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New general integral transform on time scales

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Abstract. In this paper, we introduce a single integral transform that defines all known time scales generalized integral transforms in the family of Laplace transform as the new general integral transform on time scales. As a result, a unified approach is developed for the use of integral transforms representing the family of Laplace transform for solving problems on continuous and discrete cases dynamics. The convergence conditions and some principal properties accompanying the convolution theorem are given. It is shown that all generalized integral transforms on time scales included in the family of the Laplace transform are special cases of a new general integral transform. The applicability of this transform is demonstrated by solving certain ordinary dynamic equations and integral equations.

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1 Introduction

Integral transforms are important mathematical tools for solving ordinary, partial differential, and integral equations. Using integration, they map a function from its original function space to another function space where the properties of the transformed function can be more easily characterized and exploited. A particularly chosen integral transform converts complicated differential equations into simple algebraic equations which are easy to solve. Over, the last two decades, various integral transforms representing the family of Laplace transform such as Sumudu, Elzaki, Natural, Aboodh, Pourreza, Mohad, G-transform, Sawi, Kamal, Shehu transform, etc., have been introduced in [1, 2, 7, 9, 10, 13–16, 18, 23–29].

Simply, time scale is a nonempty closed subset of real numbers. Calculus on time scales provides a unified approach for solving with problems involving continuous and discrete dynamics within a single mathematical framework. Integral transform on time scales is a generalization of the classical integral transform because the functions involved in the given integral transform are the specific functions defined

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on given time scale $\mathscr{T}$. The integral transform technique on time scales is beneficial because it provides a unified approach for solving problems that involve continuous and discrete dynamics. In recent years, there has been a growing interest in developing new integral transforms on time scales because they can be used to solve a wide variety of problems in science and engineering. In 2002, Martin Bohner and Allen Peterson [4,5] generalized the Laplace transform on a time scale that coincides with the classical Laplace transform in the continuous domain ($\mathscr{T} = \mathbb{R}$) and the $Z-$transform in the discrete domain ($\mathscr{T} = \mathbb{Z}$). Subsequently, in [3, 20–22] Sumudu, Shehu, $\nabla-$Shehu, and $\alpha-$Laplace transforms were defined on time scale that were useful for solving ordinary, partial, and integro-dynamic equations with given initial conditions. In 2013, Hassan Eltayeb et al. [8] defined the double Laplace transform on time scales and applied it to solve partial dynamic equations with relatively fewer computations. Consequently in 2023 T.G. Thange and Sneha Chhatraband [19]combined Laplace and Sumudu transforms on time scales and as its application solved partial-integro dynamic and partial dynamic equations on time scales more efficiently.

The primary motivation for this study was to develop a single transform representing, at most, all transforms on time scales in the family of Laplace transform. This method provides a unified approach for using integral transforms to solve problems in both continuous and discrete dynamics. This paper is organized as follows. The next section discusses the preliminary concepts required to establish the foundation of our work. In Section 3, a new general integral transform on time scales is proposed. In addition, some conditions of its convergence with some fundamental results are given. The relationship between our defined transform and other time scale generalized transforms is given. Further, in Section 4, using this transform dynamic, integral, and fractional integral equations are solved. In the last section, salient conclusions are given about our work.

2 Preliminary Results

All results in this section are taken from [3–6, 8, 11, 12, 17, 22, 30]. Here, we assume that the time scale $\mathscr{T}$ under consideration is unbounded above and $t^0 \in \mathscr{T}$ is fixed. For $t \in \mathscr{T}$, the forward jump operator $\sigma(t) : \mathscr{T} \to \mathscr{T}$ is given as $\sigma(t) := inf\{t' \in \mathscr{T} : t' > t\}$ while the backward jump operator is $\rho : \mathscr{T} \to \mathscr{T}$ is $\rho(t) := \sup\{t' \in \mathscr{T} : t' < t\}$. Furthermore, the forward graininess function $\mu : \mathscr{T} \to [0,\infty)$ is given as $\mu(t) := \sigma(t) - t$.

Definition 1. *If a function g is continuous at right dense points in $\mathscr{T}$, and has a finite limit at the left dense points in $\mathscr{T}$, then g is said to be rd-continuous.*

Definition 2. *We denote the set of all rd-continuous functions by $\mathscr{C}_{rd}(\mathscr{T},\mathbb{C})$.*

Definition 3. *A function $g \in \mathscr{C}_{rd}(\mathscr{T},\mathbb{C})$ is called regressive if $1+\mu g \neq 0$. For $1+\mu g \geq 0$ and $1+\mu g \leq 0$, g is said to be positively regressive and negatively regressive, respectively.*

We denote the set of all regressive, positively regressive and negatively regressive functions by $\mathcal{R}(\mathscr{T},\mathbb{C})$, $\mathcal{R}^+(\mathscr{T},\mathbb{C})$, $\mathcal{R}^-(\mathscr{T},\mathbb{C})$, respectively.

