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Dynamics and bifurcations of a discrete-time neural network model with a single delay

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Abstract. In the present study, we analyze dynamics and bifurcations of a discrete-time Hopfield neural network based on two neurons and the same time delay. We determine stability and bifurcations of the system consisting flip, pitchfork and Neimark-Sacker bifurcations. The normal form coefficients for the all bifurcations are calculated using reducing to the corresponding center manifold, then these coefficients are numerically obtained using MatContM. Numerical analysis validates our analytical results and reveals more complex dynamical behaviors.

Keywords: Delay system, stability, normal form, pitchfork bifurcation, flip bifurcation, Neimark-Sacker bifurcation.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

In 1989, Hopfield reviewed a simplified neural network model. In this system, each neuron was connected to another neuron by a linear circuit consisting of a resistor and a capacitor. He modeled neural networks for continuous values such that the electrical output of each neuron was between 0 and 1. According to his results, this type of network can store preserved states.

For the first time, Markus and Westervelt 1989 applied a time delay in the Hopfield model and observed stable oscillations caused by this time delay. Usually, the time delay arise in some artificial neural networks, such as associative memory, because of information processing. Also, in electronic neural networks, the time delay is caused by the limited switching speed of the amplifiers. In biological networks, it result from the propagation time of signals. Based on this, most neural network models can be described by delay differential equations [1, 5, 7, 20, 25].

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Some dynamical characteristics of time-delay neural networks, such as stability, instability, oscillation, and chaotic behavior, can be identified in many scientific fields such as optimization, intelligent control, associative memory, pattern recognition, etc, and favorable results have also been reported in this field by researchers [2, 9, 23, 26].

The stability of Hopfield neural network (HNN) plays an important role in its applications such as associative addressable memories, pattern recognition, and optimization. For example, information is stored in content-addressable memories as a stable fixed point. Usually, initial studies on stability are first conducted on models without time delay [13–15] and then on models with time delay.

Neural networks are nonlinear and large-scale dynamical systems, which make their dynamics very complex. If a system is accompanied by a time delay, its complexity will increase, so most researchers focus on systems that are structurally simple, such as a delayed neural network with two, three, or four neurons [6, 10, 12, 17, 21, 27]. In systems with multiple delays, a simple structure can be achieved by applying the same time delays. Therefore, in the present study, the HNN with two neurons and the same time delay is considered as follows [16]:

$$\begin{aligned} x_{n+1} &= \gamma x_n + l_{11} r_1(x_{n-k}) + l_{12} r_2(y_{n-k}), \qquad \forall n \geq k, \\ y_{n+1} &= \gamma y_n + l_{21} r_1(x_{n-k}) + l_{22} r_2(y_{n-k}), \qquad \forall n \geq k. \end{aligned} \tag{1}$$

In this system, we let

$$r_1 = r_2 = r, \quad l_{11} = l_{12} = l_{21} = l_{22} = l,$$

and $\gamma \in (0,1)$ which is the internal decay of neurons. Let $L = (l_{ij})_{2*2}$ be the weight matrix, $r : \mathbb{R} \to \mathbb{R}$ shows the output and input activation of the neurons, and it is a function from class $\mathbb{C}^3$ with $r(0) = 0$, and $k \in \mathbb{N}$ represents the delay. The first and second neurons are shown by x and y respectively, l is the connection weights and

$$r(0) = 0, \quad r'(0) \neq 0, \quad r''(0) = 0, \quad r'''(0) \neq 0.$$

By converting system (1) into a system without delay, we will extract the dynamics of the system.

System (1) can be converted to a $2k+2$ dimension system without delay as follows:

$$\begin{cases} x \mapsto \gamma x + l\, r(x_k) + l\, r(y_k), \\ x_1 \mapsto x, \\ x_2 \mapsto x_1, \\ x_j \mapsto x_{j-1}, & j = 3, 4, \ldots, k, \\ y \mapsto \gamma y + l\, r(x_k) + l\, r(y_k), \\ y_1 \mapsto y, \\ y_2 \mapsto y_1, \\ y_j \mapsto y_{j-1}, & j = 3, 4, \ldots, k. \end{cases} \tag{2}$$

Since system (1) is equivalent to system (2), we examine the latter system. System (2) can be rewritten as follows:

$$\begin{pmatrix} x \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ y \\ y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix} \mapsto f(X) = \begin{pmatrix} \gamma x + l\,r(x_k) + l\,r(y_k) \\ x \\ x_1 \\ \vdots \\ x_{k-1} \\ \gamma y + l\,r(x_k) + l\,r(y_k) \\ y \\ y_1 \\ \vdots \\ y_{k-1} \end{pmatrix},$$

where $X = (x, x_1, x_2, \ldots, x_k, y, y_1, y_2, \ldots, y_k, \gamma, l)$.

