\|^2+k\|\varphi_{hx}^n+\psi_h^n+lw_h^n\|^2\right.\\ &\left.+b\|\psi_{hx}^n\|^2+k_0\|w_{hx}^n-l\varphi_h^n\|^2+\rho_3\|\theta_h^n\|^2+\tau\|q_h^n\|^2\right). \end{aligned} \tag{20}$$

Then, the decay property

$$\frac{\mathscr{E}_h^n - \mathscr{E}_h^{n-1}}{\triangle t} \leq 0, \tag{21}$$

holds for n=1,2,...,N, where $\|\cdot\|$ *represents the norm in the space* $L^2(0,1)$.

Proof. Let $\zeta_h = \widetilde{\varphi}_h^n$, $\chi_h = \widetilde{\psi}_h^n$, $\xi_h = \widetilde{w}_h^n$, $\eta_h = \theta_h^n$ and $\alpha_h = q_h^n$ in (17). By (18) and (19), we deduce that

$$\begin{aligned}\tfrac{\rho_1}{2\triangle t}\left(\|\widetilde{\varphi}_h^n - \widetilde{\varphi}_h^{n-1}\|^2 + \|\widetilde{\varphi}_h^n\|^2 - \|\widetilde{\varphi}_h^{n-1}\|^2\right) + k\left(\varphi_{hx}^n + \psi_h^n + lw_h^n, \widetilde{\varphi}_{hx}^n\right)\\ -lk_0\left(w_{hx}^n - l\varphi_h^n, \widetilde{\varphi}_h^n\right) + \delta\left(\theta_{hx}^n, \widetilde{\varphi}_h^n\right) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}\tfrac{\rho_2}{2\triangle t}\left(\|\widetilde{\psi}_h^n - \widetilde{\psi}_h^{n-1}\|^2 + \|\widetilde{\psi}_h^n\|^2 - \|\widetilde{\psi}_h^{n-1}\|^2\right) + \tfrac{b}{2\triangle t}\left(\|\psi_{hx}^n - \psi_{hx}^{n-1}\|^2 + \|\psi_{hx}^n\|^2\right.\\ \left. - \|\psi_{hx}^{n-1}\|^2\right) + k\left(\varphi_{hx}^n + \psi_h^n + lw_h^n, \widetilde{\psi}_h^n\right) = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}\tfrac{\rho_1}{2\triangle t}\left(\|\widetilde{w}_h^n - \widetilde{w}_h^{n-1}\|^2 + \|\widetilde{w}_h^n\|^2 - \|\widetilde{w}_h^{n-1}\|^2\right) + k_0\left(w_{hx}^n - l\varphi_h^n, \widetilde{w}_{hx}^n\right)\\ +lk\left(\varphi_{hx}^n + \psi_h^n + lw_h^n,\ \widetilde{w}_h^n\right) = 0,\end{aligned} \tag{24}$$

$$\frac{\rho_3}{2\triangle t}\left(\|\theta_h^n - \theta_h^{n-1}\|^2 + \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right) - (q_h^n, \theta_{hx}^n) - \delta\left(\widetilde{\psi}_h^n, \theta_{hx}^n\right) = 0, \tag{25}$$

$$\frac{\tau}{2\triangle t}\left(\|q_h^n - q_h^{n-1}\|^2 + \|q_h^n\|^2 - \|q_h^{n-1}\|^2\right) + \beta\|q_h^n\|^2 + (\theta_{hx}^n, q_h^n) = 0. \tag{26}$$

Using again (18) and (19), we obtain

$$\begin{aligned}(u^n, \widetilde{u}^n) = \left(u^n, \frac{u^n - u^{n-1}}{\Delta t}\right) &= \frac{1}{2\Delta t}\left(\|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2\right)\\ &\geq \frac{1}{2\Delta t}\left(\|u^n\|^2 - \|u^{n-1}\|^2\right),\end{aligned} \tag{27}$$

it results that

$$k\left(\varphi_{hx}^n + \psi_h^n + lw_h^n, \widetilde{\varphi}_{hx}^n + \widetilde{\psi}_h^n + l\widetilde{w}_h^n\right) \geq \frac{k}{2\triangle t}\left(\|\varphi_{hx}^n + \psi_h^n + lw_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1} + lw_h^{n-1}\|^2\right), \tag{28}$$

and

$$k_0\left(w_{hx}^n - l\varphi_h^n, \widetilde{w}_h^n - l\widetilde{\varphi}_h^n\right) \geq \frac{k_0}{2\triangle t}\left(\|w_{hx}^n - l\varphi_h^n\|^2 - \|w_{hx}^{n-1} - l\varphi_h^{n-1}\|^2\right), \tag{29}$$

