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# Complexity analysis of primal-dual interior-point methods for convex quadratic programming based on a new twice parameterized kernel function

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**Abstract.** In this paper, we present primal-dual interior-point methods (IPMs) for convex quadratic programming (CQP) based on a new twice parameterized kernel function (KF) with a hyperbolic barrier term. To our knowledge, this is the first KF with a twice parameterized hyperbolic barrier term. By using some conditions and simple analysis, we derive the currently best-known iteration bounds for large- and small-update methods, namely, $\mathbf{O}\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ and $\mathbf{O}\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$, respectively, with special choices of the parameters. Finally, some numerical results regarding the practical performance of the new proposed KF are reported.

*Keywords*: Convex quadratic programming, kernel function, interior-point methods, large- and small-update methods.
*AMS Subject Classification 2010*: 90C20, 90C25, 90C51.

## 1 Introduction

In this paper, we deal with primal-dual IPMs for solving the standard CQP problem

$$\min\left\{c^T x+\frac{1}{2}x^T Qx\,:\,Ax=b,\;x\geq 0\right\},\tag{P}$$

and its dual as

$$\max\left\{b^T y-\frac{1}{2}x^T Qx\,:\,A^T y-Qx+s=c,\;s\geq 0\right\},\tag{D}$$

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where $Q \in \mathbb{S}_+^n$, $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A) = m, b \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, and $x, c, s \in \mathbb{R}^n$ with $m \leq n$. If $Q = 0$, we see that our problem reduces to linear programming (LP), so, we can say that CQP is a generalization of LP.

Throughout the paper, we assume that the matrix $A$ has full row rank, i.e., rank($A$)= $m \leq n$ and both problems (P) and (D) satisfy the interior point condition (IPC), i.e., there exists a strictly feasible point, namely, $(x^0, y^0, s^0)$ such that

$$Ax^0 = b,\ x^0 > 0, \quad A^T y^0 - Qx^0 + s^0 = c,\ s^0 > 0.$$

CQP appears in many areas of applications, for example in optimal control, economics, finance, agriculture, geometric problems and also as sub-problems in sequential quadratic programming. In the last two decades, powerful mathematical approaches have been suggested for solving CQP problems. IPMs are one of the effective tools for this purpose. The revolution of IPMs started with the seminal work of Karmarkar [14], who proposed an efficient polynomial-time IPM for LP and invented his famous algorithm for LP.

Then, the concept of primal-dual IPMs was first introduced by Kojima et al. [16] and Megiddo [18]. After that, Nestrov et al. [19] developed the IPMs from LP to convex programming problems such as CQP, semidefinite programming (SDP) and second order cone programming (SOCP).

KFs play an important role in the analysis of IPMs and the determination of the search directions. Peng et al. in [20,21] introduced and analyzed for the first time primal-dual IPMs for LP problems based on the so-called self-regular (SR) KFs. They successively improved the theoretical complexity bound from $\mathbf{O}\big(n\log\frac{n}{\varepsilon}\big)$ to the currently best-known iteration bound, namely, $\mathbf{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big)$ for large-update methods.

Since then, several KFs have been introduced, they differs from the types of their barrier term, polynomial [3,6], exponential [9,10,26,30], trigonometric [4,5,13,15] and the last introduced type, that is the hyperbolic [8,23–25,28,29].

The first primal-dual IPM for CQP, based on a KF, was proposed by Wang et al. in [27]. The proposed KF already used for LP and SDP, covers the classical logarithmic KF and the SR function. Later on, Cai et al. [9] presented a primal-dual IPM for CQP based on a finite barrier term. The theoretical results show that their algorithms have the best-known iteration bound. Very recently, Boudjellal et al. also suggested other KFs for CQP in [6,7].

The main goal of this paper is to introduce a new twice parametric KF with a hyperbolic barrier term. For the development of primal-dual IPMs to solve CQP problems, we analyze the properties of the proposed KF. More precisely, we prove that, for large-update methods, the corresponding algorithm has $\mathbf{O}\big((p+q)\big(2+\frac{1}{q}\log n\big)^2\sqrt{n}\log\frac{n}{\varepsilon}\big)$ iteration bound. An interesting choice, where $p = q = \log(n)$, leads to the best iteration complexity bound for large-update methods, namely, $\mathbf{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big)$. Table 1 gathers all twice KFs existing in the literature, to our knowledge.

This paper is organized as follows; In Section 2, we briefly describe the generic primal-dual IPMs for solving CQP problems and its algorithm. In Section 3, we present our new twice KF and some of its properties, we also compute the growth of the proximity function. In Section 4, we talk about the step size. In Section 5, we provide the worst-case iteration complexity bounds for large- and small-update methods. Section 6 contains some numerical experiments and commentaries. Finally, we finish the paper with some concluding remarks.

Throughout the paper, we use the following notations. The non-negative and positive orthants are denoted by $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$, respectively. $\|\cdot\|$ denotes the Euclidian norm in $\mathbb{R}^n$, $\langle\cdot,\cdot\rangle$ denotes the inner

Table 1: All twice KFs existing in the literature with their complexity bounds for large-update methods.