Lemma 1. *The set of all regressive functions $\mathcal{R}$ form an abelian group under the addition $\oplus$ defined by $p \oplus q = p+q+\mu pq$. The additive inverse of p this group is denoted by $\ominus\frac{-p}{1+\mu p}$.*

Theorem 1. *For $f,g : \mathscr{T} \to \mathbb{R}$ with both f,g twice differentiable, then $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$.*

Theorem 2. *If $a,b \in \mathscr{T}$ and $f,g \in \mathscr{C}_{rd}$ then,*

(i) $\displaystyle\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.$

(ii) $\displaystyle\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t.$

(iii) $\displaystyle\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t.$

(iv) $\displaystyle\int_a^b f(t)g^{\Delta}(t) = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t.$

In order to define an exponential function for time scales, the following concepts are important. For $h > 0$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq \frac{-1}{h}\}$ and $\mathbb{Z}_h := \{z \in \mathbb{C} : \frac{-\pi}{h} < Im(z) \leq \frac{\pi}{h}\}$. The Hilger real and imaginary parts of a complex number are $\mathcal{R}e_h(z) := \frac{1}{h}(|1+hz|-1)$ and $\mathcal{I}m_h := \frac{1}{h}\mathrm{Arg}(1+hz)$, respectively with $\mathcal{R}e_0(z) := \mathrm{Re}(z)$ and $\mathcal{I}m_0(z) = \mathrm{Im}(z)$.

Further, the cylinder transformation $\zeta_h : \mathbb{C}_h \to \mathbb{Z}_h$ is given by $\zeta_h(z) = \frac{1}{h}\log(1+zh)$, where log is the principal logarithmic function.

Definition 4. *For $f \in \mathcal{R}$, the time scale exponential function is defined as*

$$e_f(t_1,t_2) = \exp\Big(\int_{t_2}^{t_1} \zeta_{\mu(s)}(f(s))\Delta s\Big), \quad for\ \ t_1,t_2 \in \mathscr{T}.$$

Theorem 3. *Suppose $f(t)$ is regressive and fix $t_0 \in \mathscr{T}$. Then $e_f(\cdot,t_0)$ is a solution of the initial value problem $y^{\Delta} = f(t)y,\ \ y(t_0) = 1$.*

Theorem 4. *If $p,q : \mathscr{T} \to \mathbb{R}$ are regressive and rd-continuous, then the following hold:*

(i) $e_p(t,s)e_p(s,r) = e_p(t,r).$

(ii) $e_p(t,s)e_{\ominus q}(t,t_0) = e_{p\ominus q}(t,s).$

(iii) $e_p^{\sigma}(t,s) = (1+\mu(t)p(t))e_p(t,s).$

Corollary 1. *If $p : \mathscr{T} \to \mathbb{R}$ is regressive, then $e^{\sigma}_{\ominus p}(t,s) = \frac{e_{\ominus p}(t,s)}{1+\mu(t)p}$.*

Definition 5. *A function $g : \mathscr{T} \to \mathbb{C}$ is said to be of exponential order $\mathbb{k}$ provided there exist constants, $C > 0, \mathbb{k} \in \mathbb{R} \cap \mathcal{R}^+$ such that $|g(t)| \leq Ce_{\mathbb{k}}(t,t_0)$ for all t in interval $[t_0,\infty) \in \mathscr{T}$ with $t_0 \in \mathscr{T}$.*

Next theorem is important regarding decay property of the exponential function.

Theorem 5. *For $\sup\mathscr{T} = \infty$. Let $t' \in \mathscr{T}$ and $\upsilon \in \mathcal{R}_c^+\big([t',\infty)_{\mathscr{T}},\mathbb{R}\big)$. Then for minimal graininess $\mu_*(t')$ and $z \in \mathbb{C}_{\mu_*(t')}(\upsilon)$, we have the following properties.*

(i) $|e_{\upsilon\ominus z}(t,t')| \leq e_{\upsilon\ominus\mathcal{R}e_{\mu_*(t')}(z)}(t,t')$ *for all $t \in [t',\infty)_{\mathscr{T}}$.*

(ii) $\displaystyle\lim_{t\to\infty} e_{\upsilon\ominus\mathcal{R}e_{\mu_*(t')}(z)}(t,t') = 0.$

(iii) $\lim_{t\to\infty} e_{\upsilon\ominus z}(t,t') = 0.$

Definition 6. *The generalized polynomial is a function $h_k : \mathscr{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows: $h_0(t,s) \equiv 1$ for all $s,t \in \mathscr{T}$, and $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau$ for all $s,t \in \mathscr{T}$, with given h_k for $k \in \mathbb{N}_0$.*

With the knowledge of the above results, we proceed to the next section.

3 Main Results

In this section, we introduce a new general integral transform on time scales and its properties.

Definition 7 (New General Integral Transform On Time Scales)**.** *Let $g : \mathscr{T} \to \mathbb{C}$ is a rd-continuous function with $p_1(z), p_2(z) : \mathbb{R} \to \mathbb{C}$ are positively regressive functions. We define the new general integral transform $\mathcal{G}(z)$ of $g(t)$ by the formula*

$$\mathscr{N}\{g(t)\}(z) = \mathcal{G}(z) = p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g(t)\Delta t, \tag{1}$$

provided that the integral exists for some $p_2(z)$ and $p_1(z) \neq 0$.

The following theorem concerns the convergence of the transform.

