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# Convergence analysis of compact finite difference method for the solution of anti-periodic boundary value problems

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**Abstract.** The main objective of this paper is to introduce the fourth and sixth-order compact finite difference methods for solving anti-periodic boundary value problems. Compact finite difference formulas can approximate the derivatives of a function more accurately than the standard finite difference formulas for the same number of grid points. The convergence analysis of the proposed method is also investigated. This analysis shows how the error between the approximate and exact solutions decreases as the grid space is reduced. To validate the proposed method's accuracy and efficiency, some computational experiments are provided. Moreover, a comparison is performed between the standard and compact finite difference methods. The experiments indicate that the compact finite difference method is more accurate and efficient than the standard one.

*Keywords*: Anti-periodic boundary value problems, finite difference method, compact finite difference method, convergence analysis.
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## 1 Introduction

Engineering and scientific applications of anti-periodic boundary value problems (BVPs) include optimal control, physics, neural network, etc [1, 14]. Therefore, the study of anti-periodic BVPs has been an interesting and attractive topic for researchers in recent years. Our goal in this research is to focus on the following linear anti-periodic BVP:

$$\begin{cases} y''(t) + p(t)y(t) = f(t), \quad t \in [a,b], \\ y(a) + y(b) = 0, \\ y'(a) + y'(b) = 0, \end{cases} \tag{1}$$

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where $p(t)$ is a continuous function on $(a,b)$ and $f(t)$ is a source function.

Many researchers investigated the existence and uniqueness of the solution to these problems. Aftabizada et al. [4], are introduced the existence and uniqueness of a special form of anti-periodic BVPs. Sun et al. [17] and Zang et al. [20] studied the existence and stability of solutions for systems of anti-periodic fractional BVPs.

Due to the widespread use of anti-periodic BVPs, researchers have made many efforts to obtain efficient numerical approximations of the problem. For instance, the least squares method and spline [2], the repeating shooting method [4], and the Galerkin method [3]. However it is clear that the convergence rate of the least squares method by spline is constant, and also in the repeating shooting method, it is pretty difficult to calculate the initial guesses like $y(a)$ and $y'(a)$.

The compact finite difference (CFD) method, as a modified form of the standard finite difference (FD) method, has attracted the attention of researchers in recent decades. Some of them are reviewed in the sequel. Spotz and Carey [16] developed the CFD method using governing equations and estimated the truncation error terms. Deriaz [7] applied the CFD method of arbitrary order to the Poisson equation in arbitrary dimensions. To solve the Helmholtz equation, Sutmann [18] used the sixth-order CFD method. Turkle et al. [19] applied the CFD approach for computing the solutions of the two and three-dimensional Helmholtz equation. Kumar et al. [9] developed an optimal new sixth-order accurate CFD method for two and three dimensional Helmholtz equation. Li et al. [13] employed a fourth-order CFD method to solve two-dimensional convectiondiffusion equation. Biazar and Asayesh [5] used the fourth and sixth-order CFD methods to solve the Helmholtz equation based on the Sine transform. Gatiso et al. [8] utilized the Sine transform of the CFD method to solve the Poisson equations with Dirichlet boundary conditions. Yang and Zhao [14] solved differential equations by the fourth-order CFD method with some free parameters. In the field of numerical modeling for fully wet porous fins with different profile shapes, Hashemi et al. [10] utalized iterative CFD method, also Hashemi et al. [11] used iterative FD approach to compute the dual solutions for the problem of mixed convection flow through a porous medium. Malele et al. [15] considered some differential equations with Neumann and Robin boundary conditions and solved them using a high-order compact approach.

Motivated by the above discussion, the main purpose of this paper is to investigate an efficient approximation for the solution of anti-periodic BVP (1) using the FD and CFD methods. We will explore the accuracy and efficiency of these methods in solving anti-periodic BVPs and compare their performance to the second and fourth-order traditional FD methods. In addition, the convergence analysis of the proposed method is studied in detail. The outline of the paper is as follows. Section 2 is specified for elaborating the traditional central FD of the second and fourth-order accuracy, and CFD methods. In Section 3, the convergence analysis of the proposed method is discussed. In Section 4, some benchmark examples of anti-periodic BVPs are solved to demonstrate the accuracy and efficiency of the methods and confirm the theoretical results. Finally, the conclusions are presented in Section 5.

## 2 Description of method

In this section, the FD and CFD methods are respectively introduced to approximate anti-periodic BVP (1). Let the dependent variable $y(t)$ be defined on the interval $[a,b]$. We consider the partition $a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N = b$ with grid space $h = (b-a)/N$, for $N \in \mathbb{N}$, and suppose that $y_i$ and $y''_i$ are the approximate values of $y(t_i)$ and $y''(t_i)$, respectively.

### 2.1 Second and fourth-order FD methods to solve BVP (1)

Here, we intend to solve the linear BVP (1) based on the central second and fourth-order difference formula to approximate the $y''(t_i)$. For this goal, we consider the discretized form of BVP (1) as follows:

$$y''(t_i) + p(t_i)y(t_i) = f(t_i), \quad i = 1,2,\ldots,N-1. \tag{2}$$

Employing the second-order central FD formula

$$y''(t_i) = \frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} - \frac{h^2}{12}y^4(\zeta_i), \quad \zeta_i \in (t_{i-1}, t_{i+1}), \tag{3}$$

and substituting it in (2), one can obtain the following discrete form

$$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} + p(t_i)y(t_i) = f(t_i) + \frac{h^2}{12}y^4(\zeta_i), \quad i = 1,2,\ldots,N-1. \tag{4}$$

