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# Estimate of the fractional advection-diffusion equation with a time-fractional term based on the shifted Legendre polynomials

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**Abstract.** In this paper, we present a well-organized strategy to estimate the fractional advection-diffusion equations, which is an important class of equations that arises in many application fields. Thus, Lagrange square interpolation is applied in the discretization of the fractional temporal derivative, and the weighted and shifted Legendre polynomials as operators are exploited to discretize the spatial fractional derivatives of the space-fractional term in multi-term time fractional advection-diffusion model. The privilege of the numerical method is the orthogonality of Legendre polynomials and its operational matrices which reduces time computation and increases speed. A second-order implicit technique is given, and its stability and convergence are investigated. Finally, we propose three numerical examples to check the validity and numerical results to illustrate the precision and efficiency of the new approach.

*Keywords*: Advection-diffusion model, multi-term time fractional term, collocation method, Legendre polynomial, stability, convergence.
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## 1 Introduction

Advection-diffusion is a physical concept that describes the movement of a substance in a fluid. It combines two important mechanisms: advection, which is the transport of a substance due to the flow of the fluid, and diffusion, which is the random movement of the substance due to its thermal energy [5]. Advection-diffusion occurs in a wide range of natural and man-made systems, such as the transport of pollutants in the atmosphere, the spread of chemicals in water bodies, and the exchange of gases in

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biological tissues. The behavior of advection-diffusion is influenced by various factors, including the properties of the fluid and the substance, the geometry of the system, and the boundary conditions [3]. Understanding advection-diffusion is crucial for predicting and controlling the transport of substances in different environments.

In recent years, several physical phenomena have been observed in media with complicated inner designs. This has led to conclusions about mathematical samples and differential equations that include fractional derivatives [13, 16].

In this paper, a type of multi-term time-fractional advection-diffusion equation (MTFADE) is being evaluated as below

$$\mathrm{P}(D_\tau)v(x,\tau) = K_1\frac{\partial^{2\beta_1}v(x,\tau)}{\partial|x|^{2\beta_1}} + K_2\frac{\partial^{2\beta_2}v(x,\tau)}{\partial|x|^{2\beta_2}} + q(x,\tau),\ 0<x<1,\ \ 0<\tau<T, \tag{1}$$

with the boundary conditions and initial condition as

$$\begin{aligned} &v(0,\tau) = r(\tau), \quad v(0,\tau) = s(\tau), \ \ 0\le\tau\le T,\\ &v(x,0) = z(x), \quad 0\le x\le 1, \end{aligned}$$

respectively. In Eq. (1), the fractional factor $\mathrm{P}(D_\tau)$ is described as

$$\mathrm{P}(D_\tau)v(x,\tau) = \sum_{e=0}^{\varepsilon} b_e {}_0^C D_\tau^{\alpha_e} v(x,\tau),$$

with $0<\alpha_\varepsilon<\cdots<\alpha_1<\alpha_0\le 1$ and $b_e\ge 0$. Terms $\frac{\partial^{2\beta_i}v(x,\tau)}{\partial|x|^{2\beta_i}}, i=1,2$ are the Riesz fractional derivative with respect to $x$ are described as

$$\frac{\partial^{2\beta_i}v(x,\tau)}{\partial|x|^{2\beta_i}} = \frac{-1}{2\cos(\beta_i\pi)}\left({}_xD_L^{2\beta_i}v(x,\tau) + {}_xD_R^{2\beta_i}v(x,\tau)\right),$$

where ${}_xD_L^{2\beta_i}v(x,\tau)$ and ${}_xD_R^{2\beta_i}v(x,\tau)$ describe the left and right fractional Riemann-Liouville derivatives specified as below

$$\begin{aligned} {}_xD_L^{2\beta_i}v(x,\tau) &= \frac{1}{\Gamma(2-2\beta_i)}\left(\frac{d}{dx}\right)^n\int_0^x (x-\zeta)^{n-2\beta_i}v(\zeta,\tau)d\zeta,\\ {}_xD_R^{2\beta_i}v(x,\tau) &= \frac{1}{\Gamma(2-2\beta_i)}\left(-\frac{d}{dx}\right)^n\int_x^1 (\zeta-x)^{n-2\beta_i}v(\zeta,\tau)d\zeta, \end{aligned}$$

where $n-1<2\beta_i\le n, n\in\mathbb{N}$.

