Journal of Mathematical Modeling
Vol. 12, No. 3, 2024, pp. 451–462. Research Article

On differential-integral optimal control problems

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Abstract. In this paper, we will study the optimal control problem of a system containing a differential integral (D-I) operator. We will deduce the necessary optimality conditions and apply it first to the problem of minimum energy to find the lowest energy for an electrical circuit containing a resistor, a coil and a capacitor (RLC circuit), and second to the problem of the minimum time to transfer electrical current in RLC circuit from one state to another in the shortest possible time.

Keywords: Optimal control, differential-integral equations, minimum energy problem, minimum time problem.
AMS Subject Classification 2020: 49J15, 49K15, 34H05.

1 Introduction

The classical differentiable optimal control problem: Determine the control signals that satisfy

$$\begin{aligned} \text{minimize:} \quad & \int_{t_0}^{t_f} g(t,x,u)\,dt, \\ \text{subject to:} \quad & \frac{dx}{dt} = f(t,x,u). \end{aligned}$$

It has one major shortcoming despite its great success, it only deals with functionals containing derivatives. Many phenomena in nature can be modeled more accurately using equations involving differential-integral operator

$$L\frac{dx}{dt} + C\int_{t_0}^{t} x(\tau)\,d\tau.$$

It finds its applications in many scientific fields, ranging from mathematics, physics and engineering to biomedical and management sciences [2, 5, 12, 14, 15, 17].

Optimal control problems with differential-integral state constraint naturally arise in many engineering applications [3, 4, 6–8, 10, 13, 16]. The two most important problems of optimal control are the

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Received: 20 January 2024 / Revised: 29 March 2024 / Accepted: 31 March 2024
DOI: 10.22124/jmm.2024.26520.2338

problem of minimum energy and the problem of minimum time [1, 11, 18]. In [13], an algorithm has been constructed for computing the exact solutions for the quadratic optimal control problem with integral constraints and it has been used to find the optimal solution for single and coupled RC electrical circuits. It is not worthwhile in applications to convert integrals into differentials, especially if there are many integrals. In this paper, we generalize the result in [13] to find the necessary conditions for the optimal control of differential- integral state as well as we apply the generalization problem to find the minimum energy and the shortest time to transfer the electrical current from one state to another in an RLC electrical circuit.

2 Necessary conditions of D-I-control problem

Consider the system described by the following nonlinear differential integral equations:

$$L\frac{dx(t)}{dt}+C\int_{t_0}^{t}x(\tau)d\tau = f(t,x(t),v(t)), \tag{1}$$

where L and C are $n\times n$ real matrices and $x(t)$ an n-vector function is determined by $v(t)$ an m-vector function, with $x\in\mathbb{R}^n$, $v\in\mathbb{R}^m$.

Consider a performance index of the form

$$J(v)=\Phi(x(t_f))+\int_{t_0}^{t_f} g(t,x(t),v(t))\,dt. \tag{2}$$

The problem is to find the functions $v(t),\, t\in[t_0,t_f]$ that minimize (or maximize)) $J(v)$. It is assumed that $\Phi(x),\, f(t,x,v),\, g(t,x,v)$ are continuous for all $t\in[t_0,t_f]$ and , $x,\, v\in\mathbb{R}^n$ and have continuous derivative up to the second order.

The necessary conditions of the constrained optimal control problem (1)-(2) are obtained by converting into an unconstrained optimal control problem using the Lagrange multiplier function $\lambda(t)\in\mathbb{R}^n$:

$$\begin{aligned}\overline{J}(v)=\ &\Phi(x(t_f))+\int_{t_0}^{t_f} g(t,x(t),v(t))\,dt\\ &+\int_{t_0}^{t_f}\lambda^T(t)\left[f(t,x(t),v(t))-L\frac{dx(t)}{dt}-C\int_{t_0}^{t}x(\tau)d\tau\right]dt.\end{aligned}$$

Let us define the Hamiltonian function

$$H(t,x(t),\lambda(t),v(t)):=g(t,x(t),v(t))+\lambda^T(t)f(t,x(t),v(t)).$$

Thus

$$\overline{J}(u)=\Phi(x(t_f))+\int_{t_0}^{t_f}\left\{[H(t,x(t),\lambda(t),v(t))-\lambda^T(t)\left[L\frac{dx(t)}{dt}+C\int_{t_0}^{t}x(\tau)d\tau\right]\right\}dt.$$