| KFs | Complexity bound | The best complexity bound | Ref. |
|---|---|---|---|
| $\frac{t^2-1}{2}-\int_1^t e^{p(x^{-q}-1)}dx, p\geq 1, q\geq 1$ | $\boldsymbol{O}\big(pq\sqrt{n}\big(1+\frac{1}{p}\log n\big)^{1+\frac{1}{q}}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big)$ $p=\log(n), q=1$ | [26] |
| $\frac{t^{1+p}-1}{p+1}+\frac{e^{q(1-t)}-1}{q}, p\in[0,1], q\geq 1$ | $\boldsymbol{O}\big(pn^{\frac{1}{1+q}}\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), p=1, q=\log(n)$ | [11] |
| $\frac{t^{p+1}-1}{p+1}+\frac{t^{1-q}-1}{q-1}, p\in[0,1], q>1$ | $\boldsymbol{O}\big(qn^{\frac{p+q}{q(p+1)}}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), p=1, q=\log(n)$ | [3] |
| $\frac{t^{p+1}-1}{p+1}+q(e^{t^{-\frac{1}{q}}-1}-1), 0\leq p\leq 1, 0\leq q\leq 1$ | $\boldsymbol{O}\big(\frac{(1+2\kappa)}{q}n^{\frac{1}{1+p}}(\log n)^{1+q}\log\frac{n\mu_0}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log(n)\log(\log(n))\log(\frac{n\mu_0}{\varepsilon})\big)$ | [10] |
| $\frac{p(t^2-1)}{2}+\frac{t^{-pq}-1}{q(q+1)}-\frac{pq((t-1))}{q+1}, p\geq 1, q>0$ | $\boldsymbol{O}\big(p(pq+1)(q+1)^{\frac{1}{pq+1}}(pn)^{\frac{pq+2}{2(pq+1)}}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), p=1, q=\frac{\log(n)}{2}-1$ | [17] |
| $t^2-1+\frac{t^{1-p}-1}{p-1}+\frac{t^{1-q}-1}{q-1}, p,q\geq 1$ | $\boldsymbol{O}\big((p+1)n^{\frac{p+1}{2(p-q)}}\log\frac{n}{\varepsilon}\big)$ | – | [1] |
| $t^2+\frac{t^{1-p}}{q-1}-\frac{q}{q-1}+\frac{4}{\pi p}\Big(\tan\big(\frac{\pi}{2t+2}\big)-1\Big), p\geq 2, q\geq 1$ | $\boldsymbol{O}\big(pn^{\frac{(p+2)(q+1)}{2((p+1)q)}}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), p=q=\log(n)$ | [4] |
| $\frac{p(t^2-1)}{2}+\frac{p(t^{-pq+1}-1)}{(pq-1)(q+1)}-\frac{pq\log(t)}{(q+1)}, p\geq 1, q\geq 1$ | $\boldsymbol{O}\big(pq(p+1)(q+1)^{\frac{1}{pq}}n^{\frac{pq+1}{2pq}}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), p=1, q=\log(n)$ | [31] |
| $\frac{p}{2}(t^2-1)+\frac{e^{t^{-pq}-1}-1}{pq}-(p-1)(t-1), p,q\geq 1$ | $\boldsymbol{O}\big(\sqrt{n}(\log(n))^{\frac{pq+1}{pq}}\log\frac{n}{\varepsilon}\big)$ | – | [30] |
| $\frac{t^2-1}{2}+\frac{(q+2)^2\Big(\cot^p\big(\frac{\pi}{q+2}\big)\tan^p\big(\frac{\pi}{qt+2}\big)-1\Big)}{\pi pq\big(\cot\big(\frac{\pi}{q+2}\big)+\tan\big(\frac{\pi}{q+2}\big)\big)}, p\geq 2, q>0$ | $\boldsymbol{O}\big(\frac{n^{\frac{p+2}{2(p+1)}}}{g(p,q)}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), p=q=\log(n)$ | [5] |
| $\frac{t^2-1}{2}+\int_1^t\Big(\frac{p\tanh x-\coth^q x}{\coth^q 1-p\tanh 1}\Big)dx, p\in[0,1], q\geq\frac{\sinh 2}{2}$ | $\boldsymbol{O}\big(qn^{\frac{q+1}{2q}}\log\frac{n}{\varepsilon}\big)$ | $\boldsymbol{O}\big(\sqrt{n}\log n\log\frac{n}{\varepsilon}\big), q=\frac{\log n}{2}$ | [8] |

product and $e$ denotes the $n$-dimensional vector of ones. The set of all $(m \times n)$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$, $\mathbb{S}^n_+$ denotes the cone of positive semidefinite matrices in the real space of $(n \times n)$ symmetrical matrices $\mathbb{S}^n$. For a vector $x \in \mathbb{R}^n$, $X = \mathrm{diag}(x)$ denotes the $n \times n$ diagonal matrix whose diagonal entries are the components of $x$. For given vectors $x$ and $s$, the vectors $xs$ and $\frac{x}{s}$ denote the coordinate-wise operations on the vectors $x, s$.

For two functions $f,\, g : \mathbb{R}^n_{++} \to \mathbb{R}$, we say that, $f(x) = \mathbf{O}(g(x))$ if $f(x) \leq Cg(x)$ for some positive constant $C$ and $f(x) = \Theta(g(x))$ if $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for some positive constants $C_1$ and $C_2$.

## 2 Preliminaries

In this section, we describe the main idea of IPMs based on KFs for solving CQP problems. We also provide the structure of the generic primal-dual interior-point algorithm.

Finding an optimal solution of (P) and (D) is equivalent to solve system of the Karush-Kuhn-Tucker (KKT) optimality conditions

$$\begin{aligned} Ax &= b, \quad x \geq 0,\\ A^T y - Qx + s &= c, \quad s \geq 0,\\ xs &= 0. \end{aligned}$$

The main idea behind primal-dual IPMs is to replace the complementarity condition $xs = 0$ with the equation $xs = \mu e$, where $\mu > 0$ is a parameter, which we obtain the following perturbed system

$$\begin{aligned} Ax &= b, \quad x > 0,\\ A^T y - Qx + s &= c, \quad s \geq 0,\\ xs &= \mu e. \end{aligned} \tag{1}$$

Since the matrix $A$ is full rank and the IPC holds, system (1) has a unique solution $(x(\mu), y(\mu), s(\mu))$ for any $\mu > 0$, where $x(\mu)$ is called the $\mu$-center of (P) and $(y(\mu), s(\mu))$ is called the $\mu$-center of (D). The set of $\mu$-centers gives a homotopy path which is called the central path of the problems (P) and (D). When $\mu \to 0$, the limit of the central path exists and converges to an analytic center of the optimal solutions set of (P) and (D) for CQP.