by summing equations (22)-(26), we have

$$\begin{aligned}0 \geq\ & \tfrac{\rho_1}{2\triangle t}\left(\|\widetilde{\varphi}_h^n\|^2 - \|\widetilde{\varphi}_h^{n-1}\|^2\right) + \tfrac{\rho_2}{2\triangle t}\left(\|\widetilde{\psi}_h^n\|^2 - \|\widetilde{\psi}_h^{n-1}\|^2\right)\\ &+ \tfrac{\rho_1}{2\triangle t}\left(\|\widetilde{w}_h^n\|^2 - \|\widetilde{w}_h^{n-1}\|^2\right) + \tfrac{\rho_3}{2\triangle t}\left(\|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right) + \tfrac{\tau}{2\triangle t}\left(\|q_h^n\|^2 - \|q_h^{n-1}\|^2\right)\\ &+ \tfrac{b}{2\triangle t}\left(\|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right) + \tfrac{\rho_1}{2\triangle t}\|\widetilde{\varphi}_h^n - \widetilde{\varphi}_h^{n-1}\|^2 + \tfrac{\rho_2}{2\triangle t}\|\widetilde{\psi}_h^n - \widetilde{\psi}_h^{n-1}\|^2\\ &+ \tfrac{\rho_1}{2\triangle t}\|\widetilde{w}_h^n - \widetilde{w}_h^{n-1}\|^2 + \tfrac{\tau}{2\triangle t}\|q_h^n - q_h^{n-1}\|^2 + \tfrac{b}{2\triangle t}\|\psi_{hx}^n - \psi_{hx}^{n-1}\|^2 + \tfrac{\rho_3}{2\triangle t}\|\theta_h^n - \theta_h^{n-1}\|^2\\ &+ \beta\|q_h^n\|^2 + k\left(\varphi_{hx}^n + \psi_h^n + lw_h^n, \widetilde{\varphi}_{hx}^n + \widetilde{\psi}_h^n + l\widetilde{w}_h^n\right) + lk_0\left(w_{hx}^n - l\varphi_h^n, \widetilde{w}_h^n - l\widetilde{\varphi}_h^n\right) \geq\\ & \tfrac{\rho_1}{2\triangle t}\left(\|\widetilde{\varphi}_h^n\|^2 - \|\widetilde{\varphi}_h^{n-1}\|^2\right) + \tfrac{\rho_2}{2\triangle t}\left(\|\widetilde{\psi}_h^n\|^2 - \|\widetilde{\psi}_h^{n-1}\|^2\right) + \tfrac{b}{2\triangle t}\left(\|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right)\\ &+ \tfrac{\rho_1}{2\triangle t}\left(\|\widetilde{w}_h^n\|^2 - \|\widetilde{w}_h^{n-1}\|^2\right) + \tfrac{\rho_3}{2\triangle t}\left(\|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right) + \tfrac{\tau}{2\triangle t}\left(\|q_h^n\|^2 - \|q_h^{n-1}\|^2\right)\\ &+ \tfrac{k}{2\triangle t}\left(\|\varphi_{hx}^n + \psi_h^n + lw_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1} + lw_h^{n-1}\|^2\right)\\ &+ \tfrac{k_0}{2\triangle t}\left(\|w_{hx}^n - l\varphi_h^n\|^2 - \|w_{hx}^{n-1} - l\varphi_h^{n-1}\|^2\right) = \tfrac{\mathscr{E}_h^n - \mathscr{E}_h^{n-1}}{\triangle t},\end{aligned} \tag{30}$$

which implies that $\frac{\mathscr{E}_h^n-\mathscr{E}_h^{n-1}}{\triangle t}\leq 0$, and the theorem is proved using the definition of the discrete energy. □

As a consequence, the following stability estimates are derived.

Remark 1. *Note that, to find the solution of (17) a square linear system of algebraic equations needs to be solved. It follows from the above proof that when all data are zero, the solution* $\left\{\widetilde{\varphi}_h^n,\widetilde{\psi}_h^n,\widetilde{w}_h^n,\theta_h^n,q_h^n\right\}$ *is zero. Then, (17) admits a unique solution.*

2.2 Error Estimate

Now, we will give some estimates for the difference between the exact solution and the numerical solution.

Theorem 3. *There exists a positive constant C, independent of the discretization parameters h and* $\triangle t$ *such that for all* $\{\zeta_h,\ \alpha_h\}_{i=0}^N\in S_0^h$, *and* $\{\chi_h,\ \xi_h,\ \eta_h\}_{i=0}^N\in S_0^h$

$$\begin{aligned}\max_{0\leq n\leq N}\ &\{\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2+\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\|^2+\|\widetilde{w}^n-\widetilde{w}_h^n\|^2+\|\widetilde{\psi}_x^n-\widetilde{\psi}_{hx}^n\|^2\\&+\|\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n)\|^2+\|w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n)\|^2+\|\theta^n-\theta_h^n\|^2\\&+\|q^n-q_h^n\|^2\}\leq C\triangle t\sum_{i=1}^N\left\|\widetilde{\varphi}_t^i-\tfrac{\widetilde{\varphi}^i-\widetilde{\varphi}^{i-1}}{\triangle t}\right\|^2+\left\|\widetilde{\psi}_t^i-\tfrac{\widetilde{\psi}^i-\widetilde{\psi}^{i-1}}{\triangle t}\right\|^2+\left\|\widetilde{w}_t^i-\tfrac{\widetilde{w}^i-\widetilde{w}^{i-1}}{\triangle t}\right\|^2\\&+\left\|\theta_t^i-\tfrac{\theta^i-\theta^{i-1}}{\triangle t}\right\|^2+\left\|q_t^i-\tfrac{q^i-q^{i-1}}{\triangle t}\right\|^2+\|\widetilde{\varphi}^i-\zeta_h^i\|^2+\|\widetilde{\varphi}_x^i-\zeta_{hx}^i\|^2+\|\widetilde{\psi}^i-\chi_h^i\|^2\\&+\|\widetilde{\psi}_x^i-\chi_{hx}^i\|^2+\|\widetilde{w}^i-\xi_h^i\|^2+\|\widetilde{w}_x^i-\xi_{hx}^i\|^2+\|\theta_x^i-\eta_{hx}^i\|^2+\|q_x^i-\alpha_{hx}^i\|^2+\|\theta^i-\eta_h^i\|^2\\&+\|q^i-\alpha_h^i\|^2+\tfrac{C}{\triangle t}\sum_{i=1}^{N-1}\left(\|\widetilde{\varphi}^i-\zeta_h^i-(\widetilde{\varphi}^{i+1}-\zeta_h^{i+1})\|^2+\|\widetilde{\psi}^i-\chi_h^i-(\widetilde{\psi}^{i+1}-\chi_h^{i+1})\|^2\right.\\&\left.+\|\widetilde{w}^i-\xi_h^i-(\widetilde{w}^{i+1}-\xi_h^{i+1})\|^2+\|\theta^i-\eta_h^i-(\theta^{i+1}-\eta_h^{i+1})\|^2+\|q^i-\alpha_h^i-(q^{i+1}-\alpha_h^{i+1})\|^2\right)\\&+C\left(\|\varphi^1-\widetilde{\varphi}_h^0\|^2+\|\psi^1-\widetilde{\psi}_h^0\|^2+\|\psi_x^0-\psi_{hx}^0\|^2+\|w^1-\widetilde{w}_h^0\|^2+\|\theta^0-\theta_h^0\|^2+\|q^0-q_h^0\|^2\right.\\&\left.+\|\varphi_{hx}^0+\psi_h^0+lw_h^0-(\varphi_x^0+\psi^0+lw^0)\|^2+\|w_x^0-l\varphi^0-(w_{hx}^0-l\varphi_h^0)\|^2\right).\end{aligned}\tag{31}$$