The necessary condition for optimality is that the variation $\delta\overline{J}$ of the modified cost with respect to all feasible variations $\delta x(t), \delta\lambda(t)$ and $\delta v(t)$ should vanish.

$$\begin{aligned}\delta \bar{J} =& \frac{\partial \Phi}{\partial x(t_f)}\delta x(t_f) + \int_{t_0}^{t_f}\left[\frac{\partial H}{\partial x(t)}\delta x(t) + \delta\lambda^T(t)\frac{\partial H}{\partial \lambda(t)} + \frac{\partial H}{\partial v(t)}\delta v(t)\right] dt\\ &- \int_{t_0}^{t_f}\left[\delta\lambda^T(t)L\frac{dx(t)}{dt} + \lambda^T(t)L\delta\left(\frac{dx(t)}{dt}\right)\right] dt\\ &- \int_{t_0}^{t_f}\left[\left(\int_{t_0}^{t}\delta\lambda^T(t)Cx(\tau)d\tau\right) + \lambda^T(t)C\delta\left(\int_{t_0}^{t}x(\tau)d\tau\right)\right] dt.\end{aligned}$$

Now

(i) $\delta\left(\frac{dx(t)}{dt}\right) = \frac{d\delta x(t)}{dt}$ and $\delta\left(\int_{t_0}^{t}x(\tau)d\tau\right) = \int_{t_0}^{t}\delta x(\tau)d\tau.$

(ii) By changing the order of integration, we have

$$\int_{t_0}^{t_f}\lambda^T(t)C\left(\int_{t_0}^{t}\delta x(\tau)d\tau\right) dt = \int_{t_0}^{t_f}\left(\int_{t}^{t_f}\lambda^T(\tau)d\tau\right)C\delta x(t)\,dt.$$

(iii) By integration by parts, we have

$$\int_{t_0}^{t_f}\lambda^T(t)L\frac{d\delta x(t)}{dt}dt = \lambda^T(t_f)L\delta x(t_f) - \int_{t_0}^{t_f}\frac{d\lambda^T(t)}{dt}L\delta x(t)dt.$$

From, (i), (ii) and (iii), we get

$$\begin{aligned}\delta\bar{J} =& \left[\frac{\partial \Phi}{\partial x(t_f)} - \lambda^T(t_f)L\right]\delta x(t_f) + \int_{t_0}^{t_f}\left[\frac{\partial H}{\partial x(t)} + \frac{d\lambda^T(t)}{dt}L - \left(\int_{t}^{t_f}\lambda^T(\tau)d\tau\right)C\right]\delta x(t)\,dt\\ &+ \int_{t_0}^{t_f}\delta\lambda^T(t)\left[\frac{\partial H}{\partial \lambda(t)} - L\frac{dx(t)}{dt} - C\int_{t_0}^{t}x(\tau)d\tau\right] dt + \int_{t_0}^{t_f}\frac{\partial H}{\partial v(t)}\delta v(t)\,dt.\end{aligned}$$

Setting the terms that multiply variations to be zero yield:

$$\begin{aligned}&L\frac{dx(t)}{dt} + C\int_{t_0}^{t}x(\tau)d\tau = \frac{\partial H}{\partial \lambda(t)} = f(t, x(t), v(t)),\\ &-L^T\frac{d\lambda(t)}{dt} + C^T\int_{t}^{t_f}\lambda(\tau)d\tau = \left(\frac{\partial H}{\partial x(t)}\right)^T,\\ &0 = \frac{\partial H}{\partial v(t)},\\ &\lambda(t_f) = L^{-1^T}\left(\frac{\partial \Phi}{\partial x(t_f)}\right)^T.\end{aligned}$$

Thus, we obtained the following theorem.