Then, the new iterate is computed as

$$x_+ = x + \alpha\Delta x, \quad s_+ = s + \alpha\Delta s, \quad y_+ = y + \alpha\Delta y, \tag{2}$$

where $0 < \alpha \leq 1$ is the step size, then the new iterate satisfies $(x^+, s^+) > 0$. In fact, we may assume that $x^0 = s^0 = e$ and $\mu^0 = 1$.

For a fixed $\mu > 0$, applying Newton's method on (1) provides the following system for the search directions $(\Delta x, \Delta y, \Delta s)$

$$\begin{aligned} A\Delta x &= b,\\ A^T\Delta y - Q\Delta x + \Delta s &= 0,\\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \tag{3}$$

System (3) has a unique solution $(\Delta x, \Delta y, \Delta s)$, called Newton's direction.

Therefore, we define the following scaled vector $v$ and the scaled search directions $d_x$ and $d_s$ by

$$v = \sqrt{\frac{xs}{\mu}}, \quad d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s}. \tag{4}$$

After some elementary calculations, system (3) can be represented as follows

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y - \bar{Q}d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{5}$$

whereat,

$$\bar{A} = \frac{1}{\mu}AV^{-1}X = AS^{-1}V, \quad \bar{Q} = \mu VS^{-1}QVS^{-1}.$$

It can be easily seen that $d_x = d_s = 0$ if and only if $v^{-1} - v = 0$ if and only if $x = e$ if and only if $x = x(\mu)$, $s = s(\mu)$.

We can observe that the right-hand side in the third equation in (5) equals minus the gradient of the classical logarithmic barrier function $\Psi_c(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ defined as

$$\Psi_c(v) := \Psi_c(x,s;\mu) = \sum_{i=1}^{n} \psi_c(v_i) = \sum_{i=1}^{n} \left( \frac{v_i^2 - 1}{2} - \log v_i \right), \; v \in \mathbb{R}^n_{++}.$$

That is to say

$$v^{-1} - v = -\nabla\Psi_c(v),$$

we say that $\psi_c(t)$ is the KF of the logarithmic barrier function $\Psi_c(v)$.

We replace the right-hand-side of the last equation in (5) by $-\nabla\Psi(v)$: the negative gradient of the barrier function $\Psi(v)$ whose KF is defined in (6). Then, system (5) becomes

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y - \bar{Q}d_x + d_s &= 0, \\ d_x + d_s &= -\nabla\Psi(v). \end{aligned}$$

This system has a unique solution $(dx, \Delta y, ds)$ for each $\mu > 0$. If $(x,y,s) \neq (x(\mu), y(\mu), s(\mu))$, then $(\Delta x, \Delta y, \Delta s)$ is nonzero.

The only difference comparing with the LP case is that, in the CQP analysis, we loose the orthogonality of the search direction vectors $d_x$ and $d_s$. Since $\bar{Q}$ is symmetric positive semidefinite matrix, we have

$$d_x^T d_s = d_s^T d_x = \left(\bar{Q}d_x - \bar{A}^T\Delta y\right)^T d_x = d_x^T \bar{Q} d_x \geq 0.$$

Despite this difference, almost all theoretical results on iterate directions developed for LP hold for CQP.

Now, we can outline the above procedure in the following primal-dual interior-point algorithm. Algorithm 1 has an inner and an outer iteration. Each outer iteration consists of an update of the parameter $\mu$ and a sequence of (one or more) inner iterations. At the start of an outer iteration of Algorithm 1 and just before updating the parameter $\mu$ with the factor $1-\theta$, one has $\Psi(v) \leq \tau$, for given $\tau$. Due to the

**Algorithm 1** A generic primal-dual interior-point algorithm for CQP.

**Input**

a threshold parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta \in ]0,1[$;
$(x^0, y^0, s^0)$ satisfy the IPC and $\mu^0 = 1$ such that $\Psi(x^0, y^0; \mu^0) := \Psi(v^0) \leq \tau$.

**begin**

$x := x^0; s := s^0; \mu := \mu^0$;

**while** $n\mu \geq \varepsilon$ **do**

**begin** (outer iteration )

$\mu := (1-\theta)\mu$;

**while** $\Psi(x, y; \mu) := \Psi(v) > \tau$ **do**

**begin** (inner iteration)

$x = x + \alpha\Delta x$;
$y = y + \alpha\Delta y$;
$s = s + \alpha\Delta s$;
$v := \sqrt{\frac{xs}{\mu}}$

**end** (inner iteration)

**end** (outer iteration)

$\mu$-update, the vector $v$ is divided by a factor $\sqrt{1-\theta}$, which leads to an increase in the value of $\Psi(v)$, in general. The subsequent inner iterations are performed in order to decrease the values of $\Psi(v)$ until it passes the threshold $\tau$ again, i.e., $\Psi(v) \leq \tau$. So, the largest values of the proximity function $\Psi(v)$ occur just after $\mu$-update. At this stage, we have found an $\varepsilon-$ optimal solution of problems (P) and (D).

# 3 The new twice parameterized KF and its properties

In this section, we introduce the following new KF and investigate its properties

$$\psi_{pq}(t) = \frac{t^2-1}{2} - \tanh^p(1) \int_1^t \coth^p(x) e^{qc(\coth(x)-\coth(1))} dx, \quad t > 0, \quad p \geq 1, \quad q \geq 1, \tag{6}$$

where

$$c = \frac{1}{\coth^2(1) - 1}.$$

Now, we need to compute the first three derivatives of function (6). In fact, we have for all $t > 0$

$$\psi'_{pq}(t) = t - \tanh^p(1) \coth^p(t) e^{qc(\coth(t)-\coth(1))}, \tag{7}$$

$$\psi''_{pq}(t) = 1 + \tanh^p(1) K(t) \coth^{p-1}(t) \Big( p + qc\coth(t) \Big) e^{qc(\coth(t)-\coth(1))}, \tag{8}$$

$$\psi_{pq}'''(t) = -\tanh^p(1)K(t)\Big(\big(p + qc\coth(t)\big)\big(2\coth^2(t) + K(t)((p-1) + qc\coth(t)\big) + qc\coth(t)K(t)\Big)e^{qc(\coth(t)-\coth(1))},$$

where

$$K(t) = \coth^2(t) - 1 = \frac{1}{\sinh^2(t)}.$$

We can see that $\psi_{pq}''(t) > 1$ and $\psi_{pq}'''(t) < 0$, for all $t > 0$.