Proof. **Step 1:** For a continuous function $g(t)$, let $g^n=g(t_n)$. Subtracting equation $(14)_1$ at time t_n for $\zeta=\zeta_h\in S_0^h$ and the discrete variational equation $(17)_1$, we obtain

$$\begin{aligned}&\rho_1\left(\widetilde{\varphi}_t^n-\frac{\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1}}{\triangle t},\zeta_h\right)+k\left((\varphi_x^n+\psi^n+lw^n)-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\zeta_{hx}\right)\\&-lk_0\left(w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n),\zeta_h\right)-\delta\left(\theta^n-\theta_h^n,\zeta_{hx}\right)=0.\end{aligned}\tag{32}$$

Thus, for all $\zeta\in S_0^h$, we obtain

$$\begin{aligned}&\rho_1\left(\widetilde{\varphi}_t^n-\frac{\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1}}{\triangle t},\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+k\left(\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n\right)\\&-lk_0\left(w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n),\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)-\delta\left(\theta^n-\theta_h^n,\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n\right)\\&=\rho_1\left(\widetilde{\varphi}_t^n-\frac{\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1}}{\triangle t},\widetilde{\varphi}^n-\zeta_h\right)+k\left(\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\widetilde{\varphi}_x^n-\zeta_{hx}\right)\\&-lk_0\left(w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n),\widetilde{\varphi}^n-\zeta_h\right)-\delta\left(\theta^n-\theta_h^n,\widetilde{\varphi}_x^n-\widetilde{\zeta}_{hx}\right).\end{aligned}\tag{33}$$

Similarly, from equations (14)$_2$−(14)$_5$ and (17)$_2$-(17)$_5$ we deduce, for all χ_h, ξ_h and $\eta_h \in S_1^h,\ \alpha_h \in S_0^h$:

$$\begin{aligned}
&\rho_2\left(\widetilde{\psi}_t^n-\frac{\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1}}{\triangle t},\widetilde{\psi}^n-\widetilde{\psi}_h^n\right)+b\left(\psi_x^n-\psi_{hx}^n,\ \widetilde{\psi}_x^n-\widetilde{\psi}_{hx}^n\right)\\
&\quad+k\left(\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\widetilde{\psi}^n-\widetilde{\psi}_h^n\right)\\
&=\rho_2\left(\widetilde{\psi}_t^n-\frac{\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1}}{\triangle t},\widetilde{\psi}^n-\chi_h\right)+b\left(\psi_x^n-\psi_{hx}^n,\ \widetilde{\psi}_x^n-\chi_{hx}\right)\\
&\quad+k\left(\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\widetilde{\psi}^n-\chi_h\right),
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\rho_1\left(\widetilde{w}_t^n-\frac{\widetilde{w}_h^n-\widetilde{w}_h^{n-1}}{\triangle t},\widetilde{w}^n-\widetilde{w}_h^n\right)+k_0\left(w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n),\widetilde{w}_x^n-\widetilde{w}_{hx}^n\right)\\
&\quad+lk\left(\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\widetilde{w}^n-\widetilde{w}_h^n\right)\\
&=\rho_1\left(\widetilde{w}_t^n-\frac{\widetilde{w}_h^n-\widetilde{w}_h^{n-1}}{\triangle t},\widetilde{w}^n-\xi_h\right)+k_0\left(w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n),\widetilde{w}_x^n-\xi_{hx}\right)\\
&\quad+lk\left(\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n),\widetilde{w}^n-\xi_h\right),
\end{aligned}\tag{35}$$

$$\begin{aligned}
&\rho_3\left(\theta_t^n-\frac{\theta_h^n-\theta_h^{n-1}}{\triangle t},\theta^n-\theta_h^n\right)+\left(q_x^n-q_{hx}^n,\theta^n-\theta_h^n\right)+\delta\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx},\theta^n-\theta_h^n\right)\\
&=\rho_3\left(\theta_t^n-\frac{\theta_h^n-\theta_h^{n-1}}{\triangle t},\theta^n-\eta_h\right)+\left(q_x^n-q_{hx}^n,\theta^n-\eta_h\right)+\delta\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx},\theta^n-\eta_h\right),
\end{aligned}\tag{36}$$

$$\begin{aligned}
&\tau\left(q_t^n-\tfrac{q_h^n-q_h^{n-1}}{\triangle t},q^n-q_h^n\right)+\beta\left(q^n-q_h^n,q^n-q_h^n\right)+\left(\theta_x^n-\theta_{hx}^n,q^n-q_h^n\right)\\
&=\tau\left(q_t^n-\tfrac{q_h^n-q_h^{n-1}}{\triangle t},q^n-\alpha_h\right)+\beta\left(q^n-q_h^n,q^n-\alpha_h\right)+\left(\theta_x^n-\theta_{hx}^n,q^n-\alpha_h\right).
\end{aligned}\tag{37}$$

Step 2 : Using (19), the first term in equation (33) become

$$\begin{aligned}
\left(\widetilde{\varphi}_t^n-\tfrac{\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1}}{\triangle t},\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right) &= \left(\widetilde{\varphi}_t^n-\tfrac{\widetilde{\varphi}^n-\widetilde{\varphi}^{n-1}}{\triangle t},\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+\tfrac{1}{\triangle t}\left(\widetilde{\varphi}^n-\widetilde{\varphi}^{n-1}-\left(\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1}\right),\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)\\
&= \left(\widetilde{\varphi}_t^n-\tfrac{\widetilde{\varphi}^n-\widetilde{\varphi}^{n-1}}{\triangle t},\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+\tfrac{1}{2\triangle t}\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n-\left(\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right)\right\|^2\\
&\quad+\tfrac{1}{2\triangle t}\left(\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right\|^2-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right).
\end{aligned}\tag{38}$$

Then

$$\left(\widetilde{\varphi}_t^n-\tfrac{\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1}}{\triangle t},\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)\geq \left(\widetilde{\varphi}_t^n-\tfrac{\widetilde{\varphi}^n-\widetilde{\varphi}^{n-1}}{\triangle t},\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+\tfrac{1}{2\triangle t}\left(\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right\|^2-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right).$$