Theorem 1. *A necessary conditions for the pair* (v^*, x^*) *to satisfy* (1)-(2) *are*
State equations

$$\begin{cases} L\dfrac{dx^*(t)}{dt} + C\displaystyle\int_{t_0}^{t} x^*(\tau)d\tau = f(t, x^*(t), v^*(t)), \\ x^*(t_f) = x_0. \end{cases}$$

Adjoint equations

$$\begin{cases} -L^T\dfrac{d\lambda(t)}{dt} + C^T\displaystyle\int_{t}^{t_f} \lambda(\tau)d\tau = \left(\dfrac{\partial g}{\partial x^*(t)}\right)^T + \left(\dfrac{\partial f}{\partial x^*(t)}\right)^T \lambda(t), \\ \lambda(t_f) = L^{-1^T}\left(\frac{\partial \Phi}{\partial x^*(t_f)}\right)^T. \end{cases}$$

Minimum principle

$$\frac{\partial g}{\partial v^*(t)} + \lambda^T(t)\frac{\partial f}{\partial v^*(t)} = 0.$$

3 Minimum energy problem

Consider the quadratic optimal control problem:

$$J(u(\cdot)) = \frac{1}{2}x^T(t_f).x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} v^T(t).v(t)\,dt \to \min, \tag{3}$$

under the constraint

$$L\frac{dx(t)}{dt} + C\int_{t_0}^{t} x(\tau)d\tau = Ax(t) + Bv(t), \qquad t \in [t_0, t_f], \tag{4}$$

where L, C and A are $n \times n$ matrices, L is invertible and B is $n \times m$ matrix.

By applying the necessary conditions of the general optimal control given in the above sections to (3)-(4), we have the following result.

Theorem 2. *The optimal control of problem* (3)-(4) *is characterized by:*

$$v^*(t) = -B^T\lambda_2(t),$$

where $\lambda(t)$ *and* $x^*(t)$ *satisfy the following equations:*
State equations

$$\begin{cases} L\dfrac{dx^*(t)}{dt} + C\displaystyle\int_{t_0}^{t} x^*(\tau)d\tau = Ax^*(t) - BB^T\lambda(t), \\ x^*(t_0) = x_0. \end{cases} \tag{5}$$

Adjoint equations

$$\begin{cases} -L^T\dfrac{d\lambda(t)}{dt} + C^T\displaystyle\int_{t}^{t_f} \lambda(\tau)d\tau = A^T\lambda(t), \\ \lambda(t_f) = L^{-1^T}x^*(t_f). \end{cases} \tag{6}$$

Remark 1. *Equations* (5) *and* (6) *provide the solution for the problem(if it exists). They constitute* $2n$ *second order differential-integral equations whose solution contains* $4n$ *constants of integration. To evaluate these, we have n-equation* $x^*(t_0) = x_0$, *n-equation* $\lambda(t_f) = L^T x^*(t_f)$, *n -equation* $\int_{t_0}^{t} x^*(\tau) d\tau = 0$ *at* $t = t_0$ *and n-equation* $\int_t^{t_f} \lambda(\tau) d\tau = 0$ *at* $t = t_f$. *So, we can solve this problem by sweeps method as follows.*

Step 1. First we solve the adjoint equation (6). Let

$$\lambda_1(t) = \int_t^{t_f} \lambda(\tau) d\tau, \qquad \lambda_2(t) = \frac{d\lambda_1(t)}{dt} = -\lambda(t),$$

then (6) can be written in the following matrix form

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -\left(L^T\right)^{-1} C^T & -\left(L^T\right)^{-1} A^T \end{bmatrix} \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix},$$

with final conditions:

$$\begin{bmatrix} \lambda_1(t_f) \\ \lambda_2(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ -L^T x^*(t_f) \end{bmatrix},$$

which has the solution

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = e^{M_\lambda (t-t_f)} \begin{bmatrix} 0 \\ -L^T x^*(t_f) \end{bmatrix}, \tag{7}$$

with unknown $x^*(t_f)$, where $M_\lambda = \begin{bmatrix} 0 & I \\ -\left(L^T\right)^{-1} C^T & -\left(L^T\right)^{-1} A^T \end{bmatrix}$.

Step 2. We solve the state equation (5). Let

$$x_1(t) = \int_0^t x^*(\tau) d\tau, \qquad x_2^*(t) = \frac{dx_1(t)}{dt} = x^*(t),$$

then (5) can be written in the following nonhomogeneous matrix form

$$\begin{bmatrix} x_1(t) \\ x_2^*(t) \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -L^{-1}C & L^{-1}A \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2^*(t) \end{bmatrix} + \begin{bmatrix} 0 \\ BB^T \end{bmatrix} \lambda_2(t),$$

with initial conditions:

$$\begin{bmatrix} x_1(t_0) \\ x_2^*(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ x_0 \end{bmatrix},$$

which has the solution

$$\begin{bmatrix} x_1(t) \\ x_2^*(t) \end{bmatrix} = e^{M_x t} \begin{bmatrix} 0 \\ x_0 \end{bmatrix} + \int_{t_0}^{t} e^{M_x(t-\tau)} \begin{bmatrix} 0 \\ BB^T \end{bmatrix} \lambda_2(\tau) d\tau, \tag{8}$$

with unknown $x^*(t_f)$, where $M_x = \begin{bmatrix} 0 & I \\ -L^{-1}C & L^{-1}A \end{bmatrix}$.