Moreover, to treat the boundary conditions of the problem, using the second-order forward and backward FD formulas

$$y'(t_0) = \frac{-3y(t_0) + 4y(t_1) - y(t_2)}{2h} + 2\frac{h^2}{3!}y'''(t_0) + \mathscr{O}(h^3), \tag{5}$$

$$y'(t_N) = \frac{3y(t_N) - 4y(t_{N-1}) + y(t_{N-2})}{2h} + 2\frac{h^2}{3!}y'''(t_N) + \mathscr{O}(h^3), \tag{6}$$

we have

$$\begin{cases} y(t_0) + y(t_N) = 0, \\ \dfrac{-3y(t_0) + 4y(t_1) - y(t_2)}{2h} + \dfrac{3y(t_N) - 4y(t_{N-1}) + y(t_{N-2})}{2h} = 2\dfrac{h^2}{3!}\Big(y'''(t_0) + y'''(t_N)\Big) + \mathscr{O}(h^3). \end{cases} \tag{7}$$

Eliminating the local truncation errors from (4) and (7), and using $y_i$ as an approximating value of $y(t_i)$, we obtain

$$\begin{cases} y_0 + y_N = 0, \\ y_{i+1} - (2 - h^2 p(t_i))y_i + y_{i-1} = h^2 f(t_i), \quad i = 1,2,3\ldots,N-1, \\ -3y_0 + 4y_1 - y_2 + 3y_N - 4y_{N-1} + y_{N-2} = 0. \end{cases} \tag{8}$$

The matrix form of the linear system (8) can be expressed as:

$$A^{\mathrm{FD2}}Y = B^{\mathrm{FD2}}, \tag{9}$$

where

$$A^{\mathrm{FD2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & \upsilon_1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \upsilon_2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \upsilon_{n-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \upsilon_{N-1} & 1 \\ -3 & 4 & -1 & 0 & \cdots & 0 & 1 & -4 & 3 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B^{\mathrm{FD2}} = \begin{pmatrix} 0 \\ h^2 f(t_1) \\ h^2 f(t_2) \\ \vdots \\ h^2 f(t_{N-2}) \\ h^2 f(t_{N-1}) \\ 0 \end{pmatrix}_{(N+1)\times 1}, \quad Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix}_{(N+1)\times 1},$$

and $\upsilon_i = -(2 - h^2 p(t_i))$ for $i = 1, 2, \ldots, N-1$. Solving the resulting linear algebraic system, a discrete approximation of the solution to the problem is computed.

Similarly, to increase the order of accuracy of the standard FD method, we can employ the fourth-order FD formulas as follows:

$$y''(t_1) = \frac{10y(t_0) - 15y(t_1) - 4y(t_2) + 14y(t_3) - 6y(t_4) + y(t_5)}{12h^2} - \frac{52}{720} h^4 y^{(6)}(t_0) + \mathscr{O}(h^5), \tag{10}$$

$$y''(t_i) = \frac{-y(t_{i+2}) + 16y(t_{i+1}) - 30y(t_i) + 16y(t_{i-1}) - y(t_{i-2})}{12h^2} + \frac{1}{30} h^4 y^{(5)}(\zeta_i),$$
$$\zeta_i \in (t_{i-2}, t_{i+2}), \quad 2 \le i \le N-2, \tag{11}$$

$$y''(t_{N-1}) = \frac{10y(t_N) - 15y(t_{N-1}) - 4y(t_{N-2}) + 14y(t_{N-3}) - 6y(t_{N-4}) + y(t_{N-5})}{12h^2} - \frac{52}{720} h^4 y^{(6)}(t_N) + \mathscr{O}(h^5). \tag{12}$$

Also, to satisfy the boundary conditions of the problem, utilizing the sixth-order forward and backward FD formulas

$$y'(t_0) = \frac{1}{h}\Big(\frac{-137}{60} y(t_0) + 5y(t_1) - 5y(t_2) + \frac{10}{3} y(t_3) - \frac{5}{4} y(t_4) + \frac{1}{5} y(t_5)\Big) - 5h^5 y^{(5)}(t_0) + \mathscr{O}(h^6), \tag{13}$$

$$y'(t_N) = \frac{1}{h}\Big(\frac{137}{60} y(t_N) - 5y(t_{N-1}) + 5y(t_{N-2}) - \frac{10}{3} y(t_{N-3}) + \frac{5}{4} y(t_{N-4}) - \frac{1}{5} y(t_{N-5})\Big) - 5h^5 y^{(5)}(t_N) + \mathscr{O}(h^6), \tag{14}$$

we can get

$$\begin{cases} y(t_0) + y(t_N) = 0, \\ \frac{1}{h}\Big(\frac{137}{60}(y(t_N) - y(t_0)) - 5(y(t_{N-1}) - y(t_1)) + 5(y(t_{N-2}) - y(t_2)) - \frac{10}{3}(y(t_{N-3}) - y(t_3)) \\ \qquad + \frac{5}{4}(y(t_{N-4}) - y(t_4)) - \frac{1}{5}(y(t_{N-5}) - y(t_5))\Big) = -5h^5\Big(y^{(5)}(t_0) + y^{(5)}(t_N)\Big) + \mathscr{O}(h^6). \end{cases} \tag{15}$$

Substituting (12)-(14) in the discrete form (2) and employing (15), we obtain the following linear algebraic system to find the approximate solution of the problem:

$$A^{\mathrm{FD4}} Y = B^{\mathrm{FD4}}, \tag{16}$$

where

$$A^{\mathrm{FD4}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 10 & \hat{\upsilon}_1 & -4 & 14 & -6 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 16 & \hat{\upsilon}_2 & 16 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 16 & \hat{\upsilon}_3 & 16 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 16 & \hat{\upsilon}_{N-3} & 16 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 16 & \hat{\upsilon}_{N-2} & 16 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -6 & 14 & -4 & \hat{\upsilon}_{N-1} & 10 \\ -\frac{137}{60} & 5 & -5 & \frac{10}{3} & -\frac{5}{4} & \frac{1}{5} & 0 & \cdots & 0 & -\frac{1}{5} & \frac{5}{4} & -\frac{10}{3} & 5 & -5 & \frac{137}{60} \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B^{\mathrm{FD4}} = \begin{pmatrix} 0 \\ 12h^2 f(t_1) \\ 12h^2 f(t_2) \\ \vdots \\ 12h^2 f(t_{N-2}) \\ 12h^2 f(t_{N-1}) \\ 0 \end{pmatrix}_{(N+1)\times 1},$$

with

$$\hat{\upsilon}_i = \begin{cases} -15 + 12h^2 p(t_i), & \text{for} \quad i = 1, N-1, \\ -30 + 12h^2 p(t_i), & \text{for} \quad i = 2, 3, \ldots, N-2. \end{cases}$$

Finally, a discrete approximation of the solution to the problem is calculated by solving the linear algebraic system (16).