Note that a well-known Eq. (1) consists of some classic fractional instances. Using $e=0$ in the fractional factor $\mathrm{P}(D_\tau)$ results in traditional advection-diffusion equation [20, 23]. Taking $K_1=0$ and $\beta_2=1$ gives you the multi-term time-fractional diffusion equations [10, 19]. Taking $\alpha_0=2, \alpha_1=1$, and $K_1=0$ in the case $1<\alpha_e<\alpha_{e-1}<\cdots<\alpha_0\le 2$ results in a spatially fractional Telegraph equation [8].

Many studies have been done about the space-fractional, time-fractional, and space-time-fractional generalization of the advection-diffusion model [15, 17, 18, 21]. In addition to the physical aspects of this equation, it has numerical aspects as well which are used for its solution and were vastly investigated in many articles [24].

To solve space-time fractional models, such as the fractional advection-diffusion model, productive implicit numerical schemes have been proposed in [4, 25], and stability and convergence have been considered [11, 12, 20]. Many papers have been designed based on numerical and analytical solutions of MTFADEs [22]. Analytical solutions of these equations with just the diffusion term have also been examined in [6].

The remainder of this paper deals with the following sections. The numerical procedure and construction to discretize the instance are presented in Section 2. Moreover, the approximating operator of the fractional Legendre polynomials is described in this section. The analysis of the convergence and stability of the time scheme of the numerical strategy is also presented in Section 3. Section 4 deals with some numerical examples and their comparison with other results.

## 2 Discretization of the model

This section consults the discretization of Eq. (1). To do so, let $\delta h$ and $\delta\tau = \frac{T}{M}$ denote the sizes of the spatial and temporal discretization steps, respectively. We describe the partition in space as $\{x_i\}_{i=0}^{M}$ that is the root of the basis polynomials and the partition in time as $\tau_j = j\delta\tau$, $j = 0, 1, \ldots, M$. We denote the discrete values of the variables $v(x,\tau)$ and $q(x,\tau)$ as

$$v(x,\tau_k) = v^k(x), \quad v(x_i,\tau_k) = v_i^k, \quad q(x_i,\tau_k) = q_i^k, \;\; k = 0, 1, \ldots, M.$$

Since $0 < \alpha_e \leq 1$, the Caputo fractional derivative can be used instead of the Riemann-Liouville.

### 2.1 Time-discrete scheme

By applying $S_2$ method in paper [9] in which the unknown function is approximated by Lagrange linear interpolation for the case that $j = 0$ in the interval $[\tau_0, \tau_1]$ and by Lagrange square interpolation for $j > 0$ in the interval $[\tau_{j-1}, \tau_j]$ to discretize the temporal variable. Practically, we have three nodes to achieve the approximation.

$${}_0^C D_\tau^{\alpha_e} v^k(x) = \frac{\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)} \sum_{j=0}^{k} \mathscr{S}_{k,j}^{\alpha_e} v^j(x) + \mathscr{O}(\delta\tau^{3-\alpha_e}), \tag{2}$$

where for $k = 1$ and $k = 2$ the coefficients $\mathscr{S}_{k,j}^{\alpha_e}$ is as

$$\mathscr{S}_{1,j}^{\alpha_e} = \begin{cases} -\mathscr{A}_1, & j = 0, \\ \mathscr{A}_1, & j = 1, \end{cases} \qquad \mathscr{S}_{2,j}^{\alpha_e} = \begin{cases} -\mathscr{A}_2 + \mathscr{B}_{2,2}, & j = 0, \\ \mathscr{A}_2 + \mathscr{C}_{2,2}, & j = 1, \\ \mathscr{D}_{2,2}, & j = 2, \end{cases}$$

and for $k \geq 3$, we have

$$\mathscr{S}_{k,j}^{\alpha_e} = \begin{cases} -\mathscr{A}_k + \mathscr{B}_{k,j+2}, & j = 0, \\ \mathscr{A}_k + \mathscr{B}_{k,j+2} + \mathscr{C}_{k,j+1}, & j = 1, \\ \mathscr{B}_{k,j+2} + \mathscr{C}_{k,j+1} + \mathscr{D}_{k,j}, & 2 \leq j \leq k-2, \\ \mathscr{C}_{k,j+1} + \mathscr{D}_{k,j}, & j = k-1, \\ \mathscr{D}_{k,j}, & j = k, \end{cases}$$