In the same way, for (34)-(37) we find

$$\begin{aligned}
&\left(\widetilde{\psi}_t^n - \tfrac{\widetilde{\psi}_h^n - \widetilde{\psi}_h^{n-1}}{\triangle t}, \widetilde{\psi}^n - \widetilde{\psi}_h^n\right) \geq \left(\widetilde{\psi}_t^n - \tfrac{\widetilde{\psi}^n - \widetilde{\psi}^{n-1}}{\triangle t}, \widetilde{\psi}^n - \widetilde{\psi}_h^n\right) + \tfrac{1}{2\triangle t}\left(\left\|\widetilde{\psi}^n - \widetilde{\psi}_h^n\right\|^2 - \left\|\widetilde{\psi}^{n-1} - \widetilde{\psi}_h^{n-1}\right\|^2\right),\\
&\left(\widetilde{w}_t^n - \tfrac{\widetilde{w}_h^n - \widetilde{w}_h^{n-1}}{\triangle t}, \widetilde{w}^n - \widetilde{w}_h^n\right) \geq \left(\widetilde{w}_t^n - \tfrac{\widetilde{w}^n - \widetilde{w}^{n-1}}{\triangle t}, \widetilde{w}^n - \widetilde{w}_h^n\right) + \tfrac{1}{2\triangle t}\left(\left\|\widetilde{w}^n - \widetilde{w}_h^n\right\|^2 - \left\|\widetilde{w}^{n-1} - \widetilde{w}_h^{n-1}\right\|^2\right),\\
&\left(\theta_t^n - \tfrac{\theta_h^n - \theta_h^{n-1}}{\triangle t}, \theta^n - \theta_h^n\right) \geq \left(\theta_t^n - \tfrac{\theta^n - \theta^{n-1}}{\triangle t}, \theta^n - \theta_h^n\right) + \tfrac{1}{2\triangle t}\left(\left\|\theta^n - \theta_h^n\right\|^2 - \left\|\theta^{n-1} - \theta_h^{n-1}\right\|^2\right),\\
&\left(q_t^n - \tfrac{q_h^n - q_h^{n-1}}{\triangle t}, q^n - q_h^n\right) \geq \left(q_t^n - \tfrac{q^n - q^{n-1}}{\triangle t}, q^n - q_h^n\right) + \tfrac{1}{2\triangle t}\left(\left\|q^n - q_h^n\right\|^2 - \left\|q^{n-1} - q_h^{n-1}\right\|^2\right).
\end{aligned}$$

Using again (27) for

$$u^n = \psi_x^n - \psi_{hx}^n,\ \varphi_x^n + \psi^n + lw^n - (\varphi_{hx}^n + \psi_h^n + lw_h^n) \text{ and } w_x^n - l\varphi^n - (w_{hx}^n - l\varphi_h^n),$$

and adding (33)-(37) we obtain

$$\begin{aligned}
\tfrac{\rho_1}{2\triangle t} \quad &\left(\left\|\widetilde{\varphi}^n - \widetilde{\varphi}_h^n\right\|^2 - \left\|\widetilde{\varphi}^{n-1} - \widetilde{\varphi}_h^{n-1}\right\|^2 + \left\|\widetilde{w}^n - \widetilde{w}_h^n\right\|^2 - \left\|\widetilde{w}^{n-1} - \widetilde{w}_h^{n-1}\right\|^2\right)\\
&+ \tfrac{\rho_2}{2\triangle t}\left(\left\|\widetilde{\psi}^n - \widetilde{\psi}_h^n\right\|^2 - \left\|\widetilde{\psi}^{n-1} - \widetilde{\psi}_h^{n-1}\right\|^2\right) + \tfrac{b}{2\triangle t}\left(\left\|\psi_x^n - \psi_{hx}^n\right\|^2 - \left\|\psi_x^{n-1} - \psi_{hx}^{n-1}\right\|^2\right)\\
&+ \tfrac{k}{2\triangle t}\left(\left\|\varphi_x^n + \psi^n + lw^n - \left(\varphi_{hx}^n + \psi_h^n + lw_h^n\right)\right\|^2\right.\\
&\left. - \left\|\varphi_x^{n-1} + \psi^{n-1} + lw^{n-1} - \left(\varphi_{hx}^{n-1} + \psi_h^{n-1} + lw_h^{n-1}\right)\right\|^2\right)\\
&+ \tfrac{k_0}{2\triangle t}\left(\left\|w_x^n - l\varphi^n - \left(w_{hx}^n - l\varphi_h^n\right)\right\|^2 - \left\|w_x^{n-1} - l\varphi^{n-1} - \left(w_{hx}^{n-1} - l\varphi_h^{n-1}\right)\right\|^2\right)\\
&+ \tfrac{\rho_3}{2\triangle t}\left(\left\|\theta^n - \theta_h^n\right\|^2 - \left\|\theta^{n-1} - \theta_h^{n-1}\right\|^2\right) + \tfrac{\tau}{2\triangle t}\left(\left\|q^n - q_h^n\right\|^2 - \left\|q^{n-1} - q_h^{n-1}\right\|^2\right)\\
\leq C &\left(\left\|\widetilde{\varphi}_t^n - \tfrac{\widetilde{\varphi}^n - \widetilde{\varphi}^{n-1}}{\triangle t}\right\|^2 + \tfrac{1}{\triangle t}\left(\widetilde{\varphi}^n - \widetilde{\varphi}^{n-1} - \left(\widetilde{\varphi}_h^n - \widetilde{\varphi}_h^{n-1}\right), \widetilde{\varphi}^n - \zeta_h\right)\right.\\
&+ \left\|\widetilde{\varphi}^n - \widetilde{\varphi}_h^n\right\|^2 + \left\|\widetilde{\varphi}^n - \zeta_h\right\|^2 + \left\|\widetilde{\varphi}_x^n - \zeta_{hx}\right\|^2 + \left\|w_x^n + l\varphi^n - \left(w_{hx}^n + l\varphi_h^n\right)\right\|^2\\
&+ \left\|\widetilde{\psi}_t^n - \tfrac{\widetilde{\psi}^n - \widetilde{\psi}^{n-1}}{\triangle t}\right\|^2 + \left\|\widetilde{\psi}^n - \chi_h\right\|^2 + \left\|\widetilde{\psi}^n - \widetilde{\psi}_h^n\right\|^2 + \left\|\psi_x^n - \chi_{hx}\right\|^2 + \left\|\psi_x^n - \psi_{hx}^n\right\|^2\\
&+ \left\|\widetilde{w}_t - \tfrac{\widetilde{w}^n - \widetilde{w}^{n-1}}{\triangle t}\right\|^2 + \tfrac{1}{\triangle t}\left(\widetilde{\psi}^n - \widetilde{\psi}^{n-1} - \left(\widetilde{\psi}_h^n - \widetilde{\psi}_h^{n-1}\right), \widetilde{\psi}^n - \chi_h\right)\\
&+ \tfrac{1}{\triangle t}\left(\widetilde{w}^n - \widetilde{w}^{n-1} - \left(\widetilde{w}_h^n - \widetilde{w}_h^{n-1}\right), \widetilde{w}^n - \xi_h\right) + \left\|\widetilde{w}^n - \widetilde{w}_h^n\right\|^2 + \left\|\widetilde{w}^n - \xi_h\right\|^2 + \left\|\widetilde{w}_x^n - \xi_{hx}\right\|^2\\
&+ \left\|\theta_t - \tfrac{\theta^n - \theta^{n-1}}{\triangle t}\right\|^2 + \left\|\theta^n - \theta_h^n\right\|^2 + \left\|\theta_x^n - \eta_x^n\right\|^2\\
&+ \tfrac{1}{\triangle t}\left(\theta^n - \theta^{n-1} - \left(\theta_h^n - \theta_h^{n-1}\right), \theta^n - \eta_h\right)\\
&+ \left\|\theta^n - \eta_h^n\right\|^2 + \left\|\theta_x^n - \eta_{hx}^n\right\|^2 + \left\|q_t - \tfrac{q^n - q^{n-1}}{\triangle t}\right\|^2\\
&+ \tfrac{1}{\triangle t}\left(q^n - q^{n-1} - \left(q_h^n - q_h^{n-1}\right), q^n - \alpha_h\right)\\
&\left. + \left\|q^n - q_h^n\right\|^2 + \left\|q^n - \alpha_h^n\right\|^2 + \left\|q_x^n - q_{hx}^n\right\|^2 + \left\|q_x^n - \alpha_{hx}^n\right\|^2\right).
\end{aligned} \tag{39}$$