## 2.2 Fourth and sixth-order CFD methods to solve BVP (1)

A CFD method is a numerical technique for approximating solutions to differential equations. It is an enhanced version of FD methods, specifically developed to offer improved accuracy and efficiency compared to conventional FD methods. From [12], the CFD formulation for the second-order derivative is defined as:

$$\begin{aligned} \beta y''_{i+2} + \alpha y''_{i+1} + y''_i + \alpha y''_{i-1} + \beta y''_{i-2} = & \frac{a}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{b}{4h^2}(y_{i+2} - 2y_i + y_{i-2}) \\ & + \frac{c}{9h^2}(y_{i+3} - 2y_i + y_{i-3}), \end{aligned} \tag{17}$$

where $a, b, c, \alpha$ and $\beta$ are some coefficient constants to be determined. To attain the desired accuracy, employing Taylor series expansion yields the following results:

$$1 + 2\alpha + 2\beta - a - b - c = 0, \qquad \text{(Second-order)} \tag{18}$$

$$12(\alpha + 2^2\beta) - a - 2^2 b - 3^2 c = 0, \quad \text{(Fourth-order)} \tag{19}$$

$$30(\alpha + 2^4\beta) - a - 2^4 b - 3^4 c = 0, \quad \text{(Sixth-order)} \tag{20}$$

$$56(\alpha + 2^6\beta) - a - 2^6 b - 3^6 c = 0, \quad \text{(Eighth-order)} \tag{21}$$

$$90(\alpha + 2^8\beta) - a - 2^8 b - 3^8 c = 0. \quad \text{(Tenth-order)} \tag{22}$$

Clearly, we can construct various sub-systems by extracting some or all of the equations from (18) to (22), which helps us to find out the values of $a, b, c, \alpha$ and $\beta$. For example, a system having the first two equations has five unknowns, which gives us a three parameters family of infinitely many solutions. The CFD formulas with different orders of accuracy can be derived by solving systems of equations established from (18) to (22) as reported in [12]. Setting $\beta = c = 0$ in equations (18) to (20), we obtain

$$a = \frac{4(1-\alpha)}{3}, \qquad b = \frac{10\alpha - 1}{3}. \tag{23}$$

Here, the local truncation error of the difference formula (17) is $-\frac{4(11\alpha-2)}{6!} h^4 y^{(6)}(\tau_i)$ where $\tau_i \in (t_{i-1}, t_{i+1})$. When $\alpha = \frac{1}{10}$ is fixed in (23), we have $a = \frac{6}{5}$, and $b = 0$. Therefore, the fourth-order CFD method is derived by substituting these constants into (17) as follows:

$$\frac{1}{10} y''_{i+1} + y''_i + \frac{1}{10} y''_{i-1} = \frac{6}{5h^2}(y_{i+1} - 2y_i + y_{i-1}). \tag{24}$$

Similarly, choosing $\alpha = \frac{2}{11}$ and using (23) we find that $a = \frac{12}{11}$ and $b = \frac{3}{11}$. Substituting these values into (17), yields the following sixth-order CFD method

$$\frac{2}{11} y''_{i+1} + y''_i + \frac{2}{11} y''_{i-1} = \frac{12}{11h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{3}{44h^2}(y_{i+2} - 2y_i + y_{i-2}). \tag{25}$$

### 2.2.1 Fourth-order CFD method based on (24)

To describe the method based on the CFD formula (24), using the boundary condition $y_0 + y_N = 0$, we can conclude that

$$y_0 = -y_N = \lambda, \tag{26}$$

where $\lambda$ is unknown value that is determined at the end using the boundary condition $y'_0 + y'_N = 0$. Here, we consider the relation (24) to calculate the interior nodes $t_i$ for $i = 2, 3, \ldots, N-2$, and modify (24) such that they are suitable to use in the method for nodes $t_1$ and $t_{N-1}$. For this aim, the first and last equations in (24) are reevaluated by the following fourth-order formulas:

$$y''_1 + \alpha y''_2 = \frac{1}{h^2}(\hat{a}_0 y_0 + \hat{a}_1 y_1 + \hat{a}_2 y_2 + \hat{a}_3 y_3 + \hat{a}_4 y_4 + \hat{a}_5 y_5), \tag{27}$$

$$y''_{N-2} + \alpha y''_{N-1} = \frac{1}{h^2}(\hat{a}_0 y_N + \hat{a}_1 y_{N-1} + \hat{a}_2 y_{N-2} + \hat{a}_3 y_{N-3} + \hat{a}_4 y_{N-4} + \hat{a}_5 y_{N-5}), \tag{28}$$

in which $\hat{a}_0 = \frac{-1}{12}\alpha + \frac{5}{6}$, $\hat{a}_1 = \frac{4}{3}\alpha - \frac{5}{4}$, $\hat{a}_2 = \frac{-5}{2}\alpha - \frac{1}{3}$, $\hat{a}_3 = \frac{4}{3}\alpha + \frac{7}{6}$, $\hat{a}_4 = \frac{-1}{12}\alpha - \frac{1}{2}$, $\hat{a}_5 = \frac{1}{12}$. For $\alpha = \frac{1}{10}$, the relations (24), (27) and (28) lead to the following linear system:

$$\begin{cases} y''_1 + \frac{1}{10} y''_2 = \frac{1}{h^2}\left(\frac{33}{40}\lambda + \frac{-67}{60} y_1 + \frac{-7}{12} y_2 + \frac{13}{10} y_3 + \frac{-61}{120} y_4 + \frac{1}{12} y_5\right), \\ \frac{1}{10} y''_{i+1} + y''_i + y''_{i-1} \frac{1}{10} = \frac{6}{5h^2}(y_{i+1} - 2y_i + y_{i-1}), \quad i = 2, 3, 4, \ldots, N-2, \\ y''_{N-2} + \frac{1}{10} y''_{N-1} = \frac{1}{h^2}\left(-\frac{33}{40}\lambda + \frac{-67}{60} y_{N-11} + \frac{-7}{12} y_{N-2} + \frac{13}{10} y_{N-3} + \frac{-61}{120} y_{N-4} + \frac{1}{12} y_{N-5}\right). \end{cases} \tag{29}$$

According to the above discussions, the matrix form of the relation (29) can be expressed as:

$$\hat{M} Y'' = \frac{1}{h^2}(\hat{A} Y + \hat{B}), \tag{30}$$

where

$$\hat{M}=\begin{pmatrix} 1 & \frac{1}{10} & 0 & \cdots & 0\\ \frac{1}{10} & 1 & \frac{1}{10} & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \frac{1}{10} & 1 & \frac{1}{10}\\ 0 & \cdots & \cdots & \frac{1}{10} & 1 \end{pmatrix}_{(N-1)\times(N-1)},\quad Y=\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_{N-2}\\ y_{N-1}\end{pmatrix}_{(N-1)\times 1},\quad Y''=\begin{pmatrix} y''_1\\ y''_2\\ \vdots\\ y''_{N-2}\\ y''_{N-1}\end{pmatrix}_{(N-1)\times 1},$$

$$\hat{B}=\begin{pmatrix} \frac{33}{40}\lambda\\ 0\\ \vdots\\ 0\\ -\frac{33}{40}\lambda\end{pmatrix}_{(N-1)\times 1},\quad \hat{A}=\begin{pmatrix} \frac{-67}{60} & \frac{-7}{12} & \frac{13}{10} & \frac{-61}{120} & \frac{1}{12} & \cdots & 0\\ \frac{6}{5} & \frac{-12}{5} & \frac{6}{5} & \cdots & \cdots & \cdots & 0\\ 0 & \frac{6}{5} & \frac{-12}{5} & \frac{6}{5} & \cdots & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & \cdots & \frac{6}{5} & \frac{-12}{5} & \frac{6}{5} & 0\\ 0 & \cdots & \cdots & \cdots & \frac{6}{5} & \frac{-12}{5} & \frac{6}{5}\\ 0 & \cdots & \frac{1}{12} & \frac{-61}{120} & \frac{13}{10} & \frac{-7}{12} & \frac{-67}{60} \end{pmatrix}_{(N-1)\times(N-1)}.$$

We note that according to the system (30), $\hat{B}$ is a constant $(N-1)\times 1$ column vector containing the unknown value $\lambda$. Now, for $i=1,2,\ldots,N-1$, we rewrite (2) in the matrix form as:

$$Y''+PY=F, \tag{31}$$

where the diagonal matrix $P$ and vector $F$ are defined as follows:

$$P=\mathrm{Diag}(p(t_1),p(t_2),\ldots,p(t_{N-1})),\quad F=\begin{pmatrix} f(t_1)\\ f(t_2)\\ \vdots\\ f(t_{N-1})\end{pmatrix}_{(N-1)\times 1}.$$

To avoid computing the inverse matrix, by multiplying the matrix $\hat{M}$ in (31), we have

$$\hat{M}Y''+\hat{M}PY=\hat{M}F. \tag{32}$$

Substituting (30) into (32) and performing simplifications, the following linear algebraic system can be obtained:

$$A^{\mathrm{CFD4}}Y=B^{\mathrm{CFD4}}, \tag{33}$$

where

$$A^{\mathrm{CFD4}}=\hat{A}+h^2\hat{M}P,\quad B^{\mathrm{CFD4}}=h^2\hat{M}F-\hat{B}.$$

Solving the linear system (33), a discrete approximation of the solution to the problem is computed at nodal points $t_i$ for $i=1,2,\ldots,N-1$ based on the unknown value $\lambda$. To determine the unknown $\lambda$ and thereby obtain the approximate solution to the problem, we utilize the sixth-order forward and backward approximations (13) and (14), along with boundary condition $y'_0+y'_N=0$ and solve the following linear equation:

$$-\frac{137}{30}\lambda-5(y_{N-1}-y_1)+5(y_{N-2})-y_2)-\frac{10}{3}(y_{N-3}-y_3)+\frac{5}{4}(y_{N-4}-y_4)-\frac{1}{5}(y_{N-5}-y_5)=0.$$

#### 2.2.2 Sixth-order CFD method based on (25)

Similar to the fourth-order CFD method, to introduce the method based on the CFD formula (25), we consider the relation (26). Furthermore, we take into account equation (25) to compute the interior nodes $t_i$ for $i = 2,3,\dots,N-2$. To adapt equation (24) for the nodes $t_1$ and $t_{N-1}$, we make the necessary modifications to ensure its suitability within the method. To do this, the first and last equations in (25) are recalculated by the following sixth-order formulas:

$$y_1'' + \alpha y_2'' = \frac{1}{h^2}(\tilde{a}_0 y_0 + \tilde{a}_1 y_1 + \tilde{a}_2 y_2 + \tilde{a}_3 y_3 + \tilde{a}_4 y_4 + \tilde{a}_5 y_5 + \tilde{a}_6 y_6 + \tilde{a}_7 y_7), \tag{34}$$