2 Stability of neural network model

The Jacobian matrix of system (2) at the origin is

$$A = \begin{pmatrix} \gamma & 0 & \cdots & 0 & b & 0 & 0 & \cdots & 0 & b \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & b & \gamma & 0 & \cdots & 0 & b \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where $b = lg'(0)$. We then obtain the characteristic equation of the system:

$$P(\phi) = \phi^k(\phi - \gamma)\big(\phi^k(\phi - \gamma) - 2b\big).$$

It is clear that the characteristic equation has a zero eigenvalue with multiplicity k, and has an eigenvalue $\phi = \gamma$. According to $\gamma \in (0,1)$, this eigenvalue lies inside the unit circle. To check the stability of system (2), we now present the following theorem. See the details of the theorem in reference [22].

Theorem 1. *1. System (2) has an eigenvalue of size one if and only if $b = \frac{c_j}{2}$ for $j = 0, 1, \ldots, k+1$, where $c_j = (-1)^j \sqrt{1 + \gamma^2 - \gamma cos(\psi_j)}$ for $\psi_j \in [0, \pi]$, and $\sin(k+1)\psi_j - \gamma \sin k\psi_j = 0$. Also eigenvalues of the system are not repeated, and $\frac{d|\phi(b)|}{db}\Big|_{b=\frac{c_j}{2}} \neq 0$.*

2. If $b \in (\frac{c_1}{2}, \frac{c_0}{2})$, all eigenvalues of system (2) are inside the unit circle which indicates the fixed point is stable. If $b \in (-\infty, \frac{c_1}{2})$ or $b \in (\frac{c_0}{2}, \infty)$, system (2) has at least one eigenvalue with size greater than one, which shows that the fixed point of the system is unstable.

3. *If* $b = \frac{c_0}{2} = \frac{1-\gamma}{2}$, *system* (2) *has an eigenvalue* $\phi = 1$, *and the size of the rest of the eigenvalues is less than one.*

4. *If* $b = \frac{c_{k+1}}{2} = \dfrac{(-1)^{k+1}(1+\gamma)}{2}$, *system* (2) *has an eigenvalue* $\phi = -1$, *and the size of the rest of the eigenvalues is less than one.*

3 Bifurcations of neural network model

Suppose we have

$$X \mapsto f(X,\alpha), \qquad X \in \mathbb{R}^n, \ \ \alpha \in \mathbb{R}. \tag{3}$$

If in equilibrium point $X = X_0$ for $\alpha = \alpha_0$, eigenvalues of system (3) fall on the unit circle, there is a possibility of bifurcation at these points. In single-parameter bifurcations, we examine the cases where only one eigenvalue is located on the unit circle. Therefore, we consider three situations [18, 19].
1. If we have a simple eigenvalue $\phi = +1$, the reduced system to its central manifold is as:

$$w \mapsto w + a_{fold}w^2 + b_{pitch}w^3 + \mathscr{O}(w^4), \quad w \in \mathbb{R}^1.$$

If a_{fold} becomes non-zero, there is a possibility of fold bifurcation, but if a_{fold} becomes zero and b_{pitch} is non-zero, there is a possibility of pitchfork bifurcation. If $b_{pitch} \neq 0$ and the transversality condition $\dfrac{d|\phi(\alpha)|}{d\alpha}\big|_{\alpha=\alpha_0} \neq 0$, was established, then pitchfork bifurcation occurs.
2. If we have a simple eigenvalue $\phi = -1$, the reduced system to its central manifold is as:

$$w \mapsto -w + b_{flip}w^3 + \mathscr{O}(w^4), \quad w \in \mathbb{R}^1. \tag{4}$$