Step 3. Multiplying the latter inequality by $\triangle t$ and summing over n we obtain, for all $\{\zeta_h,\ \eta_h\}_{i=0}^n \in S_0^h$, and $\{\chi_h,\ \xi_h,\ \alpha_h\}_{i=0}^{n} \in S_0^h$, we have

$$\begin{aligned}
&\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2+\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\|^2+\|\widetilde{w}^n-\widetilde{w}_h^n\|^2+\|\widetilde{\psi}_x^n-\widetilde{\psi}_{hx}^n\|^2\\
&\quad+\|\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n)\|^2+\|w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n)\|^2\\
&\quad+\|\theta^n-\theta_h^n\|^2+\|q^n-q_h^n\|^2\le C\triangle t\sum_{i=1}^{n}\Big(\big\|\widetilde{\varphi}^i-\widetilde{\varphi}_h^i\big\|^2+\big\|\widetilde{\psi}^i-\widetilde{\psi}_h^i\big\|^2+\big\|\widetilde{w}^i-\widetilde{w}_h^i\big\|^2\\
&\quad+\big\|\widetilde{\psi}_x^i-\widetilde{\psi}_{hx}^i\big\|^2+\big\|\varphi_{hx}^i+\psi_h^i+lw_h^i-\big(\varphi_x^i+\psi^i+lw^i\big)\big\|^2+\big\|w_x^i-l\varphi^i-\big(w_{hx}^i-l\varphi_h^i\big)\big\|^2\\
&\quad+\big\|\theta^i-\theta_h^i\big\|^2+\big\|q^i-q_h^i\big\|^2+\left\|\widetilde{\varphi}_t^i-\frac{\widetilde{\varphi}^i-\widetilde{\varphi}^{i-1}}{\triangle t}\right\|^2+\frac{1}{\triangle t}\left(\widetilde{\varphi}^i-\widetilde{\varphi}^{i-1}-\left(\widetilde{\varphi}_h^i-\widetilde{\varphi}_h^{i-1}\right),\widetilde{\varphi}^i-\zeta_h^i\right)\\
&\quad+\left\|\widetilde{\psi}_t^i-\frac{\widetilde{\psi}^i-\widetilde{\psi}^{i-1}}{\triangle t}\right\|^2+\frac{1}{\triangle t}\left(\widetilde{\psi}^i-\widetilde{\psi}^{i-1}-\left(\widetilde{\psi}_h^i-\widetilde{\psi}_h^{i-1}\right),\widetilde{\psi}^i-\chi_h^i\right)+\left\|\widetilde{w}_t^i-\frac{\widetilde{w}^i-\widetilde{w}^{i-1}}{\triangle t}\right\|^2\\
&\quad+\frac{1}{\triangle t}\left(\widetilde{w}^i-\widetilde{w}^{i-1}-\left(\widetilde{w}_h^i-\widetilde{w}_h^{i-1}\right),\widetilde{w}^i-\xi_h^i\right)+\left\|\theta_t^i-\frac{\theta^i-\theta^{i-1}}{\triangle t}\right\|^2+\left\|q_t^i-\frac{q^i-q^{i-1}}{\triangle t}\right\|^2\\
&\quad+\frac{1}{\triangle t}\left(\theta^i-\theta^{i-1}-\left(\theta_h^i-\theta_h^{i-1}\right),\theta^i-\eta_h^i\right)+\frac{1}{\triangle t}\left(q^i-q^{i-1}-\left(q_h^i-q_h^{i-1}\right),q^i-\alpha_h^i\right)\\
&\quad+\big\|\widetilde{\varphi}^i-\zeta_h^i\big\|^2+\big\|\widetilde{\varphi}_x^i-\zeta_{hx}^i\big\|^2+\big\|\widetilde{\psi}^i-\chi_h^i\big\|^2+\big\|\widetilde{\psi}_x^i-\chi_{hx}^i\big\|^2+\big\|\widetilde{w}^i-\xi_h^i\big\|^2+\big\|\widetilde{w}_x^i-\xi_{hx}^i\big\|^2\\
&\quad+\big\|\theta_x^i-\eta_{hx}^i\big\|^2+\big\|q_x^i-\alpha_{hx}^i\big\|^2+\big\|\theta^i-\eta_h^i\big\|^2+\big\|q^i-\alpha_h^i\big\|^2\Big)+C\Big(\big\|\varphi^1-\widetilde{\varphi}_h^0\big\|^2+\big\|\psi^1-\widetilde{\psi}_h^0\big\|^2\\
&\quad+\big\|w^1-\widetilde{w}_h^0\big\|^2+\big\|\psi_x^0-\psi_{hx}^0\big\|^2+\big\|\varphi_{hx}^0+\psi_h^0+lw_h^0-\big(\varphi_x^0+\psi^0+lw^0\big)\big\|^2\\
&\quad+\big\|w_x^0-l\varphi^0-\big(w_{hx}^0-l\varphi_h^0\big)\big\|^2+\big\|\theta^0-\theta_n^0\big\|^2+\big\|q^0-q_n^0\big\|^2\Big).
\end{aligned}$$