in which

$$
\begin{aligned}
\mathscr{A}_k &= k^{1-\alpha_e} - (k-1)^{1-\alpha_e},\\
\mathscr{B}_{k,j} &= \frac{1}{2-\alpha_e}\big[(k-j+1)^{1-\alpha_e}(k-j+\frac{\alpha_e}{2}) - (k-j)^{1-\alpha_e}(k-j-\frac{\alpha_e}{2}+1)\big],\\
\mathscr{C}_{k,j} &= \frac{2}{2-\alpha_e}\big[(k-j)^{1-\alpha_e}(k-j-\alpha_e+2) - (k-j+1)^{2-\alpha_e}\big],\\
\mathscr{D}_{k,j} &= \frac{1}{2-\alpha_e}\big[(k-j+1)^{1-\alpha_e}(k-j-\frac{\alpha_e}{2}+2) - (k-j)^{1-\alpha_e}(k-j-\frac{3\alpha_e}{2}+3)\big].
\end{aligned}
$$

Applying the discretization of the relation (2) for the left side of Eq. (1), we have

$$
\begin{aligned}
\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}^{\alpha_e}_{k,k}v^k(x) - K_1\frac{\partial^{2\beta_1}v^k(x)}{\partial|x|^{2\beta_1}} - K_2\frac{\partial^{2\beta_2}v^k(x)}{\partial|x|^{2\beta_2}} = {}&\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}^{\alpha_e}_{k-1,j}v^j(x)\\
&+q^k(x)+\mathscr{O}(\delta\tau^{3-\max\alpha_e}),\ 0<x<1,
\end{aligned}
\tag{3}
$$

Letting $V_i^k$ as the approximate solution $v_i^k$ in Eq. (3), then we have

$$
\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}^{\alpha_e}_{k,k}V^k(x) - K_1\frac{\partial^{2\beta_1}V^k(x)}{\partial|x|^{2\beta_1}} - K_2\frac{\partial^{2\beta_2}V^k(x)}{\partial|x|^{2\beta_2}} = \sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}^{\alpha_e}_{k-1,j}V^j(x)+q^k(x). \tag{4}
$$

## 2.2 Spatial discrete scheme with Legendre approximation

Now, to get the full discrete of Eq. (4), we employ the following series

$$
V^k(x) = \sum_{i=0}^{k}\sigma_i^k\mathbb{L}_i^{\circledS}(x), \quad k=0,1,\dots,N, \tag{5}
$$

where $\mathbb{L}_\iota^{\circledS}(x)$ is the shifted orthogonal polynomials in the domain $[0,1]$ that is defined in paper [14] as

$$
\mathbb{L}_i^{\circledS}(x) = \sum_{r=0}^{\lfloor\frac{i}{2}\rfloor}\sum_{\iota=0}^{i-2r}N_{i,r,\iota}x^\iota, \quad i=0,1,\dots, \tag{6}
$$

where

$$
N_{i,r,\iota} = \binom{i-2r}{\iota}\frac{(-1)^{i-r-\iota}2^{\iota-i}(2i-2r)!}{r!(i-r)!(i-2r)!}.
$$

Then the unknown coefficients in Eq. (6) are defined as below

$$
\sigma_i^k = (2i+1)\int_0^1\mathbb{L}_i^{\circledS}(x)V^k(x)dx.
$$

Here is a formula for the approximation of the fractional derivative $\mathbb{L}_i^{\circledS}$, which we denote by $\mathbb{L}_i^{\gamma,\circledS}$. Letting $\gamma>0$, by using the Caputo linearity property, we get

$$
\mathbb{L}_i^{\gamma,\circledS}(x) = \sum_{r=0}^{\lfloor\frac{i}{2}\rfloor}\sum_{\iota=\lceil\gamma\rceil}^{i-2r}N^{\gamma}_{i,r,\iota}x^{\iota-\gamma}, \quad i=0,1,\dots, \tag{7}
$$

in which $\lfloor \gamma \rfloor$ and $\lceil \gamma \rceil$ are the floor and ceiling of the fractional term $\gamma$ and

$$N_{i,r,\iota}^{\gamma} = \binom{i-2r}{\iota} \frac{(-1)^{i-r-\iota} 2^{\iota-i} (2i-2r)! \Gamma(\iota+1)}{r!(i-r)!(i-2r)! \Gamma(\iota-\gamma+1)}.$$