Theorem 6 (Convergence of the transform)**.** *Let $g : \mathscr{T} \to \mathbb{C}$ is rd-continuous function of exponential order $\Bbbk$. Then, the new general integral transform $\mathscr{N}\{g(t)\}(z)$ of $g(t)$ exists for all $p_2(z) \in \mathbb{C}_{\mu_*(z)}(\Bbbk)$ and converges absolutely.*

Proof. We have

$$\begin{aligned}
\left|p_1(z)\int_{t^0}^{\tau} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g(t)\Delta t\right| &\le p_1(z)\int_{t^0}^{\tau} |e^{\sigma}_{\ominus p_2(z)}(t,t^0)g(t)\Delta t| \\
&\le Mp_1(z)\int_{t^0}^{\tau} e^{\sigma}_{\ominus p_2(z)}(t,t^0)e_{\Bbbk}(t,t^0)\Delta t \\
&= Mp_1(z)\int_{t^0}^{\tau} \frac{e_{\ominus p_2(z)}(t,t^0)e_{\Bbbk}(t,t^0)}{1+\mu(t)p_2(t)}\Delta t \\
&= \frac{Mp_1(z)}{\Bbbk - p_2(z)}\int_{t^0}^{\tau} \frac{(\Bbbk - p_2(z))e_{\Bbbk\ominus p_2(z)}(t,t^0)}{1+\mu(t)p_2(z)}\Delta t \\
&= \frac{Mp_1(z)}{\Bbbk - p_2(z)}\int_{t^0}^{\tau} \Bbbk \ominus p_2(z)e_{\Bbbk\ominus p_2(z)}(t,t^0)\Delta t \\
&= \frac{Mp_1(z)}{\Bbbk - p_2(z)}\int_{t^0}^{\tau} e^{\Delta}_{\Bbbk\ominus p_2(z)}(t,t^0)\Delta t \\
&= \frac{Mp_1(z)}{\Bbbk - p_2(z)}[1 - e_{\Bbbk\ominus p_2(z)}(\tau,t^0)].
\end{aligned}$$

Taking $\tau \to \infty$ and using decay property of exponential function, we get

$$|\mathscr{N}\{g(t)\}(z)| = \left|p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g(t)\right| \le \frac{Mp_1(z)}{p_2(z)-\Bbbk} \quad \text{for all } \; p_2(z) \in \mathbb{C}_{\mu_*(z)}(\Bbbk). \qquad \square$$

Table 1: The new general integral transform of some elementary functions.

$g(t)$	$\mathscr{N}\{g(t)(t,t^0)\}(z)$
1	$\frac{p_1(z)}{p_2(z)}$
$e_\alpha(t,t^0)$	$\frac{p_1(z)}{p_2(z)-\alpha}$
$\sin_\alpha(t,t^0)$	$\frac{p_2(z)\alpha}{p_2^2(z)+\alpha^2}$
$\cos_\alpha(t,t_0)$	$\frac{p_1(z)p_2(z)}{p_2^2(z)+\alpha^2}$
$\sinh_\alpha(t,t^0)$	$\frac{p_1(z)\alpha}{p_2^2(z)-\alpha^2}$
$\cosh_\alpha(t,t^0)$	$\frac{p_1(z)p_2(z)}{p_2^2(z)-\alpha^2}$
$h_k(t,t^0)$	$\frac{p_1(z)}{p_2(z)^{k+1}}$

3.1 New general integral transform of some elementary functions

The new general integral transform of some elementary functions found using Definition 7 are given in Table 1.

The next theorem is related to the transform of integral of a function.

Theorem 7. *Assume $g(t)$ is a regulated function with $G(t)=\int_{t^0}^t g(s)\Delta s$, then*

$$\mathscr{N}\{G(t)\}(z)=\frac{p_1(z)}{p_2(z)}\mathscr{N}\{g(t)\}(z),$$

for all regressive $p_2(z):\mathbb{R}\to\mathbb{C},(p_2(z)\neq 0)$ satisfying,

$$\lim_{t\to\infty}\{e_{\ominus p_2(z)}(t,t^0)\int_{t^0}^t g(s)\Delta s\}=0.$$

Proof.

$$\begin{aligned}\mathscr{N}\{G(t)\}(z)&=p_1(z)\int_{t^0}^\infty e^\sigma_{\ominus p_2(z)}(t,t^0)G(t)\Delta t=p_1(z)\int_{t^0}^\infty \frac{e_{\ominus p_2(z)}(t,t^0)}{1+\mu(t)p_2(z)}G(t)\Delta t\\
&=\frac{-p_1(z)}{p_2(z)}\int_{t^0}^\infty \frac{-p_2(z)}{1+\mu(t)p_2(z)}e_{\ominus p_2(z)}(t,t^0)G(t)\Delta t\\
&=\frac{-p_1(z)}{p_2(z)}\int_{t^0}^\infty \ominus p_2(z)e_{\ominus p_2(z)}(t,t^0)G(t)\Delta t=\frac{-p_1(z)}{p_2(z)}\int_{t^0}^\infty e^\Delta_{\ominus p_2(z)}(t,t^0)G(t)\Delta t\\
&=\frac{-p_1(z)}{p_2(z)}\int_{t^0}^\infty\Big[\big(e_{\ominus p_2(z)}(t,t^0)G(t)\big)^\Delta-G^\Delta(t)e^\sigma_{\ominus p_2(z)}(t,t^0)\Big]\Delta t\\
&=\frac{-p_1(z)}{p_2(z)}\Big[\big(e_{\ominus p_2(z)}(t,t^0)G(t)\big)_{t=t^0}^\infty-\int_{t^0}^\infty e^\sigma_{\ominus p_2(z)}(t,t^0)G^\Delta(t)\Delta t\Big]\\
&=\frac{-p_1(z)}{p_2(z)}\Big[-G(t_0)-\int_{t^0}^\infty g(t)e^\sigma_{\ominus p_2(z)}(t,t^0)\Delta t\Big]\\
&=\frac{1}{p_2(z)}\mathscr{N}\{g(t)\}(z).\end{aligned}$$