If we take $t \geq 1$ in (6), we get $x \geq 1$. We know that the function $x \to \coth(x)$ is decreasing on $(0, +\infty)$, which implies that $\coth(x) \leq \coth(1)$. Thus

$$\psi_{pq}(t) \geq \frac{t^2-1}{2} - \int_1^t dx = \frac{t^2-1}{2} - (t-1),$$

which implies that $\lim_{t\to+\infty} \psi_{pq}(t) = +\infty$.

We recall from [28] that

$$2x\coth(x) - 1 > 0, \quad x > 0. \tag{9}$$

When we take $t < 1$ in (6), we have for all $x \in [t, 1]$, $\coth(x) \geq \coth(1)$. Thus,

$$\int_t^1 \coth^p(x)e^{qc(\coth(x)-\coth(1))}dx \geq \int_t^1 \coth(x)dx.$$

This gives

$$\psi_{pq}(t) > \frac{t^2-1}{2} + \frac{\tanh^p(1)}{2}\int_t^1 \frac{1}{x}dx = \frac{t^2-1}{2} - \frac{\tanh^p(1)\ln(t)}{2},$$

which leads to $\lim_{t\to 0^+} \psi_{pq}(t) = +\infty$. Thus, we can say that $\psi_{pq}$ is a KF.

**Lemma 1.** *Let $\psi_{pq}$ be the function defined in (6). Then, we have*

$$ct(\coth^2(t) - 1) - 1 > 0, \quad i.e. \quad ctK(t) - 1 > 0, \quad \forall t < 1. \tag{10}$$

*Proof.* We define

$$g(t) = ct(\coth^2(t) - 1) - 1,$$

then, we have

$$g'(t) = c(\coth^2(t) - 1)(1 - 2t\coth(t)) < 0.$$

We see from (9) that $g$ is decreasing, then for all $t < 1$, we obtain $g(t) > g(1) = 0$. □

**Lemma 2.** *For the function $\psi_{pq}$ defined by (6), we have*

$$t\,\psi_{pq}''(t) - \psi_{pq}'(t) > 0, \quad t > 0, \tag{11}$$

$$t\,\psi_{pq}''(t) + \psi_{pq}'(t) > 0, \quad t > 0, \ (i.e., \ \psi_{pq} \text{ is exponentially convex}). \tag{12}$$

*Proof.* We use the first two derivatives of $\psi_{pq}$ defined in (7) and (8). First, we have for (11)

$$t\,\psi''_{pq}(t)-\psi'_{pq}(t)=\tanh^p(1)\left(K(t)t\big(p+qc\coth(t)\big)+\coth(t)\right)e^{qc(\coth(t)-\coth(1))}>0.$$

The inequality (12) is obvious for $t\geq 1$ since in that case $\psi'_{pq}(t)\geq 0$.

For $t<1$, we have

$$t\,\psi''_{pq}(t)+\psi'_{pq}(t)=2t+\tanh^p(1)\Big(ptK(t)+\coth(t)\big(qc\,tK(t)-1\big)\Big)e^{qc(\coth(t)-\coth(1))},$$

which gives the desired inequality thanks to (10) since $q\geq 1$, this completes the proof. $\square$

In what follows, we define $\rho:[0,+\infty)\to(0,1]$ as the inverse function of $t\in(0,1]\mapsto -\frac{1}{2}\psi'_{pq}(t)$. We have the following lemma.

**Lemma 3.** *For all $z\geq 0$, we have*

$$e^{qc(\coth(\rho(z))-\coth(1))}\leq\frac{2z+1}{\tanh^p(1)\coth^p(\rho(z))}, \tag{13}$$

$$\coth(\rho(z))\leq\coth(1)+\frac{1}{q}\log(2z+1). \tag{14}$$

*Proof.* Let $z\geq 0$, thus there exists a unique $t\in]0,1]$ such that $z=-\frac{1}{2}\psi'_{pq}(t)$. By the definition of $\psi'_{pq}$, we have

$$-\left(t-\tanh^p(1)\coth^p(t)e^{qc\left(\coth(t)-\coth(1)\right)}\right)=2z,$$

then, we obtain

$$\tanh^p(1)\coth^p(t)e^{qc(\coth(t)-\coth(1))}\leq 2z+1.$$

Since $\dfrac{1}{\tanh^p(1)\coth^p(t)}\leq 1$, for all $t\leq 1$, we have

$$e^{qc(\coth(t)-\coth(1))}\leq\frac{2z+1}{\tanh^p(1)\coth^p(t)}\leq 2z+1. \tag{15}$$

Taking the logarithm of both sides and since $c>1$, we get

$$q(\coth(t)-\coth(1))\leq\log(2z+1). \tag{16}$$

Then, putting $t=\rho(z)$ in (15) and (16), respectively, we get the desired inequalities. $\square$

The following lemma gives some other properties of the new proposed KF.