Step 3. Substitute $t = t_f$ into $x^*(t) = x_2^*(t)$ that we obtained in Step 2, and then solve the algebraic equations with respect to $x^*(t_f)$ to obtain the value of $x^*(t_f)$.

Step 4. Substitute again the value of $x^*(t_f)$ that is obtained in Step 3 in the state and adjoint equations that are obtained in Step 1 and Step 2 to find the exact solution of $v^*(t) = B^T \lambda(t)$, and $x^*(t)$.

Step 5. Substitute the exact solution of $v^*(t)$ and $x^*(t)$ that are obtained in Step 4 in (3) to calculate the minimum energy.

Example 1. (**Minimum energy of RLC series circuit**) In this example, we want to find the unknown

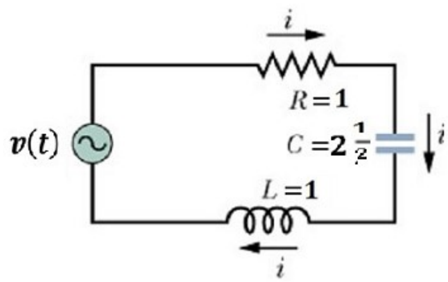

Figure 1: RLC circuit.

supplied voltage $v(t)$ for the RLC circuit in Figure 1, which minimizes the cost functional given by

$$J = \frac{1}{2}i^2(1) + \frac{1}{2}\int_0^{10} v^2(t)dt, \tag{9}$$

with $i(0) = 1$.

By applying the Kirchhoff's voltage law, we get

$$\frac{di}{dt} + \frac{5}{2}\int_0^t i(\tau)d\tau = -i(t) + v(t). \tag{10}$$

In this case, we apply Theorem 2 to the problem (9)-(10), with $t_0 = 0, t_f = 10, x_0 = 1, L = 1$, $C = \frac{5}{2}, A = -1,$ and $B = 1$. Then the optimal control given by

$$v^*(t) = -\lambda(t),$$

where $\lambda(t)$ and $i^*(t)$ satisfy the following equations:

$$\begin{cases} \dfrac{di^*(t)}{dt} + \dfrac{5}{2}\displaystyle\int_0^t i^*(\tau)d\tau = -i^*(t) - \lambda(t), \\ i(0) = 1. \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{d\lambda(t)}{dt} - \dfrac{5}{2}\displaystyle\int_t^1 \lambda(\tau)d\tau = \lambda(t), \\ \lambda(1) = i^*(1). \end{cases} \tag{12}$$

Step 1. By using formula (7), we get the solution of (12) as follows

$$\lambda(t) = i^*(1)e^{\frac{1}{2}t-5}\left[\cos\left(\frac{3}{2}t - 15\right) + \frac{1}{3}\sin\left(\frac{3}{2}t - 15\right)\right]. \tag{13}$$

Step 2. By using formula (8), we get the solution of (11) as below

$$\begin{aligned} i^*(t) = {} & e^{-\frac{1}{2}t}\left[\cos\frac{3}{2}t - \frac{1}{3}\sin\frac{3}{2}t\right] \\ & + \frac{1}{18}x^*(1)e^{-\frac{1}{2}t-5}\left[\cos\left(\frac{3}{2}t + 15\right) - 10\cos\left(\frac{3}{2}t - 15\right) + 3\sin\left(\frac{3}{2}t + 15\right)\right] \\ & + \frac{1}{6}x^*(1)e^{\frac{1}{2}t-5}\left[3\cos\left(\frac{3}{2}t - 15\right)\sin\left(\frac{3}{2}t - 15\right)\right]. \end{aligned} \tag{14}$$

Step 3. Substitute $t = 10$ into $i^*(t)$ that we obtained in Step 2, and then solve the algebraic equations with respect to $i^*(1)$. We get

$$i^*(10) = \frac{6e^5(-3\cos 15 + sin15)}{\cos 30 + 3\sin 30 - 9e^{10} - 10}.$$

Step 4. Substitute again the value of $i^*(10)$ that is obtained in Step 3 in (13) and (14) to find the exact solution of $v^*(t) = -\lambda(t)$ and $i^*(t)$ in Figure 2 and Figure 3.