If $b_{flip} \neq 0$ and the transversality condition $\dfrac{d|\phi(\alpha)|}{d\alpha}\big|_{\alpha=\alpha_0} \neq 0$, was established, then flip bifurcation occurs.
3. If we have a pair of simple complex eigenvalues $\phi = \pm e^{i\theta_0}$, the reduced system to its central manifold is as:

$$w \mapsto we^{i\theta_0}(1 + d_{NS}|w|^2) + (\mathscr{O}|w|^4), \quad w \in \mathbb{C} \ \text{ and } \ e^{im_0\theta_0} \neq 1 \ \text{ for } \ m_0 = 1,2,3,4,$$

where d_{NS} is

$$d_{NS} = \frac{1}{2}e^{-i\theta_0}\langle \eta, G(\mu,\mu,\bar{\mu}) + 2H(\mu,(I_n - J)^{-1}H(\mu,\bar{\mu}) + H(\bar{\mu},(e^{2i\theta_0}I_n - J)^{-1}H(\mu,\mu))\rangle,$$

η is the left eigenvector, μ is the right eigenvector, H and G are the components of multilinear functions.

If $\Re(d_{NS}) \neq 0$ (d_{NS} is the first Lyapunov coefficient of the Neimark-Sacker Bifurcation) and the transversality condition $\dfrac{d|\phi(\alpha)|}{d\alpha}\big|_{\alpha=\alpha_0} \neq 0$, was established, then Neimark-Sacker bifurcation occurs.

To calculate the normal forms near the bifurcation point, we need to calculate Taylor expansion of $f(x)$ around the fixed point $X^0 = (x^0, x_1^0, \ldots, x_k^0, y^0, y_1^0, \ldots, y_k^0, \gamma^0, l^0)$. So we have

$$f(x + X^0) = X^0 + Jx + \frac{1}{2}H(x,x) + \frac{1}{6}G(x,x,x) + \cdots,$$

where $J = f_x(X^0)$ and the components of multilinear functions are defined as follows:

$$H_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 f_i(X^0)}{\partial \vartheta_j \partial \vartheta_k} x_j y_k,$$

$$G_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 f_i(X^0)}{\partial \vartheta_j \partial \vartheta_k \partial \vartheta_l} x_j y_k z_l.$$

By the central manifold theorem [3], we now determine the bifurcations such as pitchfork, flip, Neimark-Sacker, their direction and their normal form coefficients [8, 18, 19, 24]. Then we examine the obtained result by numerical simulation.

3.1 Pitchfork bifurcation

In this subsection, we examined pitchfork bifurcation by deriving the following theorem.

Theorem 2. *In system (2), if $b = \frac{c_0}{2} = \frac{1-\gamma}{2}$ and $b_{pitch} \neq 0$, then pitchfork bifurcation occurs at the origin. If $b_{pitch} < 0$ ($b_{pitch} > 0$), then the pitchfork is subcritical (supercritical).*

Proof. If $b = \frac{c_0}{2} = \frac{1-\gamma}{2}$, system (2) has an eigenvalue $\phi = 1$, and the size of the rest of the eigenvalues is less than one. Therefore, in this case, the reduced system to the central manifold is as follows:

$$w \mapsto w + a_{fold} w^2 + \mathscr{O}(w^3), \quad w \in \mathbb{R}^1,$$

in which [19]

$$a_{fold} = \frac{1}{2!} \langle \eta, H(\mu, \mu) \rangle,$$

where η and μ are the left and right eigenvectors corresponding to the eigenvalue $\phi = 1$. The corresponding left and right eigenvectors are given by [19]:

$$J\mu = \mu, \qquad J^T \eta = \eta, \qquad \langle \mu, \mu \rangle = 1, \qquad \langle \eta, \mu \rangle = 1,$$

where

$$\mu = \frac{\mu_1}{\sqrt{2k+2}}, \qquad \eta = \frac{\sqrt{2k+2}\ \eta_1}{2k+2-2k\gamma},$$

and

$$\mu_1 = (\underbrace{1, 1, \ldots, 1}_{2k+2})^T, \qquad \eta_1 = (\underbrace{1, 1-\gamma, 1-\gamma, \ldots, 1-\gamma}_{k+1}, \underbrace{1, 1-\gamma, 1-\gamma, \ldots, 1-\gamma}_{k+1}).$$