**Lemma 4** (Lemma 3.4 in [22])**.** *For any $v\in\mathbb{R}^n_{++}$, one has*

1. $\dfrac{1}{2}(t-1)^2\leq\psi_{pq}(t)\leq\dfrac{1}{2}(\psi'_{pq}(t))^2, \qquad t>0.$
2. $\Psi(v)\leq 2\delta(v)^2.$

*3.* $\|v\| \leq \sqrt{n} + \sqrt{2\Psi(v)} \leq \sqrt{n} + 2\delta(v)$,

*where* $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ *is the norm-based proximity measure defined as*

$$\delta(v) := \frac{1}{2}\| - \nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'_{pq}(v_i))^2}.$$

**Remark 1.** *We assume that* $\Psi(v) \geq \tau \geq 1$. *Using the second item of Lemma 4, we conclude that*

$$\delta(v) \geq \frac{1}{2}. \tag{17}$$

## 3.1 The growth of the proximity function

In this subsection, we discuss the growth behavior of the proximity function after $\mu$-update. Note that, at the start of the algorithm and before updating the barrier parameter $\mu$, we have $\Psi(v) \leq \tau$ for a fixed parameter $\tau$. When $\mu$ is updated by factor $1-\theta$ for $\theta \in (0,1]$, the vector $v$ is updated as $v_+ = \frac{v}{\sqrt{1-\theta}}$. It causes an increase in the value of $\Psi(v)$, in general. Therefore, the subsequent inner iterations are performed in order to bring the values of $\Psi(v)$ back to the situation, where $\Psi(v) \leq \tau$. Here, we study the effect of barrier update on the proximity function. To this end, we present two technical results.

**Lemma 5.** *For given* $\beta \geq 1$, *one has*

$$\psi_{pq}(\beta t) \leq \psi_{pq}(t) + \frac{1}{2}(\beta^2 - 1)t^2, \quad t > 0.$$

*Proof.* Let us write $\psi_{pq}(t)$ as

$$\psi_{pq}(t) = \frac{t^2 - 1}{2} + \varphi_{pq}(t),$$

where

$$\varphi_{pq}(t) = \psi_{pq}(t) - \frac{t^2-1}{2} = -\tanh^p(1)\int_1^t \coth^p(x)e^{qc(\coth(x)-\coth(1))}dx.$$

One sees easily that $\varphi_{pq}$ is a decreasing function. Since $\beta \geq 1$, we conclude the desired inequality. □

**Theorem 1** (Lemma 4.2 in [22])**.** *For any* $v \in \mathbb{R}^n_{++}$ *and* $\beta \geq 1$, *one has*

$$\Psi(\beta v) \leq \Psi(v) + \frac{\beta^2 - 1}{2}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

**Corollary 1.** *For any* $0 \leq \theta < 1$ *and* $v_+ = \beta v$ *with* $\beta = \dfrac{1}{\sqrt{1-\theta}}$, *one has*

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

# 4 An estimation for the step size

In this section, we estimate the value of the step size $\alpha$. So, we need to investigate the inner iteration of the algorithm. The new points are constructed as $(x_+, s_+)$ defined in (2) and the proximity function $\Psi$ decreases sufficiently. Due to (4), we can write the new iterate as

$$x_+ = \frac{x}{v}(v + \alpha d_x), \quad s_+ = \frac{s}{v}(v + \alpha d_s).$$

We assume that throughout the paper the step size $\alpha$ satisfies

$$v + \alpha d_x > 0, \quad v + \alpha d_s > 0.$$

We define the vector $v_+ := \sqrt{\frac{x_+ s_+}{\mu}}$. Then, we have

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

Applying the exponentially convex property, we get an upper bound for the function $\Psi(v_+)$ as follows

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \frac{1}{2}\big[\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\big].$$

We define the function $f(\alpha)$ as the difference of proximities between a new iterate and a current iterate for a fixed $\mu$

$$f(\alpha) = \Psi(v_+) - \Psi(v) \leq f_1(\alpha),$$

where $f_1(\alpha)$ is the convex function

$$f_1(\alpha) := \frac{1}{2}\big[\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\big] - \Psi(v).$$

Here, we give the first derivative of the function $f_1(\alpha)$ with respect to $\alpha$ as

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\big(\psi_{pq}'(v_i + \alpha d_{x_i})d_{x_i} + \psi_{pq}'(v_i + \alpha d_{s_i})d_{s_i}\big).$$

Then, we get

$$f_1'(0) = \frac{1}{2}\langle \nabla\Psi(v), (d_x + d_s)\rangle = -\frac{1}{2}\langle \nabla\Psi(v), \nabla\Psi(v)\rangle = -2\delta(v)^2.$$

To simplify, we use the following notations; $\delta := \delta(v)$ and $\Psi := \Psi(v)$. Then, we consider some important results for estimating the step size $\alpha$.

**Lemma 6** (Lemma 4.3 in [2])**.** *Let $\rho : [0,\infty) \to (0,1]$ be the inverse of the function $t \in (0,1] \mapsto -\frac{1}{2}\psi'(t)$, the largest possible value for the step size $\bar{\alpha}$ is given by*

$$\bar{\alpha} = \frac{\rho(\delta) - \rho(2\delta)}{2\delta}. \tag{18}$$

**Lemma 7** (Lemma 4.4 in [2]). *Let $\bar{\alpha}$ and $\rho$ be defined as in Lemma 6. Then, we have*

$$\bar{\alpha} \geq \frac{1}{\psi''_{pq}(\rho(2\delta))}. \tag{19}$$

Using the above lemma, we have the following value for the step size

$$\tilde{\alpha} := \frac{1}{\psi''_{pq}(\rho(2\delta))}. \tag{20}$$

Due to (19) and (20), it is easily seen that $\tilde{\alpha} \leq \bar{\alpha}$.
Here, we demonstrate the step size during an inner iteration.