Notice that for $0 \le i < \lceil \gamma \rceil$, we have $\mathbb{L}_i^{\gamma,\circledS}(x) = 0$. Substituting the operators (6) and (7) in (4), we approximate $V^k(x)$ with respect to $x$ as below

$$\sum_{i=0}^{k} \sigma_i^k A_{e,i}(x) = \sum_{j=0}^{k-1} \sum_{i=0}^{j} \sigma_i^j B_{e,i}(x) + Q(x), \tag{8}$$

where

$$A_{e,i}(x) = \sum_{e=0}^{\varepsilon} \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\iota=0}^{i-2r} \frac{b_e \delta \tau^{-\alpha_e}}{\Gamma(2-\alpha_e)} \mathscr{S}_{k,k}^{\alpha_e} N_{i,r,\iota} x^{\iota} + \frac{K_1}{2\cos(\beta_1 \pi)} \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\iota=\lceil \beta_1 \rceil}^{i-2r} \left( N_{i,r,\iota}^{\beta_1} x^{\iota-\beta_1} + N_{i,r,\iota}^{\beta_1} (1-x)^{\iota-\beta_1} \right)$$
$$+ \frac{K_2}{2\cos(\beta_2 \pi)} \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\iota=\lceil \beta_2 \rceil}^{i-2r} \left( N_{i,r,\iota}^{\beta_2} x^{\iota-\beta_2} + N_{i,r,\iota}^{\beta_2} (1-x)^{\iota-\beta_2} \right),$$
$$B_{e,i}(x) = \sum_{e=0}^{\varepsilon} \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\iota=0}^{i-2r} \frac{b_e \delta \tau^{-\alpha_e}}{\Gamma(2-\alpha_e)} \mathscr{S}_{k-1,j}^{\alpha_e} N_{i,r,\iota} x^{\iota}, \quad Q(x) = q^k(x).$$

We use the collocation manner to obtain the coefficients $\sigma_i^k$ of relation (8). For this objective, we get the roots of the shifted Legendre polynomials, $\mathbb{L}_i^{\circledS}(x)$, as the collocation points and substitute them in Eq. (8) to obtain linear equations at each time step $k$. $\mathbb{L}_{N-1}^{\circledS}(x)$ has $N-1$ roots, which needs two more conditions to obtain a system of linear equations with $N+1$ equations, which can be obtained using the following boundary conditions for $i = 0, 1, \ldots, N$ and $k = 1, 2, \ldots, M$

$$v(0,t) = \sum_{i=0}^{k} \sigma_i^k \mathbb{L}_i^{\circledS}(0) = \sum_{i=0}^{k} \sigma_i^k (-1)^i = \mu(t_k), \quad v(1,t) = \sum_{i=0}^{k} \sigma_i^k \mathbb{L}_i^{\circledS}(1) = \sum_{i=0}^{k} \sigma_i^k = \rho(t_k).$$

To start the iterative method, we need the initial condition as

$$\sum_{i=0}^{k} \sigma_i^k \mathbb{L}_i^{\circledS}(x) = \phi(x),$$

where $\sigma_i^k = (2i+1) \int_0^1 \mathbb{L}_i^{\circledS}(x) \phi(x) dx$.

## 3 Study of the convergence approach

This section investigates the order of convergence and the method's stability. Defining the following function space in Hilbert space $L^2(\Omega)$ in $\Omega$ and standard norm $\|\theta(x)\|_2^2 = \langle \theta(x), \theta(x) \rangle$, we prove two theorems that describe the precision and efficiency of the numerical strategy explained in the earlier section.

$$H_{\Omega}^n(\theta) = \{\theta \in L^2(\Omega), D^{\alpha}\theta \in L^2(\Omega), \forall |\alpha| \le n\},$$

where $D^{\alpha}$ is the fractional derivative.