According to the fact that in this system $H(\mu, \mu)$ is zero, the generalized fold bifurcation occurs, and the pitchfork bifurcation appears. Its normal form is as follows:

$$w \mapsto w + b_{pitch} w^3 + \mathscr{O}(w^4), \quad w \in \mathbb{R}^1,$$

and its coefficient of the normal form is as follows:

$$b_{pitch} = \frac{1}{6} \langle \eta, G(\mu, \mu, \mu) \rangle = \frac{2l r'''(0)}{3(2k+2-2k\gamma)(2k+2)}.$$

If $b_{pitch} < 0$ ($b_{pitch} > 0$), the pitchfork is subcritical (supercritical). □

3.2 Flip bifurcation

In this subsection, we stated and proved the following theorem about flip bifurcation.

Theorem 3. *In system (2), if*

$$b = \frac{c_{k+1}}{2} = \frac{(-1)^{k+1}(1+a)}{2},$$

and $b_{flip} \neq 0$ then flip bifurcation occurs at the origin, and if $b_{flip} > 0$ ($b_{flip} < 0$), the flip is supercritical (subcritical) bifurcation.

Proof. If

$$b = \frac{c_{k+1}}{2} = \frac{(-1)^{k+1}(1+\gamma)}{2},$$

system (2) has an eigenvalue $\phi = -1$, and the size of the rest of the eigenvalues is less than one. Therefore, in this case, the reduced system to the central manifold is as follows:

$$w \mapsto -w + b_{flip}w^3 + \mathscr{O}(w^4), \quad w \in \mathbb{R}^1. \tag{5}$$

In equation (5), the critical coefficient of flip bifurcation is as follows:

$$b_{flip} = \frac{1}{6}\langle \eta, G(\mu,\mu,\mu) + 3H(\mu,(I-J)^{-1}H(\mu,\mu))\rangle,$$

which the left and right eigenvectors are as follows:

$$J\mu = -\mu, \qquad J^T\eta = -\eta, \qquad \langle \mu,\mu\rangle = 1, \quad \langle \eta,\mu\rangle = 1,$$

where

$$\mu = \frac{\mu_1}{\sqrt{2k+2}}, \qquad \eta = \frac{\eta_1}{(-1)^k(2+2k+2k\gamma)},$$

and

$$\mu_1 = \Big(\underbrace{(-1)^k,(-1)^{k-1},\dots,1}_{k+1},\underbrace{(-1)^k,(-1)^{k-1},\dots,1}_{k+1}\Big)^T,$$

$$\eta_1 = \Big(\underbrace{1,(-1-\gamma),(1+\gamma),\dots,(-1)^{k-1}(-1-\gamma)}_{k+1},\underbrace{1,(-1-\gamma),(1+\gamma),\dots,(-1)^{k-1}(-1-\gamma)}_{k+1}\Big)^T.$$

Then the coefficient of normal form is obtained as follows:

$$b_{flip} = \frac{4lr'''(0)}{6(-1)^k(2k+2)(2+2k+2k\gamma)}.$$

If $b_{flip} > 0$ ($b_{flip} < 0$), the fixed point of the system become unstable (stable) and the quasi-cycle of two periodicities created become stable (unstable). This phenomenon is called super-critical (sub-critical) bifurcation. □

3.3 Neimark-Sacker bifurcation

In this subsection, we prove the following theorem about Neimark-Sacker bifurcation.

Theorem 4. *In system (2), if $b = \frac{c_j}{2}$, $j = 1,\ldots,k$, and $\Re(d_{NS}) \neq 0$, (d_{NS} is the first Lyapunov coefficient of Neimark-Sacker Bifurcation) Neimark-Sacker bifurcation occurs at the origin. If $\Re(d_{NS}) < 0$ ($\Re(d_{NS}) > 0$), a stable (unstable) cycle forms, and the fixed point of the origin becomes unstable (stable). It is called supercritical (subcritical) Neimark-Sacker bifurcation.*