□

Theorem 8. *Let $\omega \in \mathscr{T},\ \omega > 0$ the unit step function $\hat{u}_\omega(t)$ is defined as*

$$\hat{u}_\omega(t) = \begin{cases} 0, & \text{if} \quad t \in \mathscr{T} \cap (-\infty, \omega) \\ 1, & \text{if} \quad t \in \mathscr{T} \cap [\omega, \infty) \end{cases}$$

then $\mathscr{N}\{\hat{u}_\omega(t)g(t)\} = e_{\ominus p_2(z)}(\omega, t^0)\mathscr{N}\{g(t)\}$.

Proof.

$$\begin{aligned}
\mathscr{N}\{\hat{u}_\omega(t)g(t)\} &= p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)\hat{u}_\omega(t)g(t)\Delta t = p_1(z)\int_{\omega}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g(t)\Delta t \\
&= p_1(z)\int_{\omega}^{\infty} \frac{e_{\ominus p_2(z)}(t,t^0)}{1+\mu(t)p_2(z)}g(t)\Delta t = p_1(z)\int_{\omega}^{\infty} \frac{e_{\ominus p_2(z)}(t,\omega)e_{\ominus p_2(z)}(\omega,t^0)}{1+\mu(t)p_2(z)}g(t)\Delta t \\
&= p_1(z)e_{\ominus p_2(z)}(\omega,t^0)\int_{\omega}^{\infty} \frac{e_{\ominus p_2(z)}(t,\omega)}{1+\mu(t)p_2(z)}g(t)\Delta t \\
&= e_{\ominus p_2(z)}(\omega,t^0)p_1(z)\int_{\omega}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,\omega)g(t)\Delta t \\
&= e_{\ominus p_2(z)}(\omega,t^0)\mathscr{N}\{g(t)\}(z).
\end{aligned}$$

□

In the next theorem, we find the general integral transform of the derivative of a function.

Theorem 9. *For a function $g(t) : \mathscr{T} \to \mathbb{C}$ with g^{Δ} and $g^{\Delta\Delta}$ rd-continuous, and for a positively regressive function $p_2(z)$ satisfying $\lim\limits_{t\to\infty} g(t)e_{\ominus p_2(z)}(t,t^0) = 0$ and $\lim\limits_{t\to\infty} g^{\Delta}(t)e_{\ominus p_2(z)}(t,t^0) = 0$, we get*

(i) $\mathscr{N}\{g^{\Delta}(t)\}(z) = p_2(z)\mathscr{N}\{g(t)\}(z) - p_1(z)g(t^0)$.

(ii) $\mathscr{N}\{g^{\Delta\Delta}(t)\}(z) = p_2^2(z)\mathscr{N}\{g(t)\}(z) - p_1(z)p_2(z)g(t^0) - p_1(z)g^{\Delta}(t^0)$.

Proof.

$$\begin{aligned}
(i) \quad \mathscr{N}\{g^{\Delta}(t)\}(z) &= p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g^{\Delta}(t)\Delta t \\
&= p_1(z)\int_{t^0}^{\infty} \Big[\big(g(t)e_{\ominus p_2(z)}(t,t^0)\big)^{\Delta} - g(t)e^{\Delta}_{\ominus p_2(z)}(t,t^0)\Big]\Delta t \\
&= p_1(z)\Big[\big(g(t)e_{\ominus p_2(z)}(t,t^0)\big)_{t=t^0}^{t\to\infty} - \int_{t^0}^{\infty} g(t)e^{\Delta}_{\ominus p_2(z)}(t,t^0)\Delta t\Big] \\
&= p_1(z)\Big[-g(t^0) - \int_{t^0}^{\infty} g(t)\ominus p_2(z)e_{\ominus p_2(z)}(t,t^0)\Delta t\Big] \\
&= p_1(z)\Big[-g(t^0) - \int_{t^0}^{\infty} g(t)\Big(\frac{-p_2(z)}{1+\mu(t)p_2(z)}\Big)e_{\ominus p_2(z)}(t,t^0)\Delta t\Big] \\
&= -p_1(z)g(t^0) + p_1(z)p_2(z)\int_{t^0}^{\infty} g(t)\frac{e_{\ominus p_2(z)}(t,t^0)}{1+\mu(t)p_2(z)}\Delta t \\
&= -p_1(z)g(t^0) + p_1(z)p_2(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g(t)\Delta t \\
&= -p_1(z)g(t^0) + p_2(z)\mathscr{N}\{g(t)\}(z) \\
&= p_2(z)\mathscr{N}\{g(t)\}(z) - p_1(z)g(t^0).
\end{aligned}$$