**Lemma 8.** *Assume that $\Psi(v) \geq \tau \geq 1$, and let $\rho$ and $\tilde{\alpha}$ be defined as in Lemma 6 and Eq. (20) respectively. Then, we have*

$$\tilde{\alpha} \geq \Theta\left(\frac{1}{(p+q)\delta\left(2+\frac{1}{q}\log(4\delta+1)\right)^2}\right). \tag{21}$$

*Proof.* Here, we use the second derivative of $\psi_{pq}$. Based on the inequalities (13), (14) with $z = 2\delta$, and the following inequalities

$$K(\rho(2\delta)) \leq K(\rho(2\delta)) + 1 = \coth^2(\rho(2\delta)),$$

$$p + qc\coth(\rho(2\delta)) \leq 2(p+q)\coth(\rho(2\delta)),$$

we obtain

$$\begin{aligned}
\psi''_{pq}(\rho(2\delta)) &= 1 + \tanh^p(1)(p + qc\coth(t))\coth^{p-1}(\rho(2\delta))K(\rho(2\delta))e^{qc(\coth(\rho(2\delta))-\coth(1))} \\
&\leq 1 + \tanh^p(1)c(p+q)\coth^{p+2}(\rho(2\delta))e^{qc(\coth(\rho(2\delta))-\coth(1))} \\
&\leq 1 + 2\tanh^p(1)(p+q)\coth^{p+2}(\rho(2\delta))e^{qc(\coth(\rho(2\delta))-\coth(1))} \\
&\leq 1 + 2(p+q)(4\delta+1)\left(\coth(1) + \frac{1}{q}\log(4\delta+1)\right)^2.
\end{aligned} \tag{22}$$

From (22) and Remark 1, we get a lower bound for $\tilde{\alpha}$

$$\begin{aligned}
\tilde{\alpha} = \frac{1}{\psi''_{pq}(\rho(2\delta))} &\geq \left(\frac{1}{2(p+q)\left(2+\frac{1}{q}\log(4\delta+1)\right)^2(4\delta+1)}\right) \\
&\geq \Theta\left(\frac{1}{(p+q)\delta\left(2+\frac{1}{q}\log(4\delta+1)\right)^2}\right),
\end{aligned}$$

which gives the result. □

The following technical lemma is crucial for obtaining the decrease of the proximity function in the inner iteration.

**Lemma 9** (Lemma 4.5 in [2])**.** *For $\alpha$ satisfies $\alpha \leq \bar{\alpha}$, we have*

$$f(\alpha) \leq -\alpha\delta^2.$$

As a consequence of Lemma 9 and the second item of Lemma 4, we conclude the following corollary.

**Corollary 2.** *Let $\tilde{\alpha}$ be given by (20). Then, we have*

$$f(\tilde{\alpha}) \leq \Theta\left(-\frac{\delta}{(p+q)\left(2+\frac{1}{q}\log(4\delta+1)\right)^2}\right) \leq \Theta\left(-\frac{\sqrt{\Psi}}{(p+q)(2+\frac{1}{q}\log(\sqrt{\Psi}+1))^2}\right). \tag{23}$$

## 5 Iteration complexity

Thanks to Corollary 1, after $\mu$-update, an upper bound for the proximity function is given by

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

Now, we need to count the number of inner iterations that are required to return the iterates back to the situation where $\Psi(v) \leq \tau$ after the $\mu$-update. Then, we denote the value of the proximity function $\Psi(v)$ after updating $\mu$ by $\Psi_0$, and the subsequent values by $\Psi_k$, for all $k = 1, \ldots, K-1$, where $K$ denotes the total number of inner iterations performed in an outer iteration. The following inequality is obtained by using Corollary 1 and the fact that $\Psi(v) \leq \tau$

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2n\tau} + n\right). \tag{24}$$

The decrease of $\Psi$ in each inner iteration is given by this inequality

$$\Psi_{k+1} \leq \Psi_k - \frac{\sqrt{\Psi_k}}{(p+q)\left(2+\frac{1}{q}\log(\sqrt{\Psi_k}+1)\right)^2} \leq \Psi_k - \frac{\sqrt{\Psi_k}}{(p+q)\left(2+\frac{1}{q}\log(\sqrt{\Psi_0}+1)\right)^2},$$

for all $0 \leq k \leq K-1$.

Applying Lemma A.2 in [2] with $t_\kappa = \Psi_\kappa$, $\beta = \dfrac{1}{(p+q)(2+\frac{1}{q}\log(\sqrt{\psi_0}+1))}$ and $\gamma = \frac{1}{2}$, we get the following lemma.

**Lemma 10.** *Let $\mu$ be updated by factor $\mu_+ = (1-\theta)\mu$ for some $\theta \in (0,1)$. Then, the total number of inner iterations in the outer iteration of Algorithm 1 is given by*

$$K \leq 2(p+q)\left(2+\frac{1}{q}\log(\sqrt{\Psi_0}+1)\right)^2\sqrt{\Psi_0}. \tag{25}$$

The following theorem presents an upper bound for the total number of iterations used by the algorithm which is computed by multiplying the total number of inner iterations given by $K$ with the number of outer iterations. An upper bound for the number of required outer iterations is given by

$$\mathbf{O}\left(\frac{1}{\theta}\log\frac{n}{\varepsilon}\right),$$

**Theorem 2** (Lemma II.17 in [21])**.** *The total number of iterations to get an $\varepsilon$-solution, i.e., a solution that satisfies $x^T s = n\mu \leq \varepsilon$, is bounded by*

$$\boldsymbol{O}\left((p+q)\big(2+\frac{1}{q}\log(\sqrt{\Psi_0}+1)\big)^2\sqrt{\Psi_0}\frac{\log\frac{n}{\varepsilon}}{\theta}\right).$$

For large-update methods, we have $\tau = \mathbf{O}(n)$ and $\Theta = \theta(1)$. Thus, we get from (24) that $\Psi_0 = \mathbf{O}(n)$. Taking $p = q = \mathbf{O}\big(\log n\big)$, we obtain the following upper bound for the total number of inner iterations in an outer iteration

$$\mathbf{O}\left(\sqrt{n}\log n \log\frac{n}{\varepsilon}\right).$$

**Remark 2.** *For small-update methods, we have $\tau = \boldsymbol{O}(1)$ and $\Theta = \theta\left(\frac{1}{\sqrt{n}}\right)$. Since we have ([12, Section 4.2])*

$$\Psi_0 \leq \frac{\theta n + 2\sqrt{2\tau n} + 2\tau}{2(1-\theta)},$$

*then, $\Psi_0 = \boldsymbol{O}(1)$, which gives the following complexity bound*

$$\boldsymbol{O}\left((p+q)\sqrt{n}\log\frac{n}{\varepsilon}\right).$$

*Letting $p = q = \boldsymbol{O}(1)$. Then the worst case iteration complexity for small-update IPMs is given by $\boldsymbol{O}(\sqrt{n}\log n)$, which matches to the currently best known iteration bound for small-update methods.*