Proof. System (2) has an eigenvalue of size one if and only if $b = \frac{c_j}{2}$, $j = 1,\ldots,k$, and these eigenvalues are not repeated, and also $\frac{d|\phi(b)|}{db}\big|_{b=\frac{c_j}{2}} \neq 0$. Therefore, system (2) has the Niemarsaker bifurcation, and the reduced system to the central manifold is as follows:

$$w \mapsto we^{i\theta_0}(1 + d_{NS}|w|^2) + (\mathscr{O}|w|^4), \quad w \in \mathbb{C} \text{ and } e^{im_0\theta_0} \neq 1 \text{ for } m_0 = 1,2,3,4,$$

that d_{NS} is a complex number which is obtained as follows:

$$d_{NS} = \frac{1}{2}e^{-i\theta_0}\langle \eta, G(\mu,\mu,\bar{\mu})\rangle = \frac{e^{-i\theta} l r'''(0)}{3\langle \mu_1,\mu_1\rangle\langle \eta_1,\mu_1\rangle},$$

which the left and right eigenvectors are as follows:

$$J\mu = e^{i\theta_0}\mu, \qquad J^T\eta = e^{-i\theta_0}\eta, \qquad \langle \mu,\mu\rangle = \langle \eta,\mu\rangle = 1,$$

where

$$\mu = \frac{\mu_1}{\sqrt{\langle \mu_1,\mu_1\rangle}}, \qquad \eta = \frac{\eta_1}{\langle \eta_1,\mu\rangle},$$

and

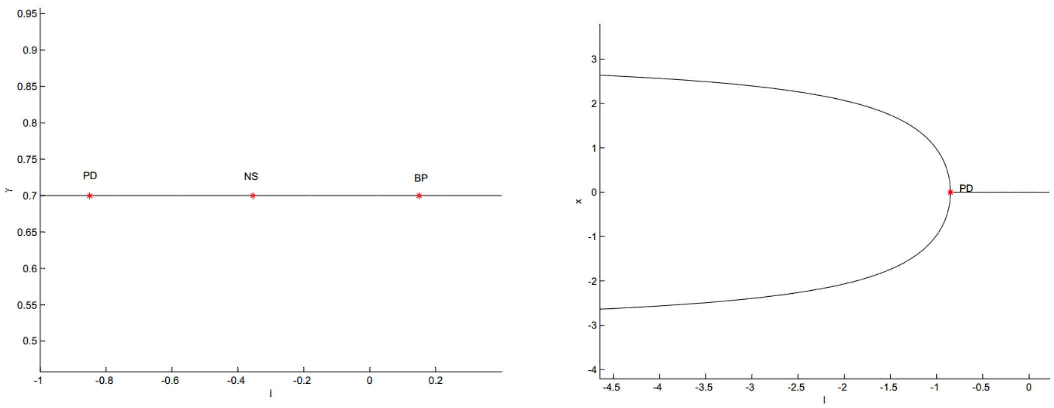
$$\begin{aligned} \mu_1 &= \left(e^{ik\theta}, e^{i(k-1)\theta},\ldots,1,e^{ik\theta},e^{i(k-1)\theta},\ldots,1\right)^T, \\ \eta_1 &= \Big(1, e^{-i\theta}-\gamma, e^{-i\theta}(e^{-i\theta}-\gamma),\ldots,e^{-(k-1)i\theta}(e^{-i\theta}-\gamma),1,e^{-i\theta}-\gamma, \\ &\qquad e^{-i\theta}(e^{-i\theta}-\gamma),\ldots,e^{-(k-1)i\theta}(e^{-i\theta}-\gamma)\Big)^T. \end{aligned}$$

If $\Re(d_{NS}) < 0$ ($\Re(d_{NS}) > 0$), a stable (unstable) cycle forms, and the fixed point of the origin becomes unstable (stable). It is called supercritical (subcritical) Neimark-Sacker bifurcation. □

4 Numerical simulation

We consider system (1) in the following special case:

$$\begin{aligned} x_{n+1} &= \gamma x_n + l\ \sin(x_{n-2}) + l\ \sin(y_{n-2}), \qquad \forall n \geq 2, \\ y_{n+1} &= \gamma y_n + l\ \sin(x_{n-2}) + l\ \sin(y_{n-2}), \qquad \forall n \geq 2, \end{aligned} \tag{6}$$



(a) (b)

Figure 1: (a) The bifurcations curve for $k = 2$ and $\gamma = 0.7$. (b) Bifurcation diagram for system (6) in near flip bifurcation point for $\gamma = 0.7$ and $l = -0.85$.

where $k = 2$ and $r(x) = \sin(x)$.