Taking into account that (with an equivalent result for similar terms) [5]

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\widetilde{\varphi}^i-\widetilde{\varphi}^{i-1}-\left(\widetilde{\varphi}_h^i-\widetilde{\varphi}_h^{i-1}\right),\widetilde{\varphi}^i-\zeta_h^i\right)=\left(\widetilde{\varphi}^n-\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\zeta_h^n\right)+\left(\widetilde{\varphi}_h^0-\widetilde{\varphi}^0,\widetilde{\varphi}^1-\zeta_h^1\right)\\
&\quad+\sum_{i=1}^{n-1}\big\|\widetilde{\varphi}_h^i-\widetilde{\varphi}_h^i,\widetilde{\varphi}^i-\zeta_h^i-\left(\widetilde{\varphi}^{i+1}-\zeta_h^{i+1}\right)\big\|\le C\Big(\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2+\|\widetilde{\varphi}^n-\zeta_h^n\|^2+\big\|\widetilde{\varphi}_h^0-\zeta_h^0\big\|^2\\
&\quad+\big\|\widetilde{\varphi}^1-\zeta_h^1\big\|^2\Big)+C\triangle t\sum_{i=1}^{n-1}\big\|\widetilde{\varphi}^i-\widetilde{\varphi}_h^i\big\|^2+\frac{C}{\triangle t}\sum_{i=1}^{n-1}\big\|\widetilde{\varphi}^i-\zeta_h^i-\left(\widetilde{\varphi}^{i+1}-\zeta_h^{i+1}\right)\big\|^2,
\end{aligned}\tag{40}$$

and applying a discrete version of Gronwall's inequality [12], the result follows. □

The linear convergence of the numerical method is summarized in the following corollary.

Corollary 1. *Assume that the solution to the continuous problem is sufficiently regular, that is*

$$\varphi,\psi,\omega\in H^3\left(0,T;L^2\left(0,L\right)\right)\cap W^{1,\infty}\left(0,T;H^1\left(0,L\right)\right)\cap H^2\left(0,T;H^1\left(0,L\right)\right),\tag{41}$$

and

$$\theta,q\in H^2\left(0,T;L^2\left(0,L\right)\right)\cap L^\infty\left(0,T;H^2\left(0,L\right)\right)\cap H^1\left(0,T;H^1\left(0,L\right)\right).\tag{42}$$

Then, there exists a positive constant C, independent of the discretization parameters h and $\triangle t$, such that

$$\begin{aligned} \|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2+\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\|^2+\|\widetilde{w}^n-\widetilde{w}_h^n\|^2+\|\widetilde{\psi}_x^n-\widetilde{\psi}_{hx}^n\|^2+\|\varphi_x^n+\psi^n+lw^n-(\varphi_{hx}^n+\psi_h^n+lw_h^n)\|^2 \\ +\|w_x^n-l\varphi^n-(w_{hx}^n-l\varphi_h^n)\|^2+\|\theta^n-\theta_h^n\|^2+\|q^n-q_h^n\|^2\le C\left(h^2+\triangle t^2\right). \end{aligned} \tag{43}$$

Proof. The result is a consequence of following estimates as in [7] and [12]:

$$\frac{1}{\triangle t}\sum_{n=1}^{N-1}\left\|\widetilde{\varphi}^n-\zeta_h^n-\left(\widetilde{\varphi}^{n+1}-\zeta_h^{n+1}\right)\right\|\le Ch^2\left\|\widetilde{\varphi}_t\right\|^2_{L^2(0,T;H^1(0,L))}.$$

□

3 Numerical simulation

The initial phase of our study focuses on validating the accuracy of our numerical approach, emphasizing the error analysis stemming from the nonhomogeneous equation (44). By choosing external forces g_i, for $i=1,2,3,4,5$, we ensure the system's exact solution is predetermined. To illustrate the system's exponential energy decay, the homogeneous equation (9) will be analyzed:

$$\begin{cases} \rho_1\varphi_{tt}-k(\varphi_x+\psi+lw)_x-lk_0(w_x-l\varphi)+\delta\theta_x=g_1, & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi+lw)=g_2, & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt}-k_0(w_x-l\varphi)_x+lk(\varphi_x+\psi+lw)=g_3, & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t+q_x+\delta\varphi_{xt}=g_4, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t+\beta q+\theta_x=g_5, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{44}$$

with the boundary conditions

$$\begin{cases} \varphi(0,t)=\ \psi_x(0,t)=\ w_x(0,t)=q(0,t)=0, & \forall t\ge 0, \\ \varphi_x(1,t)=\ \psi(1,t)=\ w(1,t)=\ \theta(1,t)=0, & \forall t\ge 0. \end{cases} \tag{45}$$

The finite element method P_1 for (44) with boundary conditions (45), using the backward Euler scheme, can be written as a linear system as follows:

$$\begin{cases} \dfrac{\rho_1}{\Delta t}M(U^n-U^{n-1})+kR\Phi^n-kC\Psi^n-klCW^n-lk_0CW^n+l^2k_0M\Phi^n+\delta C\Theta^n=G_1^n, \\ \dfrac{\rho_2}{\Delta t}M(V^n-V^{n-1})+bR\Psi^n+kC\Phi^n+kM\Psi^n+klMW^n=G_2^n, \\ \dfrac{\rho_1}{\Delta t}M(Z^n-Z^{n-1})+k_0RW^n+k_0lC\Phi^n+lkC\Phi^n+lkM\Psi^n+l^2kMW^n=G_3^n, \\ \dfrac{\rho_3}{\Delta t}M(\Theta^n-\Theta^{n-1})+CQ^n+\delta CU^n=G_4^n, \\ \dfrac{\tau}{\Delta t}M(Q^n-Q^{n-1})+\beta MQ^n+C\Theta^n=G_5^n, \end{cases} \tag{46}$$