## 6 Numerical tests

In this section, we present some numerical results of Algorithm 1 with our new twice KF and the KF introduced in [13]

$$\psi_{Fp}(t) = 2\pi\int_1^t \left(\tan\left(\frac{\pi x}{2x+2}\right) - \cot^{3p}\left(\frac{\pi x}{2x+2}\right)\right)dx, \quad p \geq 2.$$

The numerical results are obtained using MATLAB R2017b environment. All examples are taken from [6]. The values of the parameters are considered as $\tau = 5n$, $\varepsilon = 10^{-8}$, $\theta = 0.99$, $p \in \big\{2, 4, \log(n)\big\}$ and $q \in \big\{1, 2, \log(n)\big\}$.

We choose a step size $\alpha$ satisfying $0 < \alpha < \bar{\alpha}$ and $\alpha = \min(\alpha_x, \alpha_s)$ with

$$\alpha_x = \min_{i=1..n}\begin{cases} -\dfrac{x_i}{\Delta x_i}, & \text{if} \quad \Delta x_i < 0, \\ 1, & \text{elsewhere}, \end{cases}$$

$$\alpha_s = \max_{i=1..n}\begin{cases} -\dfrac{s_i}{\Delta s_i}, & \text{if} \quad \Delta s_i < 0, \\ 1, & \text{elsewhere}. \end{cases}$$

**Example 1.** $m = 2, n = 3,$

$$Q = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where the initial feasible solutions are defined as follows

$$x^0 = \left(0.3262, 1.3261, 0.3477\right)^t, \ s^0 = \left(0.7247, 0.7247, 2.0722\right)^t, \ y^0 = \left(0, 2.0721\right)^t.$$

The obtained primal-dual optimal solution is

$$x^* = \left(0.5000, 1.4999, 0.0001\right)^t, \ s^* = \left(0.0000, 0.0000, 0.9998\right)^t, \ y^* = \left(0.0000, -0.9997\right)^t.$$

**Example 2.** $m = 2, n = 4,$

$$Q = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} -4 \\ -6 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 2 \\ 5 \end{pmatrix},$$

where the initial feasible solutions are defined as follows

$$x^0 = \left(0.9683, 0.5775, 0.4543, 1.1444\right)^t, \ y^0 = \left(-0.9184, -1.1244,\right)^t,$$
$$s^0 = \left(0.7612, 0.9141, 0.9185, 1.1244\right)^t.$$

The obtained primal-dual optimal solution is

$$x^* = \left(1.1290, 0.7742, 0.0968, 0.0000\right)^t, \ s^* = \left(0.0000, 0.0000, 0.0000, 1.0322\right)^t,$$
$$y^* = \left(0.0001, -1.0322\right)^t.$$

**Example 3.** $m = 3, n = 5,$

$$Q = \begin{pmatrix} 20 & 1.2 & 0.5 & 0.5 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1.5 \\ 2 \\ 1.5 \\ 3 \end{pmatrix},$$
$$A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 9.31 \\ 5.45 \\ 6.60 \end{pmatrix},$$

where the initial feasible solutions are defined as follows

$$x^0 = \left(2.4539, 0.7875, 1.5838, 2.4038, 1.3074\right)^t, \ y^0 = \left(20.5435, 9.4781, 4.3927\right)^t,$$
$$s^0 = \left(7.1215, 7.9763, 8.3150, 6.8686, 7.9750\right)^t.$$

The obtained primal-dual optimal solution is

$$
\begin{aligned}
x^* =& \left(2.6321, 0.7019, 1.3994, 2.4643, 1.0846\right)^t,\\
s^* =& 10^{-8}\left(0.0064, 0.1056, 0.1161, 0.0037, 0.1863\right)^t,\\
y^* =& \left(25.2686, 11.7725, 5.2567\right)^t.
\end{aligned}
$$

**Example 4.** $m = 3$, $n = 10$

$$
Q = \begin{pmatrix}
30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1\\
1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1\\
1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1\\
1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1\\
1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1\\
1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1\\
1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & -1\\
1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11
\end{pmatrix}, \quad c = \begin{pmatrix} -0.5\\ -1\\ 0\\ 0\\ -0.5\\ 0\\ 0\\ -1\\ -0.5\\ -1 \end{pmatrix},
$$

$$
A = \begin{pmatrix}
1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05\\
1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1\\
1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1
\end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 11.651\\ 16.672\\ 21.295 \end{pmatrix},
$$

where the initial feasible solutions are defined as follows

$$
\begin{aligned}
x^0 =& \left(0.949,\ 0.612,\ 1.847,\ 1.811,\ 1.251,\ 2.521,\ 1.506,\ 1.565,\ 0.820,\ 1.128\right)^t,\\
y^0 =& \left(4.3800,\ 19.9367,\ 4.5679\right)^t,\\
s^0 =& \left(3.890,\ 4.462,\ 3.978,\ 3.660,\ 3.901,\ 3.556,\ 3.876,\ 3.719,\ 3.913,\ 4.339\right)^t.
\end{aligned}
$$