$$y_{N-2}'' + \alpha y_{N-1}'' = \frac{1}{h^2}(\tilde{a}_0 y_N + \tilde{a}_1 y_{N-1} + \tilde{a}_2 y_{N-2} + \tilde{a}_3 y_{N-3} + \tilde{a}_4 y_{N-4} + \tilde{a}_5 y_{N-5} + \tilde{a}_6 y_{N-6} + \tilde{a}_7 y_{N-7}), \tag{35}$$

where

$$\begin{aligned} &\tilde{a}_0 = \frac{7}{10} - \alpha\frac{11}{180}, \quad \tilde{a}_1 = \frac{-7}{18} + \alpha\frac{107}{90}, \quad \tilde{a}_2 = \frac{-27}{10} - \alpha\frac{21}{10}, \quad \tilde{a}_3 = \frac{19}{4} + \alpha\frac{13}{18}, \\ &\tilde{a}_4 = \frac{-67}{18} + \alpha\frac{17}{36}, \quad \tilde{a}_5 = \frac{9}{5} - \alpha\frac{3}{10}, \quad \tilde{a}_6 = \frac{-1}{2} + \alpha\frac{4}{45}, \quad \tilde{a}_7 = \frac{1}{1180} - \alpha\frac{1}{90}. \end{aligned}$$

For $\alpha = \frac{2}{11}$ and by using the relations (25), (34) and (35), we obtain the following linear system:

$$\begin{cases} y_1'' + \frac{2}{11}y_2'' = \frac{1}{h^2}(\tilde{a}_0 y_0 + \tilde{a}_1 y_1 + \tilde{a}_2 y_2 + \tilde{a}_3 y_3 + \tilde{a}_4 y_4 + \tilde{a}_5 y_5 + \tilde{a}_6 y_6 + \tilde{a}_7 y_7), \\ \frac{2}{11}y_{i+1}'' + y_i'' + \frac{2}{11}y_{i-1}'' = \frac{12}{11h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{3}{44h^2}(y_{i+2} - 2y_i + y_{i-2}), \quad i = 2,3,4,\dots,N-2, \\ y_1'' + \frac{2}{11}y_2'' = \frac{1}{h^2}(\tilde{a}_0 y_N + \tilde{a}_1 y_{N-1} + \tilde{a}_2 y_{N-2} + \tilde{a}_3 y_{N-3} + \tilde{a}_4 y_{N-4} + \tilde{a}_5 y_{N-5} + \tilde{a}_6 y_{N-6} + \tilde{a}_7 y_{N-7}). \end{cases} \tag{36}$$

The matrix form of the relation (36) is as follows:

$$\tilde{M}Y'' = \frac{1}{h^2}(\tilde{A}Y + \tilde{B}), \tag{37}$$

where

$$\tilde{M} = \begin{pmatrix} 1 & \frac{2}{11} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{2}{11} & 1 & \frac{2}{11} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{11} & 1 & \frac{2}{11} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{11} & 1 & \frac{2}{11} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{2}{11} & 1 & \frac{2}{11} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{2}{11} & 1 \end{pmatrix}_{(N-1)\times(N-1)}, \quad \tilde{B} = \begin{pmatrix} \frac{31}{45}\lambda \\ \frac{3}{44}\lambda \\ 0 \\ \vdots \\ 0 \\ -\frac{3}{44}\lambda \\ -\frac{31}{45}\lambda \end{pmatrix}_{(N-1)\times 1},$$

$$\tilde{A} = \begin{pmatrix} -\frac{19}{110} & \frac{-339}{110} & \frac{1933}{396} & \frac{-40}{11} & \frac{96}{55} & \frac{-479}{90} & \frac{13}{220} & \cdots & 0 \\ \frac{12}{11} & \frac{-51}{22} & \frac{12}{11} & \frac{3}{44} & 0 & \cdots & 0 & \cdots & 0 \\ \frac{3}{44} & \frac{12}{11} & \frac{-51}{22} & \frac{12}{11} & \frac{3}{44} & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \frac{3}{44} & \frac{12}{11} & \frac{-51}{22} & \frac{12}{11} & \frac{3}{44} \\ 0 & \cdots & 0 & 0 & 0 & \frac{3}{44} & \frac{12}{11} & \frac{-51}{22} & \frac{12}{11} \\ 0 & \cdots & \frac{13}{220} & \frac{-479}{90} & \frac{96}{55} & \frac{-40}{11} & \frac{1933}{396} & \frac{-339}{110} & -\frac{19}{110} \end{pmatrix}_{(N-1)\times(N-1)}.$$

We note that according to the system (36), $\tilde{B}$ is a constant $(N-1)\times 1$ column vector containing the unknown value $\lambda$. To avoid the computation of the inverse matrix, we can multiply the matrix $\hat{M}$ in (31) to obtain the desired result. So, we have:

$$\tilde{M}Y'' + \tilde{M}PY = \tilde{M}F. \tag{38}$$

Substituting (37) into (38) yields the following linear system:

$$A^{\mathrm{CFD6}}Y = B^{\mathrm{CFD6}}, \tag{39}$$

where

$$A^{\mathrm{CFD6}} = \tilde{A} + h^2\tilde{M}P, \quad B^{\mathrm{CFD6}} = h^2\tilde{M}F - \tilde{B}.$$