Let $V^k(x)$ and $\overline{V}^k(x)$ be the exact and approximate solution of Eq. (4), respectively. Multiplying $\varepsilon^k(x)$ and integrating on $\Omega$ in the relation (4) and denoting $\varepsilon^k(x)=\overline{V}^k(x)-V^k(x)$, we get the weak form of the relation (4) as

$$\begin{aligned}\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k,k}^{\alpha_e}\langle\varepsilon^k(x),\varepsilon^k(x)\rangle-K_1\langle\frac{\partial^{2\beta_1}\varepsilon^k(x)}{\partial|x|^{2\beta_1}},\varepsilon^k(x)\rangle-K_2\langle\frac{\partial^{2\beta_2}\varepsilon^k(x)}{\partial|x|^{2\beta_2}},\varepsilon^k(x)\rangle\\ =\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k-1,j}^{\alpha_e}\langle\varepsilon^j(x),\varepsilon^k(x)\rangle.\end{aligned}\tag{9}$$

Before giving the stability and convergence of the weak scheme (9), we firstly give some lemmas.

**Lemma 1** ([7])**.** *For all $f,g\in H_{\Omega}^n$ and $x\in\mathbb{R}$, we have*

$$\langle {}_xD_L^{\alpha}f(x),{}_xD_R^{\alpha}f(x)\rangle=\cos(\alpha\pi)\|{}_xD_L^{\alpha}f(x)\|^2=\cos(\alpha\pi)\|{}_xD_R^{\alpha}f(x)\|^2,\quad\forall\alpha>0,$$
$$\langle {}_xD_L^{\alpha}f(x),g(x)\rangle=\langle {}_xD_L^{\frac{\alpha}{2}}f(x),{}_xD_R^{\frac{\alpha}{2}}g(x)\rangle,\quad\langle {}_xD_R^{\alpha}f(x),g(x)\rangle=\langle {}_xD_R^{\frac{\alpha}{2}}f(x),{}_xD_L^{\frac{\alpha}{2}}g(x)\rangle,\quad\forall\alpha\in(1,2).$$

**Lemma 2.** *For all $\varepsilon\in H_{\Omega}^n$, we have*

$$\langle\frac{\partial^{2\beta_i}\varepsilon^k(x)}{\partial|x|^{2\beta_i}},\varepsilon^k(x)\rangle=-\|{}_xD_L^{\beta_i}\varepsilon^k(x)\|^2,\quad\forall i=1,2.$$

*Proof.* Using Lemma 1 and features of the interior product, we get

$$\begin{aligned}\langle\frac{\partial^{2\beta_i}\varepsilon^k(x)}{\partial|x|^{2\beta_i}},\varepsilon^k(x)\rangle&=\langle\frac{-1}{2\cos(\beta_i\pi)}\big({}_xD_L^{2\beta_i}\varepsilon^k(x)+{}_xD_R^{2\beta_i}\varepsilon^k(x)\big),\varepsilon^k(x)\rangle\\&=\frac{-1}{2\cos(\beta_i\pi)}\Big(\langle {}_xD_L^{2\beta_i}\varepsilon^k(x),\varepsilon^k(x)\rangle+\langle {}_xD_R^{2\beta_i}\varepsilon^k(x),\varepsilon^k(x)\rangle\Big)\\&=\frac{-1}{2\cos(\beta_i\pi)}\Big(\langle {}_xD_L^{\beta_i}\varepsilon^k(x),{}_xD_R^{\beta_i}\varepsilon^k(x)\rangle+\langle {}_xD_R^{\beta_i}\varepsilon^k(x),{}_xD_L^{\beta_i}\varepsilon^k(x)\rangle\Big)\\&=-\|{}_xD_L^{\beta_i}\varepsilon^k(x)\|^2.\end{aligned}$$

□

**Lemma 3** (See [1])**.** *The coefficients that are defined in Eq. (2) have the following properties*

$$1<\mathscr{S}_{k,k}^{\alpha_e}\le\frac{3}{2},\quad 1<\mathscr{D}_{k,k}\le\frac{3}{2},\quad\sum_{j=0}^{k-1}\mathscr{S}_{k,j}=-\mathscr{D}_{k-1,k-1}.$$

**Theorem 1.** *When $0<\alpha_e,\beta_i<1,\ \ e=0,1,\dots,\varepsilon,\ \ i=1,2,$ the scheme (3) is unconditionally stable.*