where

$$\Phi^n=\Delta tU^n+\Phi^{n-1},\ \ \Psi^n=\Delta tV^n+\Psi^{n-1},\ \ W^n=\Delta tZ^n+W^{n-1},$$

and

$$\Phi^n = \begin{pmatrix} \phi_0^n \\ \phi_1^n \\ \vdots \\ \phi_J^n \end{pmatrix}, \quad \Psi^n = \begin{pmatrix} \psi_0^n \\ \psi_1^n \\ \vdots \\ \psi_J^n \end{pmatrix}, \quad W^n = \begin{pmatrix} w_0^n \\ w_1^n \\ \vdots \\ w_J^n \end{pmatrix},$$
$$U^n = \begin{pmatrix} \widetilde{\varphi}_0^n \\ \widetilde{\varphi}_1^n \\ \vdots \\ \widetilde{\varphi}_J^n \end{pmatrix}, \quad V^n = \begin{pmatrix} \widetilde{\psi}_0^n \\ \widetilde{\psi}_1^n \\ \vdots \\ \widetilde{\psi}_J^n \end{pmatrix}, \quad Z^n = \begin{pmatrix} \widetilde{w}_0^n \\ \widetilde{w}_1^n \\ \vdots \\ \widetilde{w}_J^n \end{pmatrix},$$
$$\Theta^n = \begin{pmatrix} \theta_0^n \\ \theta_1^n \\ \vdots \\ \theta_J^n \end{pmatrix}, \quad Q^n = \begin{pmatrix} q_0^n \\ q_1^n \\ \vdots \\ q_J^n \end{pmatrix},$$

with

$$M = (M_{ij}) = (\nu_i, \nu_j)_{0\le i,j\le J}, \quad R = (R_{ij}) = (\nu_{ix}, \nu_{jx})_{0\le i,j\le J},$$
$$C = (C_{ij}) = (\nu_{ix}, \nu_j)_{0\le i,j\le J}, \quad D = (D_{ij}) = (\nu_i, \nu_{jx})_{0\le i,j\le J},$$
$$G_j = (g_i, \nu_j)_{0\le i,j\le J}.$$

Problem (46) consists of five uncoupled, linear systems of algebraic equations with tridiagonal matrices, each having a unique solution.

3.1 Example 1: Error estimate

First, we performed a simulation to test the numerical error estimate. We solved problem (46) where g_1, g_2, g_3, g_4, and g_5, along with the initial data, are calculated from the exact solution provided below:

$$\varphi(x,t) = \exp(t)(1-x)^2x^2, \quad \psi(x,t) = \exp(t)(1-x)^2x^2,$$
$$w(x,t) = \exp(t)(1-x)^2x^2, \quad \theta(x,t) = \exp(t)(1-x)^2x^2,$$
$$q(x,t) = \exp(t)(1-x)^2x^2.$$

A curved beam with a radius of curvature $R = 1$ and length $l = 1$ was considered, with $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 1$, $k = 1$, $k_0 = 1$, $\delta = 300$, and $b = 1$. The computed errors for $T = 1$ are shown in Tables 1 and 2, where $Error$ is defined by

$$\begin{aligned} Error = {} & \|\widetilde{\varphi}^n - \widetilde{\varphi}_h^n\|^2 + \|\widetilde{\psi}^n - \widetilde{\psi}_h^n\|^2 + \|\widetilde{w}^n - \widetilde{w}_h^n\|^2 \\ & + \|\widetilde{\psi}_x^n - \widetilde{\psi}_{hx}^n\|^2 + \|\varphi_x^n + \psi^n + lw^n - (\varphi_{hx}^n + \psi_h^n + lw_h^n)\|^2 \\ & + \|w_x^n - l\varphi^n - (w_{hx}^n - l\varphi_h^n)\|^2 + \|\theta^n - \theta_h^n\|^2 + \|q^n - q_h^n\|^2, \end{aligned} \tag{47}$$

where

$$\|\phi_h^n\| = \max_n \left(\sum_{j=0}^{J} \left|\phi_j^n\right|^2 h \right)^{1/2}. \tag{48}$$

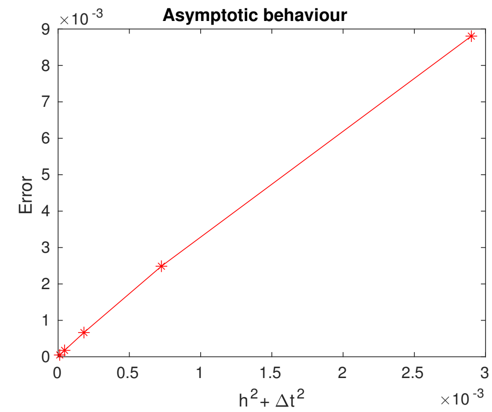

Figure 1: The evolution of $Error$.

Moreover, the convergence order p is defined by $p_X = ln(\mu_n)/ln(2)$, for $n = 1, 2, \cdots$ and $\mu_n = e_n/e_{n+1}$ where $e_n = \left\|X^n - X_h^n\right\|^2$, $X = \varphi, \psi, w$. The following table gives the error calculated between the solution and the approximate solution using our approach following the discretizations parameters.

Table 1: Numerical errors for φ, ψ, and w.

$h = \Delta t$	$\left\|\varphi^n - \varphi_h^n\right\|^2$	$\left\|\psi^n - \psi_h^n\right\|^2$	$\left\|w^n - w_h^n\right\|^2$	p_φ	p_{psi}	p_w
0.0500	1.100486649	0.333376294	0.336645966	———	———-	———
0.0250	0.156606691	0.034906201	0.036084485	2.812923885	3.255596264	3.221781607
0.0125	0.027644177	0.004616606	0.004862647	2.502098326	2.918578833	2.891564895
0.0063	0.005674667	0.000798858	0.000846337	2.284367982	2.530821625	2.522437719
0.0031	0.001277543	0.000163380	0.000172703	2.151163884	2.289707693	2.292939090

The numerical errors for some values of J and Δt is given in Table 2

Table 2: Numerical errors for some values of J and Δt for $T = 1$.