$$
\begin{aligned}
(ii)\ \ \mathscr{N}\{g^{\Delta\Delta}(t)\}(z) &= p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g^{\Delta\Delta}(t)\Delta t\\
&= p_1(z)\int_{t^0}^{\infty}\Big[\big(g^{\Delta}(t)e_{\ominus p_2(z)}(t,t^0)\big)^{\Delta} - g^{\Delta}(t)e^{\Delta}_{\ominus p_2(z)}(t,t^0)\Big]\Delta t\\
&= p_1(z)\Big[\big(g^{\Delta}(t)e_{\ominus p_2(z)}(t,t^0)\big)_{t=t^0}^{t\to\infty} - \int_{t^0}^{\infty} g^{\Delta}(t)e^{\Delta}_{\ominus p_2(z)}(t,t^0)\Big]\Delta t\\
&= p_1(z)(t,t^0)\Big[-g^{\Delta}(t^0) - \int_{t^0}^{\infty} g^{\Delta}(t)\ominus p_2(z)e_{\ominus p_2(z)}(t,t^0)\Delta t\Big]\\
&= p_1(z)\Big[-g^{\Delta}(t^0) + p_2(z)\int_{t^0}^{\infty} g^{\Delta}(t)\frac{e_{\ominus p_2(z)}(t,t^0)}{1+\mu(t)p_2(z)}\Delta t\Big]\\
&= -p_1(z)g^{\Delta}(t^0) + p_1(z)p_2(z)\int_{t^0}^{\infty} g^{\Delta}(t)e^{\sigma}_{\ominus p_2(z)}(t,t^0)\Delta t\\
&= -p_1(z)g^{\Delta}(t^0) + p_2(z)\mathscr{N}\{g^{\Delta}(t)\}(z)\\
&= -p_1(z)g^{\Delta}(t^0) + p_2(z)\big[p_2(z)\mathscr{N}\{g(t)\}(z) - p_1(z)g(t^0)\big]\\
&= p_2^2(z)\mathscr{N}\{g(t)\}(z) - p_1(z)p_2(z)g(t^0) - p_1(z)g^{\Delta}(t^0). \qquad \square
\end{aligned}
$$

More generally we have,

$$
\mathscr{N}\{g^{\Delta^m}(t)\}(z) = p_2^m(z)\mathscr{N}\{g(t)\}(z) - p_1(z)\sum_{j=0}^{m-1}(p_2(z))^{m-1-j}g^{\Delta^j}(t^0).
$$

To find new general integral transform of the fractional integral, we should prove the convolution theorem for the new general integral transform. First, we recall the definition of convolution of two functions on time scales given below.

Definition 8 ([6])**.** *For given functions $g_1, g_2 : \mathscr{T} \to \mathbb{C}$, the convolution $g_1 * g_2$ is defined by*

$$
(g_1 * g_2)(z) = \int_{t^0}^{t} \hat{g_1}(t,\sigma(s))g_2(s)\Delta s \quad t \in \mathscr{T},
$$

where $\hat{g_1}$ is the shift of the function $g_1 : [t^0,\infty) \to \mathbb{C}$ is the solution of the shifting problem

$$
\begin{aligned}
y^{\Delta_t}(t,\sigma(s)) &= -y^{\Delta_s}(t,s), \quad t,s \in \mathscr{T},\ t \geq s \geq t^0,\\
y(t,t^0) &= g_1(t), \quad t \in \mathscr{T},\ \ t \geq t^0.
\end{aligned}
$$

Theorem 10 (Convolution Theorem). *Let $g_1 : \mathscr{T} \to \mathbb{C}$ and $g_2 : \mathscr{T} \to \mathbb{C}$ have new general integral transforms $\mathscr{N}\{g_1(t)\}(z)$ and $\mathscr{N}\{g_2(t)\}(z)$ respectively. Then, the new general integral transform of the convolution of g_1 and g_2 is given by,*

$$
\mathscr{N}\{(g_1 * g_2)(t)\}(z) = \frac{1}{p_1(z)}\mathscr{N}\{g_1(t)\}(z)\ \mathscr{N}\{g_2(t)\}(z).
$$

Proof.

$$\begin{aligned}\mathscr{N}\{(g_1*g_2)(t)\}(z) &= p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)(g_1*g_2)(t)\Delta t\\ &= p_1(z)\int_{t^0}^{\infty} e_{\ominus p_2(z)}(\sigma(t),t^0)\Big[\int_{t^0}^{t}\hat{g}_1(t,\sigma(s))g_2(s)\Delta s\Big]\Delta t\\ &= p_1(z)\int_{t^0}^{\infty} e_{\ominus p_2(z)}(\sigma(s),t^0)g_2(s)\Big[\int_{\sigma(s)}^{\infty} e_{\ominus p_2(z)}(\sigma(t),\sigma(s))\hat{g}_1(t,\sigma(s))\Delta t\Big]\Delta s\\ &= \frac{p_1(z)}{p_1(z)}\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(s,t^0)g_2(s)\ \psi(\sigma(s))\ \Delta s,\end{aligned}$$

with $\psi(s) = p_1(z)\int_s^{\infty} e_{\ominus p_2(z)}(\sigma(t),s)\hat{g}_1(t,s)\Delta t$. Now from [6, Lemma 2.4] and [6, Lemma 3.3], it is observed that $\psi(s)$ is a constant function and is independent of s.
So, we get

$$\begin{aligned}\psi(t^0) &= p_1(z)\int_{t^0}^{\infty} e_{\ominus p_2(z)}(\sigma(t),t^0)\hat{g}_1(t,t^0)\Delta t\\ &= p_1(z)\int_{t^0}^{\infty} e^{\sigma}_{\ominus p_2(z)}(t,t^0)g_1(t)\Delta t\\ &= \mathscr{N}\{g_1(t)\}(z).\end{aligned}$$