We convert system (6) into a discrete system without delay, and considering l as a free parameter and $\gamma = 0.7$. We numerically study bifurcations of the system such as pitchfork, flip, and Neimark-Sacker bifurcations by using the MATLAB package MATCONTM [4,11]:

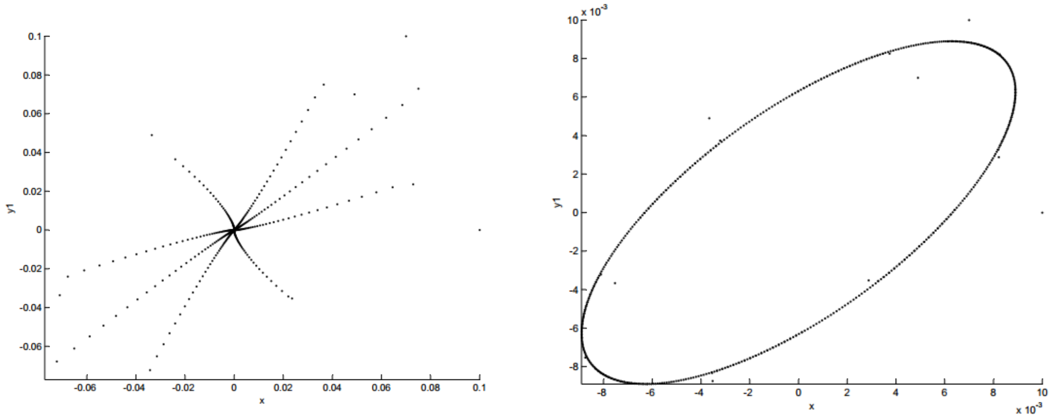
1. If $l = -0.85$, flip bifurcation occurs, and its normal form coefficient is $b_{flip} = 1.073232 \times 10^{-2}$.
2. If $l = 0.15$ pitchfork bifurcation occurs, and its normal form coefficient is $b_{pitch} = -5.2083 \times 10^{-3}$.
3. If $l = -0.354741$, Neimark-Sacker bifurcation occurs, and its normal form coefficient is $\Re(d_{NS}) = -2.5022 \times 10^{-2}$.

In case (1), the created quasi-cycle of two periodicities is stable, and the fixed point of the origin becomes unstable. This phenomenon is called the supercritical flip bifurcation. In case (2) the stable fixed point of the origin becomes three equilibrium points, and the fixed point of the origin becomes unstable. In case (3) an stable cycle occurs, and the fixed point of the origin becomes unstable. This phenomenon is called the supercritical Neimark-Sacker bifurcation.

5 Conclusions

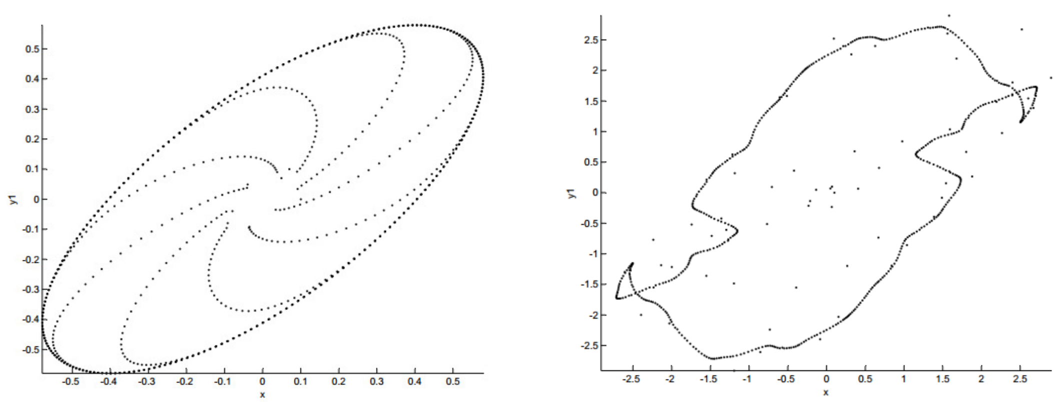
In this article, we investigated the codim-1 bifurcation such as flip, pitchfork, and Neimark-Sacker bifurcation in the two-neuron, one-delay Hopfield discrete-time neural network. We calculated the normal form coefficients bifurcations numerically and analytically by the central manifold and at the end, we determined the stability of the fixed point of origin in Neimark-Sacker and flip bifurcations according to the direction of the bifurcations. Also, we drew the Neimark-Sacker and flip bifurcation in the phase portrait and the bifurcation curve in the parameter space.