The obtained primal-dual optimal solution is

$$
\begin{aligned}
x^* &= \left(0.9639,\ 0.5096,\ 1.7399,\ 1.9049,\ 1.2376,\ 2.6266,\ 1.6170,\ 0.9043,\ 0.8240,\ 0.8976\right)^t,\\
s^* &= 10^{-9}\left(0.0872,\ 0.4924,\ 0.1398,\ 0.0633,\ 0.0768,\ 0.0326,\ 0.3057,\ 0.0530,\ 0.1010,\ 0.3057\right)^t,\\
y^* &= \left(4.2429,\ 22.3606,\ 5.1916\right)^t.
\end{aligned}
$$

**Example 5.** (Variable size)

$$
n = 2m, \quad A(i,j) = \begin{cases} 0, & \text{if} \quad i \neq j \text{ and } j \neq i+m,\\ 1, & \text{if} \quad i = j \text{ or } j = i+m, \end{cases}
$$

$c(i) = -1$, $c(i+m) = 0$, $b(i) = 2$, for $i = 1,\ldots,m$. $Q(i,j) = 0$, for $i,j = 1,\ldots,n$. The initial strictly feasible interior point is $x^0(i) = x^0(i+m) = 1$, $y^0(i) = -2$, $s^0(i) = 1$, $s^0(i+m) = 2$, for $i = 1,\ldots,m$. The obtained primal-dual optimal solution is $x^*(i) = 2$, $x^*(i+m) = 0$, $y^*(i) = -1$, $s^*(i) = 0$, $s^*(i+m) = 1$, for $i = 1,\ldots,m$.

Table 2: Number of inner iterations for $p = 2$.

| Examples | $\psi_{Fp}(t)$ | $\psi_{p1}(t)$ | $\psi_{p2}(t)$ | $\psi_{p\log(n)}(t)$ |
|---|---|---|---|---|
| Example 1 | 33 | **15** | 18 | **15** |
| Example 2 | 29 | **20** | 21 | 25 |
| Example 3 | 34 | **15** | **15** | **15** |
| Example 4 | 40 | **12** | **12** | **12** |
| Example 5 $(n,m)$ $(20,10)$ | 27 | **20** | **20** | 23 |
| $(100,50)$ | 27 | **20** | **20** | 23 |
| $(200,100)$ | 27 | **22** | **22** | 29 |
| $(400,200)$ | 29 | **22** | **22** | 29 |
| $(1000,500)$ | 29 | **22** | **22** | 29 |
| $(2000,1000)$ | 29 | **22** | **22** | 29 |

Table 3: Number of inner iterations for $p = 4$.

| Examples | $\psi_{Fp}(t)$ | $\psi_{p1}(t)$ | $\psi_{p2}(t)$ | $\psi_{p\log(n)}(t)$ |
|---|---|---|---|---|
| Example 1 | 338 | 16 | **15** | 16 |
| Example 2 | 648 | 30 | **29** | **29** |
| Example 3 | 4662 | 22 | **16** | **16** |
| Example 4 | – | **12** | **12** | **12** |
| Example 5 $(n,m)$ $(20,10)$ | 127 | **27** | 31 | 31 |
| $(100,50)$ | 148 | **27** | 31 | 31 |
| $(200,100)$ | 148 | **27** | 31 | 31 |
| $(400,200)$ | 148 | **30** | 33 | 33 |
| $(1000,500)$ | 148 | **30** | 33 | 33 |
| $(2000,1000)$ | 148 | **30** | 33 | 33 |

### Comments

For each example, we used **bold** font to highlight the best, i.e., the smallest iteration number. From Tables 2-4 we may conclude a few remarks.

• For all proposed examples, the algorithm based on the new KF requires less number of iterations to obtain the optimal solution comparing with the algorithm based on the KF proposed in [13]. This proves the efficiency of the proposed KF and evaluate its effect on the behavior of the algorithm.

• For $p = q = \log n$, $\psi_{pq}$ and $\psi_{Fp}$ meet the best theoretical complexity bound, but in Table 4, we can observe that $\psi_{pq}$ outperformed $\psi_{Fp}$ especially in Example 4 and in Example 5 for large sizes. In fact, the algorithm based on the KF $\psi_{Fp}$ requires large number of iterations to obtain the optimal solution when $n \geq 400$.

• For both functions $\psi_{pq}$ and $\psi_{Fp}$, the number of iterations clearly depends on the value of the parameter $p$. It should also be noted that the value 2 of $p$ significantly reduces the number of iterations although it does not have the best complexity bound theoretically.

Table 4: Number of inner iterations for $p = \log(n)$.

| Examples | $\psi_{Fp}(t)$ | $\psi_{p1}(t)$ | $\psi_{p2}(t)$ | $\psi_{p\log(n)}(t)$ |
|---|---|---|---|---|
| Example 1 | 20 | **16** | **16** | **16** |
| Example 2 | **24** | 25 | **24** | 27 |
| Example 3 | 19 | 16 | **15** | **15** |
| Example 4 | 213 | **12** | **12** | **12** |
| Example 5 $(n,m)$ $(20,10)$ | 29 | **23** | 24 | 27 |
| $(100,50)$ | 33 | **27** | 33 | **27** |
| $(200,100)$ | 75 | **33** | **33** | **33** |
| $(400,200)$ | – | **33** | **33** | **33** |
| $(1000,500)$ | – | **33** | **33** | **33** |
| $(2000,1000)$ | – | **33** | **33** | **33** |

## 7 Concluding remarks

This paper studied complexity analysis of primal-dual IPMs for CQP based on a new twice parameterized KF that has a hyperbolic function in its barrier term. By means of some simple analysis tools, we investigated several properties of this KF. We computed the worst-case iteration complexity bounds and proved that Algorithm 1 enjoys $\mathbf{O}\big(\sqrt{n}\log n \log \frac{n}{\varepsilon}\big)$ iterations bound for large-update methods. For small-update methods, we obtain the best-know iteration bound, namely, $\mathbf{O}\big(\sqrt{n}\log \frac{n}{\varepsilon}\big)$, by taking $(p+q) = \mathbf{O}(1)$. Finally, we present some numerical results to show the practical behavior of the new proposed KF.

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