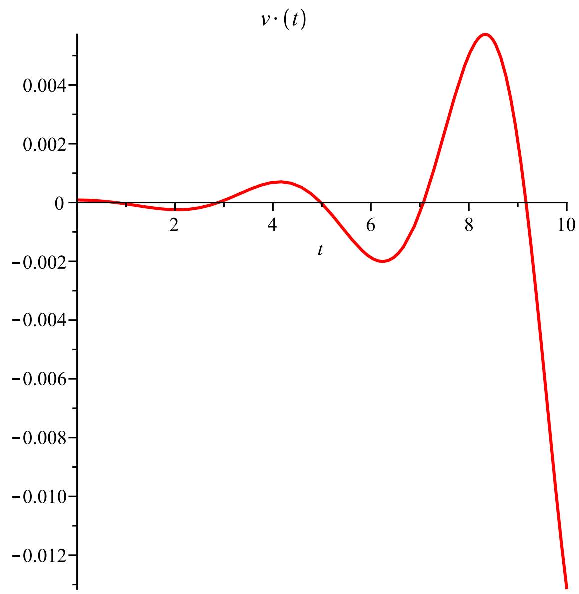

Figure 2: Optimal voltage for minimum energy problem.

Step 5. Substitute the exact solution of $v^*(t)$ and $i^*(t)$ that are obtained in Step 4 to calculate the minimum energy from (9): $\min J = 0.0001731248996$.

4 Time-optimal control problem

Let us consider the following minimum time optimization problem

$$\begin{cases} \min\limits_{v \in W} \int_{t_0}^{t_f} 1dt = \min\limits_{v \in W}(t_f - t_0), \\ L\dfrac{dx(t)}{dt} + C\displaystyle\int_{t_0}^{t} x(\tau)d\tau = Ax(t) + Bv(t), \\ x(t_0) = x_0, \quad x(t_f) \in K, \end{cases} \tag{15}$$

where the admissible class W consists of control functions $v = [v_1, v_2, \ldots, v_m]^T$ with v_i measurable on $[t_0, \infty)$ and $|v_i| \leq 1$ almost everywhere, $i = 1, \ldots, m$ and K is a closed, convex subset of $\mathbb{R}^n$.

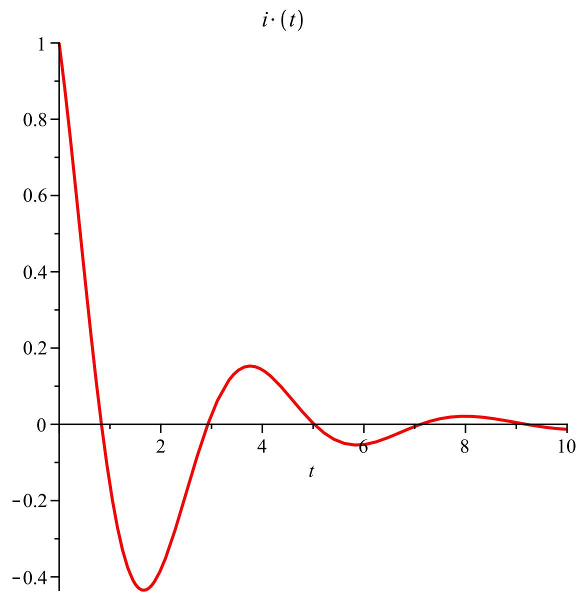

Figure 3: Optimal electrical current for minimum energy problem.

In order to consider a nontrivial problem, we will always assume that the state vector x can be brought from the initial position x_0 to the target set K in a finite amount of time using a certain control function from W.

The optimization problem (15) can be replaced by another equivalent one with a fixed time t_f, to show that we need tow auxiliary lemmas.