By solving the linear system (39), the approximate solution of the problem is obtained at nodal points $t_i$ for $i = 1,2,\ldots,N-1$ based on the unknown value $\lambda$. In a similar manner to fourth-order CFD method, the unknown $\lambda$ can be computed by utilizing the eighth-order forward and backward formulas

$$y'(t_0) = \frac{1}{h}\Big(-\frac{89}{35}y(t_0) + \frac{133}{20}y(t_1) - \frac{189}{20}y(t_2) + \frac{119}{12}y(t_3) - 7y(t_4) + \frac{63}{20}y(t_5) - \frac{49}{60}y(t_6) + \frac{13}{140}y(t_7)\Big), \tag{40}$$

$$\begin{aligned} y'(t_N) = &\frac{1}{h}\Big(\frac{89}{35}y(t_N) - \frac{133}{20}y(t_{N-1}) + \frac{189}{20}y(t_{N-2}) - \frac{119}{12}y(t_{N-3}) \\ &+ 7y(t_{N-4}) - \frac{63}{20}y(t_{N-5}) + \frac{49}{60}y(t_{N-6}) - \frac{13}{140}y(t_{N-7})\Big), \end{aligned} \tag{41}$$

along with boundary condition $y'_0 + y'_N = 0$ and solve the following linear equation:

$$\begin{aligned} -\frac{178}{35}\lambda - \frac{133}{20}(y_{N-1} - y_1) + \frac{189}{20}(y_{N-2} - y_2) - \frac{119}{12}(y_{N-3} - y_3) + 7(y_{N-4} - y_4) \\ -\frac{63}{20}(y_{N-5} - y_5) + \frac{49}{60}(y_{N-6} - y_6) - \frac{13}{140}(y_{N-7} - y_7) = 0. \end{aligned}$$

## 3 Convergence analysis

In this section, the convergence analysis of the fourth-order CFD method is discussed. To facilitate this analysis, we first introduce two lemmas that play a crucial role in the subsequent discussion.

**Lemma 1** ([4]). *If $f \in L^2[0,\pi]$ and $p \in C[0,\pi]$ satisfies one of the following conditions:*

*(i)* $(2n-1-\delta)^2 \le p(t) \le (2n+1+\delta)^2$ *for some positive integer $n$ and some $\delta \in (0,1)$,*

*(ii)* $0 \le p(t) \le (1-\delta)^2$ *for some $\delta \in (0,1)$,*

*(iii)* $p(t) \le 0$,

*then the anti-periodic BVP (1) has a unique solution.*

**Lemma 2** ([6]). *Let $a_n$ and $f_n$ be real-valued functions defined for $n \in \mathbb{N}$ and suppose that $f_n \ge 0$ for every $n \in \mathbb{N}$. If*

$$a_n \le x_0 + \sum_{s=0}^{n-1} f_s a_s, \qquad n \in \mathbb{N},$$

*where $x_0$ is a non-negative constant, then*

$$a_n \leq x_0 \sum_{s=0}^{n-1} [1+f_s], \qquad n \in \mathbb{N}.$$

In what follow, we intend to show that the fourth-order CFD approximation of BVP (1) converges to the exact solution when $h$ tends to zero.

Let $y(t)$ be the exact solution of (1) and $y_i$ be the approximate solution obtained using the fourth-order CFD method. Suppose $e_i = y(t_i) - y_i$, then from (24) we have the following discrete error relation:

$$\alpha e''_{i+1} + e''_i + \alpha e''_{i-1} = \frac{6}{5h^2}(e_{i+1} - 2e_i + e_{i-1}) + h^4 g_i, \quad i = 1,2,\dots,N-1, \tag{42}$$

where $g_i = -\frac{4(11\alpha-2)}{6!} y^{(6)}(\tau_i)$ and $\tau_i \in (t_{i-1}, t_{i+1})$ and $\alpha = \frac{1}{10}$. Also, from the anti-periodic BVP (1), we have

$$e''_i = -p(t_i)e_i + \mathscr{O}(h^4), \quad i = 1,2,\dots,N-1. \tag{43}$$

Substituting (43) into (42) yields

$$\Big(\frac{12}{5} - h^2 p(t_i)\Big)e_i = \Big(h^2 \alpha p(t_{i+1}) + \frac{6}{5}\Big)e_{i+1} + \Big(h^2 \alpha p(t_{i-1}) + \frac{6}{5}\Big)e_{i-1} + h^6 g_i, \ \ i = 1,2,\dots,N-1. \tag{44}$$

Now, let $\mathbf{e} = [e_1 \ \ e_2 \ \ e_3 \ \ \dots \ \ e_{N-1}]^T$ be the vector of errors at node points $t_i$. From relation (44), we have

$$\Big|\frac{12}{5} - h^2 p(t_i)\Big||e_i| \leq \Big|h^2 \alpha p(t_{i+1}) + \frac{6}{5}\Big||e_{i+1}| + \Big|h^2 \alpha p(t_{i-1}) + \frac{6}{5}\Big||e_{i-1}| + h^6|g_i|, \quad i = 1,2,\dots,N-1. \tag{45}$$

**Case 1:** If $p(t) < 0$ and $h$ is sufficiently small such that $h^2 \alpha p(t_i) < \frac{5}{6}$, then

$$\Big(\frac{12}{5} - h^2 p(t_i)\Big)|e_i| \leq \Big(h^2 \alpha p(t_{i+1}) + \frac{6}{5}\Big)\|\mathbf{e}\|_\infty + \Big(h^2 \alpha p(t_{i-1}) + \frac{6}{5}\Big)\|\mathbf{e}\|_\infty + h^6\|g\|_\infty, \quad i = 1,2,\dots,N-1. \tag{46}$$

Therefore, we get

$$-(1+2\alpha)h^2(\max_{t\in[0,\pi]} p(t))\|\mathbf{e}\|_\infty \leq h^6||g||_\infty, \tag{47}$$

and consequently

$$\|\mathbf{e}\|_\infty \leq \frac{\|g\|_\infty}{-(1+2\alpha)\max_{t\in[0,\pi]} p(t)} h^4, \tag{48}$$

which implies that if $h$ tends to zero, then $\|\mathbf{e}\|_\infty$ tends to zero as well.
**Case 2:** If $p(t) \geq 0$ and $h$ is sufficiently small, then from relation (44), we have

$$\Big|h^2 \alpha p(t_{i+1}) + \frac{6}{5}\Big||e_{i+1}| \leq \Big|-\frac{12}{5} + h^2 p(t_i)\Big||e_i| + \Big|h^2 \alpha p(t_{i-1}) + \frac{6}{5}\Big||e_{i-1}| + h^6|g_i|, \quad i = 1,2,\dots,N-1, \tag{49}$$

and therefore

$$|e_{i+1}| \leq \frac{\big|-\frac{12}{5} + h^2 p(t_i)\big|}{\big|h^2 \alpha p(t_{i+1}) + \frac{6}{5}\big|}|e_i| + \frac{\big|h^2 \alpha p(t_{i-1}) + \frac{6}{5}\big|}{\big|h^2 \alpha p(t_{i+1}) + \frac{6}{5}\big|}|e_{i-1}| + \frac{h^6|g_i|}{\big|h^2 \alpha p(t_{i+1}) + \frac{6}{5}\big|}, \quad i = 1,2,\dots,N-1. \tag{50}$$