*Proof.* Lemmas 1 and 2 allow us to remove the second and third terms of Eq. (9). Then we have

$$\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k,k}^{\alpha_e}\langle\varepsilon^k(x),\varepsilon^k(x)\rangle\le\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k-1,j}^{\alpha_e}\langle\varepsilon^j(x),\varepsilon^k(x)\rangle,$$

and using the Cauchy Schwarz inequality, one get

$$\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k,k}^{\alpha_e}\|\varepsilon^k(x)\|^2\leq\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k-1,j}^{\alpha_e}\|\varepsilon^j(x)\|\|\varepsilon^k(x)\|,$$

$$\Rightarrow\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k,k}^{\alpha_e}\|\varepsilon^k(x)\|\leq\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k-1,j}^{\alpha_e}\|\varepsilon^j(x)\|,$$

Since $\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}>0$ and $1<\mathscr{S}_{k,k}^{\alpha_e}\leq\frac{3}{2}$ from Lemma 3 we denote the above relation as

$$\|\varepsilon^k(x)\|\leq\varepsilon\mathscr{S}_{k,k}^{\alpha_e}\|\varepsilon^k(x)\|\leq\sum_{e=0}^{\varepsilon}\mathscr{S}_{k,k}^{\alpha_e}\|\varepsilon^k(x)\|\leq\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\Gamma_e\mathscr{S}_{k-1,j}^{\alpha_e}\|\varepsilon^j(x)\|,$$

where $\Gamma_e=\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}$. Applying the theory of induction applies to all $k$, we know that the above relation satisfies as below

$$\|\varepsilon^k(x)\|\leq\sum_{e=0}^{\varepsilon}\Gamma_e\sum_{j=0}^{k-1}\mathscr{S}_{k-1,j}^{\alpha_e}\|\varepsilon^0(x)\|.$$

Using the properties of the expansion coefficients in the Lemma 3 that describe the characteristics of approximation coefficient $1<\mathscr{D}_{k,k}\leq\frac{3}{2}$ and $\sum_{j=0}^{k-1}\mathscr{S}_{k,j}=-\mathscr{D}_{k-1,k-1}$, it can be written as

$$\|\varepsilon^k(x)\|\leq\sum_{e=0}^{\varepsilon}\Gamma_e|\mathscr{D}_{k,k}|\|\varepsilon^0(x)\|=c\|\varepsilon^0(x)\|,$$

where $c=\sum\limits_{e=0}^{\varepsilon}\Gamma_e|\mathscr{D}_{k,k}|$ is the positive constant. We have shown that the discrete scheme (4) is unconditionally stable. □

The convergence of the scheme (4) is given in the following theorem.

**Theorem 2.** *Let $v^k(x)\in H_\Omega^2$ avd $V^k(x)\in H_\Omega^2$ be the exact solution of* (3) *and the solution of the semi-scheme* (4)*, respectively. Considering $\xi^M(x)=|v^k(x)-V^k(x)|$ and $C$ as a positive constant, we have*

$$\|\xi^M(x)\|\leq C\mathscr{O}(\delta\tau^{3-\max\alpha_e}).$$

*Proof.* Subtracting (3) from (4) and denoting $\xi^k(x)=v^k(x)-V^k(x),k=1,2,\ldots,M$, we get

$$\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k,k}^{\alpha_e}\xi^k(x)-K_1\frac{\partial^{2\beta_1}\xi^k(x)}{\partial|x|^{2\beta_1}}-K_2\frac{\partial^{2\beta_2}\xi^k(x)}{\partial|x|^{2\beta_2}}=\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k-1,j}^{\alpha_e}\xi^j(x)+CR^k,$$

where $R^k=\mathscr{O}(\delta\tau^{3-\max\alpha_e})$. From Lemmas 1 and 2, we know that the second and third terms of the above relation are negative, so it can be rewritten as

$$\sum_{e=0}^{\varepsilon}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k,k}^{\alpha_e}\|\xi^k(x)\|\leq\sum_{e=0}^{\varepsilon}\sum_{j=0}^{k-1}\frac{b_e\delta\tau^{-\alpha_e}}{\Gamma(2-\alpha_e)}\mathscr{S}_{k-1,j}^{\alpha_e}\|\xi^j(x)\|+C\|R^k\|.$$