It can be seen in the example mentioned in the numerical simulation section that, at the origin equilibrium point, if $l = 0$, all the eigenvalues are inside the unit circle. When the first eigenvalue is placed on the unit circle and the bifurcation occurs, stability of the origin equilibrium point changes, so at $l = 0.15$ and $l = -0.354741$, where the pitchfork and Neimark-Sacker bifurcations occur, stability of the origin equilibrium point changes. Therefore, the origin becomes unstable. The cycle created in the Neimark-



(c) (d)

Figure 2: (c) For $l = -0.34$ and initial point $(0.1, 0, 0, 0.1, 0, 0)$, the fixed point of system (6) is stable. (d) Neimark-Sacker bifurcation with $l = -0.354741$ and normal form coefficient $\Re(d_{NS}) = -2.5022 \times 10^{-2}$, in phase portrait.



(e) (f)

Figure 3: (e) For $l = -0.37$ and initial point $(0.1, 0, 0, 0.1, 0, 0)$, the fixed point of system (6) is unstable. (f) Flip bifurcation with $\gamma = 0.7$ and $l = -0.85$ and normal form coefficient $b_{flip} = 1.073232 \times 10^{-2}$, in phase portrait.

Sacker bifurcation continues to grow as l decreases until it is deformed at $l = -0.85$, where the flip bifurcation occurs. After that, the cycle is broken and chaos occurs.

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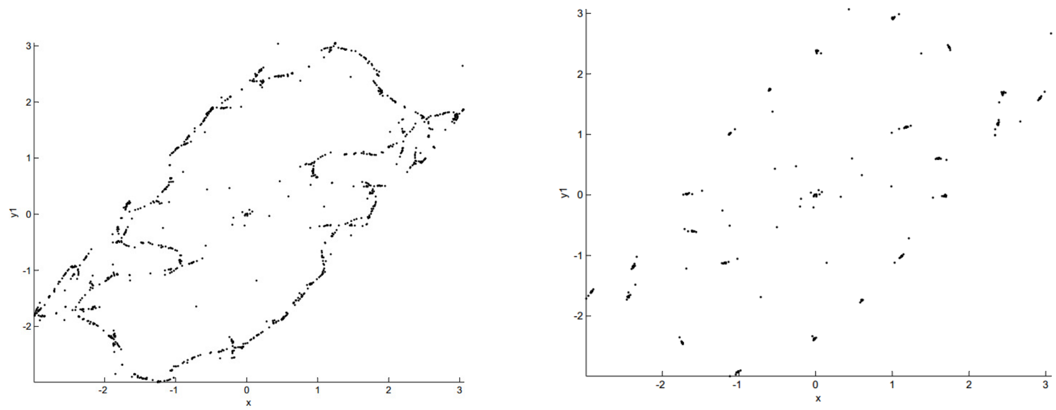

(g) (h)

Figure 4: (g) The phase portrait of the system (6) when $\gamma = 0.7$ and $l = -0.91$. (h) The phase portrait of the system (6) when $\gamma = 0.7$ and $l = -0.915$.

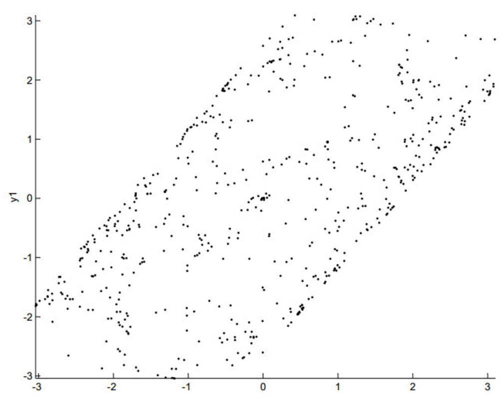

Figure 5: Breaking the quasi-cycle created in the flip bifurcation and creating chaos at $\gamma = 0.7$ and $l = -0.92$.

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