Since, $h$ is sufficiently small, then we get the following:

$$|e_{i+1}| \leq 2|e_i| + |e_{i-1}| + \frac{h^6\|g\|_\infty}{h^2\alpha \min\limits_{t\in[0,\pi]} p(t) + \frac{5}{6}} \leq 2|e_i| + |e_{i-1}| + \frac{h^4\|g\|_\infty}{\alpha \min\limits_{t\in[0,\pi]} p(t)}, \quad i = 1,2,\dots,N-1, \tag{51}$$

Finally, using Lemma 2, one can obtain

$$\|\mathbf{e}\|_\infty \leq \frac{5\|g\|_\infty}{\alpha \min\limits_{t\in[0,\pi]} p(t)} h^4. \tag{52}$$

Hence, when $h$ tends to zero, it is clear from the obtained relation (52) that $\|\mathbf{e}\|_\infty$ tends to zero.

# 4 Numerical results

In this section, numerical experiments are performed to demonstrate the efficiency and accuracy of the proposed methods for solving anti-periodic BVPs (1). In order to be able to compare the proposed methods and the exact solution, the maximum absolute error

$$\|\mathbf{e}(h)\|_\infty = \max_{0\leq i\leq N} |y(t_i) - y_i|,$$

of each method in the discretized grids is considered. Furthermore, the computational orders of convergence of methods is obtained by the following relation:

$$\text{C-order}(N,\bar{N}) = \frac{\log\left(\frac{\|\mathbf{e}(h_1)\|_\infty}{\|\mathbf{e}(h_2)\|_\infty}\right)}{\log(2)}, \tag{53}$$

where $\bar{N} = 2N$, $h_1 = \frac{b-a}{N}$, and $h_2 = \frac{b-a}{\bar{N}}$. The computational time of the proposed methods are also presented in Tables 1 and 2. All the results are attained by using Maple 2017 software on a Core(TM) i5 with 1.80 GHz of CPU and 8 GB of RAM.

**Example 1** ([3])**.** Consider the following functions:

$$\begin{aligned}
p(t) &= 3 + \sin(t), \\
f(t) &= \left(\frac{t^2 - \pi t}{2}\right)\sin(t) - \cos(t)(2 + \sin(t)) + \frac{3t^2}{2} - \frac{3\pi t}{2} + 1, \\
y(t) &= -\cos(t) + \frac{t^2}{2} - \frac{\pi t}{2}.
\end{aligned}$$

On the interval $[0,\pi]$ and for $\delta = \frac{1}{2}$ and $n = 1$, condition (i) of Lemma 1 is satisfied. Consequently, it can be inferred that there is a unique solution to this problem.

**Example 2** ([3])**.** Consider the following functions:

$$p(t) = \frac{\cos^2(t)}{2}, \quad f(t) = \frac{\cos^3(t)}{2} - \cos(t), \quad y(t) = \cos(t).$$

Choosing $\delta = \frac{1}{2}$, the function $p(t)$ satisfies the condition (ii) of Lemma 1 on the interval $[0,\pi]$. Therefore the solution to this problem exists and is unique.

Table 1: Numerical results for different values of $N$.

| $N$ | $N1=50$ | $N2=100$ | $N3=200$ | C-order$(N1,N2)$ | C-order$(N2,N3)$ | CPU time(s) $(N=200)$ |
|---|---|---|---|---|---|---|
| **Example 1** | | | | | | |
| FD2 | 1.241e-04 | 3.150e-05 | 7.942e-06 | 1.97405 | 1.98706 | 0.188 |
| FD4 | 1.544e-07 | 3.331e-09 | 2.351e-10 | 5.06778 | 5.01833 | 0.844 |
| CFD4 | 1.811e-07 | 5.103e-09 | 1.231e-10 | 5.14907 | 5.37382 | 1.110 |
| CFD6 | 3.419e-10 | 2.781e-12 | 2.256e-14 | 6.94184 | 6.94532 | 1.360 |
| **Example 2** | | | | | | |
| FD2 | 3.902e-04 | 1.154e-04 | 3.115e-05 | 1.75701 | 1.88991 | 0.375 |
| FD4 | 6.112e-07 | 1.059e-08 | 3.006e-10 | 5.84999 | 5.13955 | 0.859 |
| CFD4 | 7.665e-07 | 2.094e-08 | 4.509e-10 | 5.19414 | 5.53740 | 1.547 |
| CFD6 | 1.478e-09 | 1.215e-11 | 9.943e-14 | 6.92792 | 6.93244 | 2.750 |
| **Example 3** | | | | | | |
| FD2 | 2.379e-07 | 5.946e-08 | 1.486e-08 | 2.00016 | 2.00004 | 0.266 |
| FD4 | 8.402e-11 | 5.257e-12 | 3.285e-13 | 3.99836 | 4.00047 | 0.781 |
| CFD4 | 3.153e-11 | 1.973e-12 | 1.232e-13 | 3.99803 | 4.00127 | 1.484 |
| CFD6 | 7.584e-14 | 1.540e-16 | 2.604e-18 | 5.62172 | 5.88607 | 1.797 |

**Example 3** ([3])**.** Consider the following functions:

$$\begin{aligned}
p(t) &= -\exp(-t),\\
f(t) &= \frac{8t^2}{(16+t^2)^3} - \frac{2}{(16+t^2)^2} - \frac{2}{17} - \exp(-t)\left(\frac{1}{16+t^2} - \frac{t^2}{17}\right),\\
y(t) &= \frac{1}{16+t^2} - \frac{t^2}{17}.
\end{aligned}$$