Lemma 1. *Let t_f^* be the optimal time for problem* (15)*. If $int\,K \neq \Phi$ then $x(t_f^*) \in \partial K$(boundary of K) for any state x satisfying* (15)*.*

Proof. Any solution of (15) is continuous with respect to t. If $x(t_f^*) \in \partial K$ is not true, then there exists an admissible state x such that the observation $x(t_f^*) \in int\,K$. Thus a $t_f < t_f^*$ exists so that $x(t_f) \in \partial K$. This contradicts the optimality of t_f^*. □

Lemma 2. *Let t_f^* be the optimal time for problem* (15)*. Let v^* and x^* be an optimal control and corresponding state, respectively. Then there exists a nontrivial vector $\eta \in \mathbb{R}^n$ so that the pair (v^*, x^*) is optimal for the following control problem with the fixed time*

$$\begin{cases} \bar{J}(u) = \eta^T x(t^*) \to \min, \\ L\dfrac{dx(t)}{dt} + C\displaystyle\int_{t_0}^{t} x(\tau)d\tau = Ax(t) + Bv(t), \\ x(t_0) = x_0. \end{cases} \tag{16}$$

Proof. The linearity of equations (15) implies that the endpoints $x(t^*)$ of all admissible states $x(t^*)$ form a convex set X_{t^*}. From Lemma 1, we have $X_{t^*} \cap int\,K = \Phi$ and $x(t^*) \in \partial K$. Since $int\,K = \Phi$, there exists a closed hyperplane separating X_{t^*} and K containing $x(t^*)$, i.e. there is a nonzero vector $\eta \in \mathbb{R}^n$ such that [9]

$$\sup_{x\in X_{t^*}} \eta^T x(t^*) \leq \eta^T x^*(t^*) \leq \inf_{x\in K} \eta^T x(t^*).$$

This completes the proof. □

Now, we can apply the general theorem(Theorem 1) to the equivalent problem (16) to get the following result.

Theorem 3. *Let t_f^* be the optimal time for problem* (15)*. Let v^* and x^* be an optimal control and corresponding state, respectively. Then there exist a nontrivial vector $\eta \in \mathbb{R}^n$ so that the pair (v^*, x^*) satisfy the following equations*

State equations

$$\begin{cases} L\dfrac{dx^*(t)}{dt} + C\displaystyle\int_{t_0}^{t} x^*(\tau)d\tau = Ax^*(t) + B(t)v^*(t), \\ x^*(t_0) = x_0. \end{cases}$$

Adjoint equations

$$\begin{cases} -L^T\dfrac{d\lambda(t)}{dt} + C^T\displaystyle\int_{t}^{t_f} \lambda(\tau)d\tau = A^T\lambda(t), \\ \lambda(t_f) = L^{-1^T}\eta. \end{cases}$$

Minimum principle

$$v^*(t^*) = -sgn\left(B^T\lambda(t)\right).$$

Remark 2. *If the set K has a special form $K = \{x \in \mathbb{R}^n : \ |x_i - x_d| \leq \varepsilon\}$ where ε and $x_d \in \Re$ are given, then η_i is known explicitly and is expressed by $\eta_i = x_i(t_f) - x_d$.*

Example 2. In this example, we want to find the unknown supplied voltage $v(t)$ for the RLC circuit in Figure 1, which minimizes the cost functional given by

$$\int_0^{t_f} dt, \tag{17}$$

with

$$i(0) = 1, \quad i(t_f) \in K, \tag{18}$$

where $K = \left\{x = [x_1, ..., x_n]^T \in \mathbb{R}^n : \ |x_i + 0.5| \leq 0.001\right\}$.

In this case, we apply Theorem 3 to problem (17)-(18) and (10), with $t_0 = 0, x_0 = 1, L = 1, C = \frac{5}{2}, A = -1,$ and $B = 1$, then the optimal control is given by

$$v^*(t) = -sgn(\lambda(t)), \tag{19}$$

where $\lambda(t)$ and $i^*(t)$ satisfy the following equations:

$$\begin{cases} \dfrac{di^*(t)}{dt} + \dfrac{5}{2}\displaystyle\int_0^t i^*(\tau)d\tau = -i^*(t) + v^*(t), \\ i^*(0) = 1. \end{cases} \tag{20}$$

$$\begin{cases} \dfrac{d\lambda(t)}{dt} - \dfrac{5}{2}\displaystyle\int_t^{t_f^*} \lambda(\tau)d\tau = \lambda(t), \\ \lambda(t_f^*) = \left(i^*(t_f^*) + 0.5\right). \end{cases} \tag{21}$$