And finally, we get

$$\mathscr{N}\{(g_1*g_2)(t)\}(z) = \frac{1}{p_1(z)}\mathscr{N}\{g_1(t)\}(z)\ \mathscr{N}\{g_2(t)\}(z). \qquad \square$$

The concept of time scales power functions as an extension of the time scales generalized polynomials [4] play an important role in the development of fractional calculus for time scales through integrals. As an attempt to this one of the definitions of time scales power functions from [12] is given below.

Definition 9. *Let $t^0, t \in \mathscr{T}$ and $\lambda_1,\ \lambda_2 > -1$. The time scale power functions $h_{\lambda_1}(t,t^0)$ are defined as a family of non-negative functions satisfying.*

(i) $\int_{t^0}^{t} h_{\lambda_1}(t,\sigma(s))\ h_{\lambda_2}(s,t^0)\ \Delta t\ = h_{\lambda_1+\lambda_2+1}(t,t^0)$ *for* $t \geq t^0$.

(ii) $h_0(t,t^0) = 1$ *for* $t \geq t^0$.

(iii) $h_{\lambda_1}(t,t) = 0$ *for* $\lambda_1 \in (0,1)$.

Using this concept, the Riemann-Liouville fractional integral for time scale nabla calculus is given in [12]. Here we will consider the definition for delta calculus as,

Definition 10. *The Riemann-Liouville fractional integral of order $\lambda > 0$ with the lower limit c is defined as*

$${}_{\Delta}\mathscr{I}_c^{-\lambda}g(t) := \int_c^t h_{\lambda-1}(t,\sigma(\tau))g(\tau)\ \Delta\tau,$$

and for $\lambda = 0$, $({}_{\Delta}\mathscr{I}_c^{0}g)(t) = g(t)$.

The next theorem concerns the transform of the Riemann-Liouville fractional integral.

Theorem 11. *For a function $g : \mathscr{T} \to \mathbb{C}$ rd-continuous on the time scale interval $[c,d]_{\mathscr{T}}$, the general integral transform of the Riemann-Liouville fractional integral of order λ is given as,*

$$\mathscr{N}\{{}_{\Delta}\mathscr{I}_c^{-\lambda} g(t)\} = \frac{1}{p_2^{\lambda}(z)} \mathscr{N}\{g(t)\}.$$

Proof. Using the concept of convolutions, the Riemann-Liouville fractional integral of g is written as,

$$ {}_{\Delta}\mathscr{I}_c^{-\lambda} g(t) = h_{\lambda-1}(t) * g(t).$$

Then using convolution theorem,

$$\begin{aligned}
\mathscr{N}\{{}_{\Delta}\mathscr{I}_c^{-\lambda} g(t)\} &= \mathscr{N}\{h_{\lambda-1}(t) * g(t)\} \\
&= \frac{1}{p_1(z)} \mathscr{N}\{h_{\lambda-1}\}\mathscr{N}\{g(t)\} \\
&= \frac{1}{p_1(z)} \frac{p_1(z)}{p_2^{\lambda}(z)} \mathscr{N}\{g(t)\} \\
&= \frac{1}{p_2^{\lambda}(z)} \mathscr{N}\{g(t)\}.
\end{aligned}$$ □

Next, we show the correlation of the new general integral transform with the generalized Laplace, Sumudu, and Shehu transform on time scales.

(1) If $p_1(z) = 1$ and $p_2(z) = z$, the new general integral transform converts into the Laplace transform [4].

$$\mathscr{L}\{f(t)\}(z) = \int_{t_0}^{\infty} e_{\ominus z}^{\sigma}(t,t_0) f(t)\,\Delta t.$$

(2) If $p_1(z) = \frac{1}{z}$ and $p_2(z) = \frac{1}{z}$, the new general integral transform converts into the Sumudu transform [3].

$$\mathscr{S}\{f(t)\}(z) = \frac{1}{z}\int_{t_0}^{\infty} e_{\ominus \frac{1}{z}}^{\sigma}(t,t_0) f(t)\,\Delta t.$$

(3) If $p_1(z) = 1$ and $p_2(z) = \frac{z}{u}$, the new general integral transform converts into the Shehu transform [20].

$$\mathcal{S}h(z) = \int_{t_0}^{\infty} e_{\ominus \frac{z}{u}}^{\sigma}(t,t_0) f(t)\,\Delta t.$$

Thus we conclude that all the above generalized integral transforms on time scales are special cases of the transform given by Eq. (1).

4 Applications

In this section, we show the efficiency of the new general integral transform for solving dynamic equations on time scales through the following examples.