By the employed method to prove the earlier theorem, we know that there exists a positive fixed term $c$ such that

$$\|\xi^k(x)\| \leq c\|\xi^0(x)\| + C\|R^k\|.$$

Since $\|\xi^0(x)\| = 0$, we deduce that

$$\|\xi^M(x)\| \leq C\mathscr{O}(\delta\tau^{3-\max\alpha_e}).$$

This concludes the proof. □

# 4 Numerical experiments

This section presents numerical experiments to demonstrate the proposed method's ability. The experiments are carried out using Wolfram Mathematica (version 11) on a personal computer with a Dell Inspiron Intel (R) Core i72630QM 2.00GHz and 8:00 GB of memory. In the experiments, the suggested new method is applied, and we also present the maximum norm error and $L_2-norm$ error between the exact solution and numerical solution at $T=1$ as below

$$L_\infty - norm = E_\infty(\delta h, \delta\tau) = \max_{0\leq i\leq N} |v_i^M - \widehat{v}_i^M|, \quad L_2 - norm = E_2(\delta h, \delta\tau) = \sqrt{\frac{1}{N}\sum_{i=0}^{N} |v_i^M - \widehat{v}_i^M|^2},$$

respectively. In which $v_i^M$ is the exact solution and $\widehat{v}_i^M$ is the numerical solution with the mesh step-size $\delta h$ and $\delta\tau$ at the grid point $(x_i, \tau_j), i = 0,1,\ldots,N,\ j = 0,1,\ldots,M$. Moreover, the convergence rate is calculated as follows

$$Rate_\infty = \log_2 \frac{E_\infty(\delta h, 2\delta\tau)}{E_\infty(\delta h, \delta\tau)}, \quad Rate_2 = \log_2 \frac{E_2(\delta h, 2\delta\tau)}{E_2(\delta h, \delta\tau)}.$$

The computing results for the maximal norm error and the convergence rate of examples are shown in tables, which are utilizing the new method to discretize Eq. (1), wherein $E_\infty(\delta h, \delta\tau)$ represents the maximal norm error and $Rate_\infty$ represents the convergence rate related to it, defined in the above relations, respectively. The results in this tables demonstrate that can achieve $\mathscr{O}(3-\max(\alpha_i)), i = 1,2$ convergence accuracy in the temporal direction. Also, we compared the results of this paper with paper [2], and the accuracy of the new method is better than the method of paper [2] and has fewer errors than it.

**Example 1.** We first consider one-dimensional Eq. (1), where $(x,\tau) \in [0,1]\times[0,1]$, $T=1$, and the diffusion coefficients are given by $K_1 = K_2 = 1$ and the source term is

$$\begin{aligned}
q(x,t) &= 200(x^2 - x^3)\Big(\frac{t^{2-\alpha_0}}{\Gamma(3-\alpha_0)} + \frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)}\Big) \\
&+ \frac{50K_1(t^2+1)}{\cos(\beta_1\pi)}\Big(\frac{2\big(x^{2-2\beta_1} + (1-x)^{2-2\beta_1}\big)}{\Gamma(3-2\beta_1)} - \frac{6\big(x^{3-2\beta_1} + (1-x)^{3-2\beta_1}\big)}{\Gamma(4-2\beta_1)}\Big) \\
&+ \frac{50K_2(t^2+1)}{\cos(\beta_2\pi)}\Big(\frac{2\big(x^{2-2\beta_2} + (1-x)^{2-2\beta_2}\big)}{\Gamma(3-2\beta_2)} - \frac{6\big(x^{3-2\beta_2} + (1-x)^{3-2\beta_2}\big)}{\Gamma(4-2\beta_2)}\Big).
\end{aligned}$$

The initial and boundary values are chosen as $v(x,0) = 100(x^2 - x^3)$ and $v(0,\tau) = v(1,\tau) = 0$, respectively. Moreover, the exact solution to the problem is $v(x,\tau) = 100(1+\tau^2)(x^2 - x^3)$. The computing

Table 1: Maximal norm, $E_2$-norm error and convergence rate with $\beta_1 = 0.2, \beta_2 = 0.7$, and $N = 5$ for Example 1 at $T = 1$.