Here, we can do steps as in Example 1.
Step 1. By using formula (7), we get the solution of (21):

$$\lambda(t) = \left(i^*(t_f^*) + 0.5\right) e^{\frac{1}{2}(t-t_f^*)} \left[\cos\frac{3}{2}(t-t_f^*) + \frac{1}{3}\sin\frac{3}{2}(t-t_f^*)\right]. \tag{22}$$

Step 2. $v^*(t) = -sgn(\lambda(t)) = \begin{cases} -1, & \text{if } t_f^* - \frac{2}{3}\tan^{-1}(3) < t \le t_f^*, \\ 0, & \text{if } t = t^* - \frac{2}{3}\tan^{-1}(3), \\ 1, & \text{if } 0 \le t < t_f^* - \frac{2}{3}\tan^{-1}(3). \end{cases}$

Step 3. By using formula (8), we get

$$i^*(t) = \begin{cases} e^{-\frac{1}{2}t}\left[\cos\frac{3}{2}t - \sin\frac{3}{2}t\right], & \text{if if } t_f^* - \frac{2}{3}\tan^{-1}(3) \le t \le t_f^*, \\ e^{-\frac{1}{2}t}\left[\cos\frac{3}{2}t + \frac{1}{3}\sin\frac{3}{2}t\right], & \text{if } 0 \le t < t_f^* - \frac{2}{3}\tan^{-1}(3). \end{cases}$$

Step 4. Calculate t_f^* :

$$\begin{aligned} t_f^* &= \min\left\{t_f : \left|i^*(t_f) + 0.5\right| = 0.001\right\} \\ &= \min\left\{t_f : \left|e^{-\frac{1}{2}t_f}\left[\cos\frac{3}{2}t_f - \sin\frac{3}{2}t_f\right] + 0.5\right| = 0.001\right\} \\ &\approx 0.9187091680. \end{aligned}$$

Step 5. Substitute t_f^* again in Step 2 and Step 3 to get $v^*(t)$ and $i^*(t)$ (Figure 4)

$$v^*(t) = \begin{cases} -1, & \text{if } 0.0860119866 \le t \le 0.9187091680, \\ 1, & \text{if } 0 \le t < 0.0860119866. \end{cases}$$

$$i^*(t) = \begin{cases} e^{-\frac{1}{2}t}\left[\cos\frac{3}{2}t - \sin\frac{3}{2}t\right], & \text{if } 0.0860119866 \le t \le 0.9187091680, \\ e^{-\frac{1}{2}t}\left[\cos\frac{3}{2}t + \frac{1}{3}\sin\frac{3}{2}t\right], & \text{if } 0 \le t < 0.0860119866. \end{cases}$$

5 Conclusion

In this paper, we have extracted the necessary conditions for the optimal control of differential integral systems, and derived two algorithms, one to find the minimum energy in electrical circuit containing a resistance, a coil, and a capacitor (RLC circuit) and the second to find the shortest time for transferring RLC electrical current from one state to another. In the future, one may develop these problems and algorithms to include more general cases.

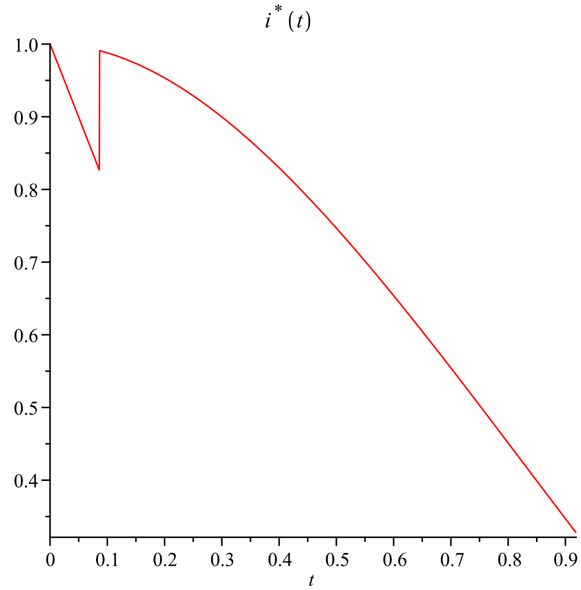

Figure 4: Optimal electrical current for minimum time problem.

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