Example 1. Consider the following second order dynamic equation

$$y^{\Delta\Delta}(t) - 6y^{\Delta}(t) + 8y(t) = e_3(t,t^0), \quad y(t^0) = 1, \ y^{\Delta}(t^0) = 0,$$

where $t, t^0 \in \mathscr{T}$. Applying the general integral transform on both sides

$$\begin{aligned}
&\mathscr{N}\{y^{\Delta\Delta}(t) - 6y^{\Delta}(t) + 8y(t)\}(z) = \mathscr{N}\{e_3(t,t^0)\}(z), \\
&\mathscr{N}\{y^{\Delta\Delta}(t)\}(z) - \mathscr{N}\{6y^{\Delta}(t)\}(z) + \mathscr{N}\{8y(t)\}(z) = \mathscr{N}\{e_3(t,t^0)\}(z),
\end{aligned}$$

applying initial conditions

$$\begin{aligned}
&(p_2(z))^2\mathscr{N}\{y(t)\}(z) - p_2(z)p_1(z)y(t^0) - p_1(z)y^{\Delta}(t^0) \\
&\qquad - 6p_2(z)\mathscr{N}\{y(t)\}(z) - 6p_1(z)y(t^0) + 8\mathscr{N}\{y(t)\}(z) = \frac{p_1(z)}{p_2(z)-3}, \\
&\mathscr{N}\{y(t)\}(z)\left(p_2^2(z) - 6p_2(z) + 8\right) - p_2(z)p_1(z) - 6p_1(z) = \frac{p_1(z)}{p_2(z)-3}, \\
&\mathscr{N}\{y(t)\}(z)\left(p_2^2(z) - 6p_2(z) + 8\right) = \frac{p_1(z)}{p_2(z)-3} + p_2(z)p_1(z) + 6p_1(z), \\
&\mathscr{N}\{y(t)\}(z) = \frac{p_1(z)}{(p_2(z)-3)(p_2^2(z) - 6p_2(z) + 8)} + \frac{p_2(z)p_1(z) + 6p_1(z)}{(p_2^2(z) - 6p_2(z) + 8)}, \\
&\mathscr{N}\{y(t)\} = \frac{-1}{2}\frac{p_1(z)}{p_2(z)-4} + \frac{5}{2}\frac{p_1(z)}{p_2(z)-2} - \frac{p_1(z)}{p_2(z)-3}.
\end{aligned}$$

Taking inverse transform on both sides using Table 1,

$$y(t) = \frac{-1}{2}e_4(t,t^0) + \frac{5}{2}e_2(t,t^0) - e_3(t,t^0).$$

Example 2. Consider the following third order dynamic equation

$$y^{\Delta\Delta\Delta}(t) + y^{\Delta}(t) = e_1(t,0)\ ; \quad y(0) = y^{\Delta}(0) = y^{\Delta\Delta} = 0.$$

Applying the general integral transform on both sides,

$$\begin{aligned}
&\mathscr{N}\{y^{\Delta\Delta\Delta}(t) + y^{\Delta}(t)\}(z) = \mathscr{N}\{e_1(t,0)\}(z), \\
&\mathscr{N}\{y^{\Delta\Delta\Delta}(t)\}(z) + \mathscr{N}\{y^{\Delta}(t)\}(z) = \mathscr{N}\{e_1(t,0)\}(z), \\
&p_2^3(z)\mathscr{N}\{y(t)\}(z) - p_2^2(z)p_1(z)y(0) - p_2(z)p_1(z)y^{\Delta}(0) \\
&\qquad + p_1(z)y^{\Delta\Delta}(0) + p_2(z)\mathscr{N}\{y(t)\}(z) - p_1(z)y(0) = \frac{p_1(z)}{p_2(z)-1},
\end{aligned}$$

applying initial conditions,

$$
\begin{aligned}
&\left(p_2^3(z)+p_2(z)\right)\mathscr{N}\{y(t)\}(z)=\frac{p_1(z)}{p_2(z)-1},\\
&\mathscr{N}\{y(t)\}(z)=\frac{p_1(z)}{(p_2(z)-1)(p_2^3(z)+p_2(z))},\\
&\mathscr{N}\{y(t)\}(z)=\frac{-p_1(z)}{p_2(z)}-\frac{1}{2}\frac{p_1(z)}{p_2^2(z)+1}+\frac{1}{2}\frac{p_1(z)p_2(z)}{p_2^2(z)+1}+\frac{1}{2}\frac{p_1(z)}{p_2(z)-1}.
\end{aligned}
$$