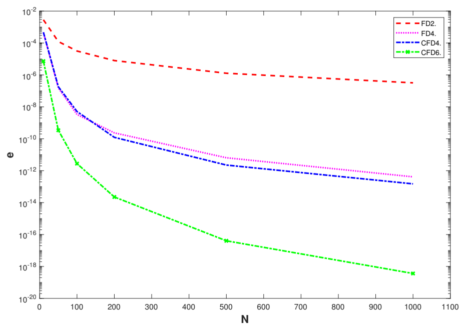
This problem is considered on the interval $[-1,1]$. If we change the interval $[-1,1]$ into $[0,\pi]$, then the shifted function $p(t)$ satisfies the condition (iii) of Lemma 1 on the interval $[0,\pi]$.

Table 1 offers a comprehensive analysis of the errors for three examples using four numerical methods. The problems are solved using both the FD method and the CFD method. It can be observed that the approximate solution obtained using the proposed methods are much closer to the exact solution due to the higher order accuracy and compactness. This demonstrates that a better approximation of the exact solution can be obtained by increasing the order of the methods and the compactness beyond this, and decreasing the grid size of the FD and CFD methods.

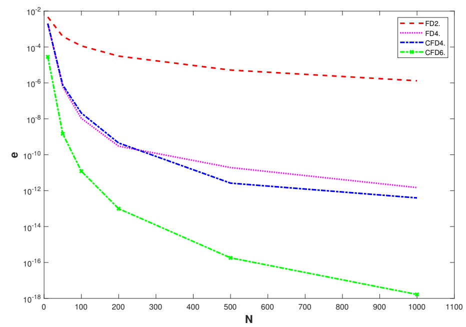
Figure 1 displays the error analysis of Examples 1 to 3. From the graph, we can observe that as the number of grid points increases, the error decreases for all methods. Additionally, the graph shows that FD2, FD4, and CFD4 methods have a lower rate of accuracy than CFD6, which suggests that the CFD6 method is more accurate for solving the problem.

**Example 4** ([3])**.** Consider the following differential equation with anti-periodic boundary conditions:

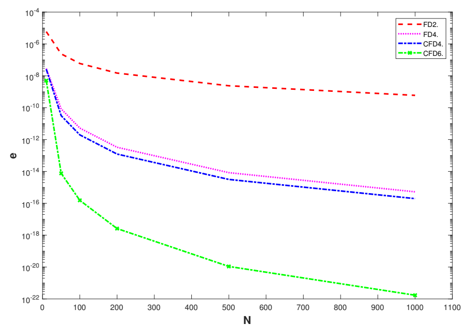
$$\begin{cases}
y''(t) - \frac{1}{t^2+3}y(t) = \frac{t^2-|t^3|}{t^2+3} + 6|t| - 2, \quad t \in [-1,1],\\
y(-1) + y(1) = 0,\\
y'(-1) + y'(1) = 0.
\end{cases}$$




(a) Example 1




(b) Example 2




(c) Example 3

Figure 1: Error analysis of proposed methods for Examples 1-3.

The exact solution is $y(t) = |t|^3 - t^2$. If we change the interval $[-1,1]$ into $[0,\pi]$, then the shifted function $p(t)$ satisfies the condition (iii) of Lemma 1.

Table 2 and Figure 2 show the errors and convergence rate of each method. The convergence rate of the method depends on the smoothness of the solution, which is typically characterized by the order of the highest derivative that is continuous and bounded. For a solution that is only twice continuously differentiable, the methods can achieve a convergence rate of $\mathscr{O}(h^2)$. This means that the error in the solution decreases by a factor of 4 as the grid spacing is divided by 2. Since we know that, $y \in C^2[-1,1]$ therefore the FD4, CFD4, and CFD6 are still applicable, but they are not achieve the full forth and sixth-order convergence rate, which are quite clear in fifth-column of the Table 2.

Table 2: Numerical results for different values of $N$.

| $N$ | $N1 = 50$ | $N2 = 100$ | $N3 = 200$ | C-order($N1,N2$) | C-order($N2,N3$) | CPU time(s) ($N = 200$) |
|---|---|---|---|---|---|---|
| FD2 | 1.447e-03 | 3.618e-04 | 9.045e-05 | 1.999908194 | 1.999977050 | 0.547 |
| FD4 | 7.282e-04 | 1.815e-04 | 4.529e-05 | 2.004487261 | 2.002272767 | 0.531 |
| CFD4 | 7.236e-04 | 1.809e-04 | 4.523e-05 | 1.999951905 | 1.999987971 | 1.969 |
| CFD6 | 7.200e-04 | 1.805e-04 | 4.517e-05 | 1.996400982 | 1.998208613 | 2.141 |

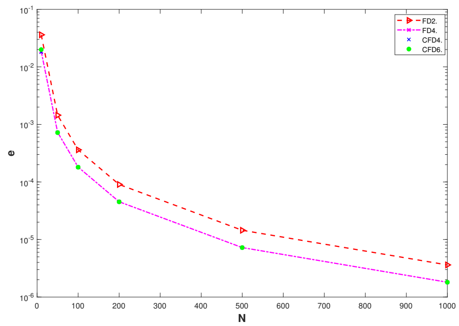



Figure 2: Error analysis of proposed methods for Example 4.

## 5 Conclusions

In this paper, we investigated that CFD methods have become famous in the numerical discretization of differential equations in recent years. We proposed CFD methods for solving the anti-periodic BVPs. Firstly, a CFD method of order 4 and 6 for anti-periodic BVP (1) are presented, and then a convergence analysis was provided to show that the theoretical order of the methods is the same in accuracy. Finally, numerical examples, are given to confirm the efficiency and accuracy of the CFD methods.

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