$J \downarrow \Delta t \rightarrow$	0.02000	0.01000	0.00500	0.00250	0.00125
20	0.008805078	0.008762398	0.008744467	0.008736530	0.008732996
40	0.002517476	0.002486375	0.002472914	0.002466716	0.002463743
80	0.000693652	0.000673492	0.000665303	0.000661690	0.000660006
160	0.000197536	0.000183505	0.000178347	0.000176242	0.000175309
320	0.000065883	0.0000549726	0.000051369	0.000050040	0.000049494

We observe that the error diminished by a factor of 2, indicating a linear convergence rate, as illustrated in Figures 1 and 2.

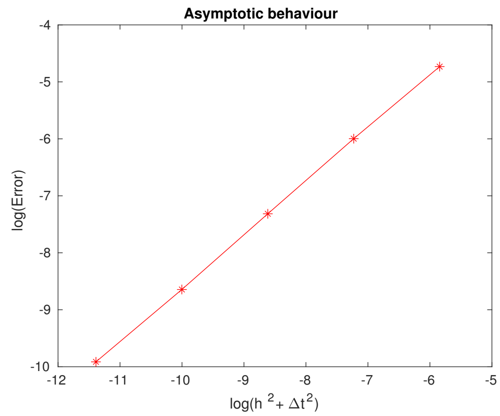

Figure 2: The evolution of $\log(Error)$.

3.2 Example 2: Energy decay

In this experiment, we investigated how energy decay evolves over time. The discrete energy is defined as follows:

$$\begin{aligned}\mathscr{E}_h^n = \frac{1}{2}\Big((\rho_1\left(\left\|\widetilde{\varphi}_h^n\right\|^2 + \left\|\widetilde{w}_h^n\right\|^2\right) + \rho_2\left\|\widetilde{\psi}_h^n\right\|^2 + k\left\|\varphi_{hx}^n + \psi_h^n + lw_h^n\right\|^2 \\ + b\left\|\psi_{hx}^n\right\|^2 + k_0\left\|w_{hx}^n - l\varphi_h^n\right\|^2 + \rho_3\left\|\theta_h^n\right\|^2 + \tau\left\|q_h^n\right\|^2\Big).\end{aligned} \tag{49}$$

The discretization parameters are $h = \frac{1}{200}$ and $\Delta t = \frac{1}{50}$, $T = 20$ and the initial data

$$\begin{aligned}\varphi_0(x) = x^2(x-1)^2,\ \ \varphi_1(x) = x^2(x-1)^2,\ \ \psi_0(x) = \psi_1(x) = \omega_0(x) = x^2(x-1)^2, \\ \omega_1(x) = x^2(x-1)^2,\ \ \theta_0(x) = x^2(x-1)^2, q_0(x) = x^2(x-1)^2.\end{aligned}$$

Case 1: $\mu_0 \neq 0$ and $\mu_1 = \mu_2 = 0$:
In this instance, we have selected the following values:

$$\begin{aligned}&\rho_1 = 1.8,\ \ \rho_2 = 1.2,\ \ \rho_3 = 1.3,\ \ k = 31.500000000000107, \\ &k_0 = 1.5,\ \ l = 1,\ \ \delta = 1,\ \ b = 21.538461538461608.\end{aligned}$$

As anticipated, the system progressed towards a zero steady-state, with the energy diminishing to zero rapidly, demonstrating the exponential decay of system energy as illustrated in Figure 3.

Case 2: $\mu_0 = 0$, $\mu_1 \neq 0$ and $\mu_2 = 0$:
For this particular case, we use the following data:

$$\rho_1 = 1,\ \ \rho_2 = 1,\ \ \rho_3 = 3,\ \ k = 1,\ \ k_0 = 1,\ \ l = 1,\ \ \delta = 10,\ \ b = 2.$$

When $\mu_0 = 0$, $\mu_1 \neq 0$, and $\mu_2 = 0$, such that conditions (6), (7), and (8) are not satisfied, the energy does not decay exponentially, as depicted in Figure 4. The numerical schemes were implemented using MATLAB on a Intel Core i5-6006U CPU @ 2.00 GHz.

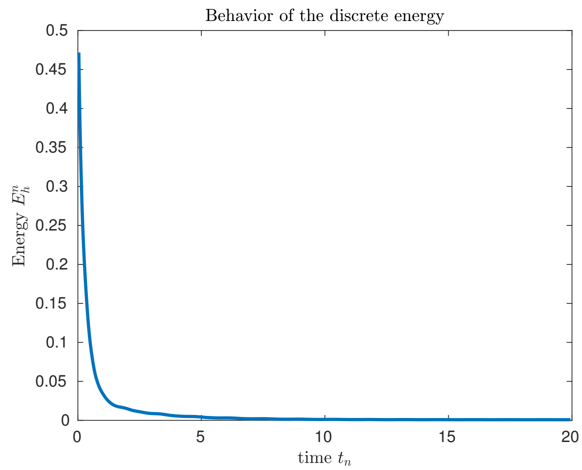

Figure 3: Numerical energy of the system in the **Case 1**.

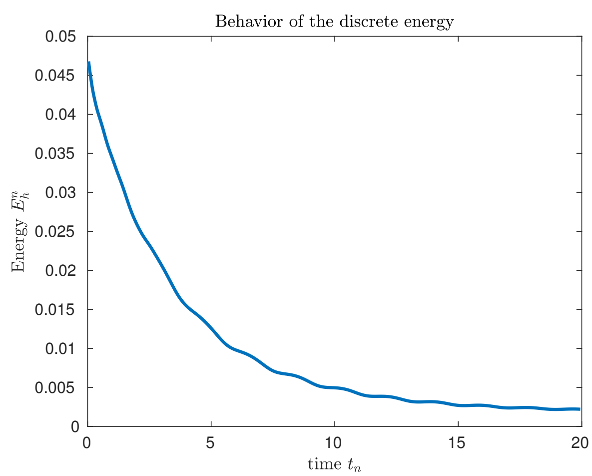

Figure 4: Numerical energy of the system in the **Case 2**.

4 Conclusions

In this article, the stability of discrete energy for a thermoelastic-Bresse system with second sound was carried out. In order to study the behavior of discrete energy, we first devised a numerical scheme based on finite elements for the spatial approximation and the implicit Euler scheme to discretize the time derivatives. The stability properties are then established, from which we deduce the energy decay of the discrete energy. The semi-discrete and completely discrete schemes a priori error estimates are then proved. Finally, several numerical tests were performed for this system, and the results show that the convergence matches what is predicted by the theories.

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