Taking inverse transform on both sides using Table 1,

$$y(t)=-1-\frac{1}{2}\sin_1(t,0)+\frac{1}{2}\cos_1(t,0)+\frac{1}{2}e_1(t,0).$$

Example 3. Consider the following integral equation

$$g(t)=e_2(t,t^0)+4\int_{t^0}^{t}g(s)\Delta s.$$

Applying the general integral transforms on both sides,

$$
\begin{aligned}
&\mathscr{N}\{g(t)\}(z)=\mathscr{N}\{e_2(t,t^0)\}(z)+4\mathscr{N}\left\{\int_{t^0}^{t}g(s)\Delta s\right\},\\
&\mathscr{N}\{g(t)\}(z)=\frac{p_1(z)}{p_2(z)-2}+4\frac{p_1(z)}{p_2(z)}\mathscr{N}\{g(t)\}(z),\\
&\mathscr{N}\{g(t)\}(z)\left(1-\frac{4p_1(z)}{p_2(z)}\right)=\frac{p_1(z)}{p_2(z)},\\
&\mathscr{N}\{g(t)\}(z)\left(\frac{p_2(z)-4p_1(z)}{p_2(z)}\right)=\frac{p_1(z)}{p_2(z)-2},\\
&\mathscr{N}\{g(t)\}(z)=\frac{p_1(z)p_2(z)}{(p_2(z)-2)(p_2(z)-4p_1(z))}\\
&\mathscr{N}\{g(t)\}(z)=\frac{-p_1(z)}{p_2(z)-2}+\frac{2p_1(z)}{p_2(z)-4}.
\end{aligned}
$$

Taking inverse transform on both sides using Table 1,

$$g(t)=-e_2(t,t^0)+2e_4(t,t^0).$$

Example 4. Consider the following Volterra integral equation

$$y(t)=t+\int_0^t y(\tau)\sin(t-\tau)\,\Delta\tau.$$

Applying the general integral transforms on both sides,

$$\begin{aligned}
\mathscr{N}\{y(t)\}(z) &= \mathscr{N}\{t\}(z) + \mathscr{N}\Big\{\int_0^t y(\tau)\sin(t-\tau)\,\Delta\tau\Big\}(z),\\
\mathscr{N}\{y(t)\}(z) &= \mathscr{N}(z) + \mathscr{N}\{y(t)*\sin(t)\}(z),\\
\mathscr{N}\{y(t)\}(z) &= \frac{p_1(z)}{p_2^2(z)} + \frac{1}{p_1(z)}\mathscr{N}\{y(t)\}\mathscr{N}\{\sin(t)\},\\
\mathscr{N}\{y(t)\}(z) &= \frac{p_1(z)}{p_2^2(z)} + \frac{1}{p_1(z)}\mathscr{N}\{y(t)\}(z)\frac{p_1(z)}{p_2^2(z)+1},\\
\mathscr{N}\{y(t)\}(z) - \frac{1}{p_2^2(z)+1}\mathscr{N}\{y(t)\}(z) &= \frac{p_1(z)}{p_2^2(z)},\\
\mathscr{N}\{y(t)\}(z)\Big(1-\frac{1}{p_2^2(z)+1}\Big) &= \frac{p_1(z)}{p_2^2(z)},\\
\mathscr{N}\{y(t)\}(z)\Big(\frac{p_2^2(z)}{p_2^2(z)+1}\Big) &= \frac{p_1(z)}{p_2^2(z)},\\
\mathscr{N}\{y(t)\}(z) &= \Big(\frac{p_2^2(z)+1}{p_2^2(z)}\Big)\Big(\frac{p_1(z)}{p_2^2(z)}\Big),\\
\mathscr{N}\{y(t)\}(z) &= \frac{p_1(z)p_2^2(z)}{p_2^4(z)} + \frac{p_1(z)}{p_2^4(z)},\\
\mathscr{N}\{y(t)\}(z) &= \frac{p_1(z)}{p_2^2(z)} + \frac{p_1(z)}{p_2^4(z)}.
\end{aligned}$$

Taking inverse transform on both sides using Table 1,

$$y(t) = h_1(t,t^0) + h_3(t,t^0).$$

Example 5. Consider the following fractional integral equation

$$y(t) = f(t) + \mathscr{I}_c^{-\lambda} y(t), \quad \text{for } \lambda > 0.$$

Applying the general integral transform on both sides

$$\begin{aligned}
\mathscr{N}\{y(t)\}(z) &= \mathscr{N}\{f(t)\}(z) + \frac{1}{p_2^\lambda(z)}\mathscr{N}\{y(t)\}(z),\\
\mathscr{N}\{y(t)\}(z)\Big(1-\frac{1}{p_2^\lambda(z)}\Big) &= \mathscr{N}\{f(t)\}(z),\\
\mathscr{N}\{y(t)\}(z) &= \Big(\frac{p_2^\lambda(z)}{p_2^\lambda(z)-1}\Big)\mathscr{N}\{f(t)\}(z),
\end{aligned}$$

thus, $y(t) = \mathscr{N}^{-1}\Big\{\Big(\frac{p_2^\lambda(z)}{p_2^\lambda(z)-1}\Big)F(z)\Big\}$, where, $\mathscr{N}\{f(t)\}(z) = F(z)$.

5 Conclusion

In this paper, a new general integral transform representing at most all integral transforms in the family of the Laplace transform generalized on time scales is introduced. Conditions for the convergence of transform and some principal properties along with the convolution theorem are proved. The relationship of this transform with all other existing integral transforms such as Laplace, Sumudu, and Shehu transforms on time scales has been revealed. On solving ordinary dynamic equations and integral equations on time scales using the new general integral transform, it is observed that all generalized integral transforms in the family of Laplace transform give the same solution, and if we choose the generalized Laplace transform, the volume of calculation will be minimized. Moreover it is interesting to note that classical integral transforms such as Elzaki, Natural, Aboodh, Sawi, and Kamal, etc. can be generalized on time scales by using the new general integral transform.

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