| | with $\alpha_1 = \alpha_2 = 0.5$ | | | | with $\alpha_1 = 0.7, \alpha_2 = 0.3$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\delta\tau$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ |
| $\frac{1}{20}$ | $2.88003E{-3}$ | – | $8.36760E{-3}$ | – | $4.65992E{-3}$ | – | $1.35337E{-2}$ | – |
| $\frac{1}{40}$ | $5.04560E{-4}$ | 2.51299 | $1.46594E{-3}$ | 2.51298 | $9.12647E{-4}$ | 2.35218 | $2.65059E{-3}$ | 2.35217 |
| $\frac{1}{80}$ | $8.88040E{-5}$ | 2.50633 | $2.58011E{-4}$ | 2.50633 | $1.80816E{-4}$ | 2.33553 | $5.25143E{-5}$ | 2.33553 |
| $\frac{1}{160}$ | $1.56647E{-5}$ | 2.50311 | $4.55123E{-5}$ | 2.50310 | $3.60866E{-5}$ | 2.32499 | $1.04806E{-6}$ | 2.32499 |
| $\frac{1}{320}$ | $2.76789E{-6}$ | 2.50066 | $8.04183E{-6}$ | 2.50066 | $7.23087E{-6}$ | 2.31926 | $2.10001E{-5}$ | 2.31926 |

Table 2: Comparing the error and convergence order of method [2] with the new method for Example 1 with $\alpha_0 = 0.5, \alpha_1 = 0.2, \beta_0 = 0.3$, and $\beta_1 = 0.8$.

| | Error and the convergence order of paper [2] | | Error and the convergence order of the new method with $N = 5$ | | | |
|---|---|---|---|---|---|---|
| $\delta\tau$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ |
| $\frac{1}{64}$ | $2.9976E{-2}$ | – | $3.91334E{-5}$ | – | $1.08214E{-4}$ | – |
| $\frac{1}{128}$ | $9.4587E{-3}$ | 1.6641 | $6.72919E{-6}$ | 2.53990 | $1.86054E{-5}$ | 2.54009 |
| $\frac{1}{256}$ | $3.1683E{-3}$ | 1.5779 | $1.16248E{-6}$ | 2.53323 | $3.21374E{-6}$ | 2.53340 |
| $\frac{1}{512}$ | $1.5910E{-3}$ | 1.5435 | $1.80841E{-7}$ | 2.68442 | $4.99025E{-7}$ | 2.68707 |
| $\frac{1}{1024}$ | $8.3916E{-4}$ | 1.5777 | $1.11978E{-8}$ | 2.64325 | $3.13224E{-8}$ | 2.65001 |

results of this model are shown in Table 1, wherein the maximal norm and $E_2$-norm errors have been provided with parametrs $\beta_1 = 0.2, \beta_2 = 0.7$, and the convergence rate has been found that it is equal to $\mathscr{O}(3 - \max(\alpha_i)), i = 1, 2$ for all temporal discretization levels. From this table, we can see that the convergence rate is 2.5 when $\alpha_1 = \alpha_2 = 0.5$ but it is 2.3 for $\alpha_1 = 0.7, \alpha_2 = 0.3$. Then the largest numbers of the parameters $\alpha_i$ are required to determine the convergence rate. Comparing the error and convergence order of the method [2] with the new method is represented in Table 2 with the parameters $\alpha_0 = 0.5, \alpha_1 = 0.2, \beta_0 = 0.3$, and $\beta_1 = 0.8$. From this table, we can see that the proposed method is much more efficient than method of [2]. Because when $\alpha_0 = 0.5$ the convergence order for the new method and method [2] is 2.5 and 1.5, respectively. Moreover, the error of the new method is much less compared to the paper of [2].

In Figure 1, we depict the curves of condition numbers of the stiffness matrix obtained for the space partitions $N = 5$ and $N = 7$ with the various values $\beta_1$ and $\beta_2$. We can observe that the two curves coincide precisely. It means the new algorithm will perform strongly to discrete the equation to be obtained the resulting systems in this situation that will decrease condition numbers by increasing the number of discretization points. And Figure 2 displays the absolute error with parameters $N = 5$, $\alpha_1 = 0.5, \alpha_2 = \beta_1 = 0.2$, and $\beta_2 = 0.7$ with different values $M$ which will decrease the absolute error by increasing the number of discretization points.

**Example 2.** This example is similar to Example 1, where the exact solution is $v(x, \tau) = 100(1 + \tau^2)x^2(1 -$

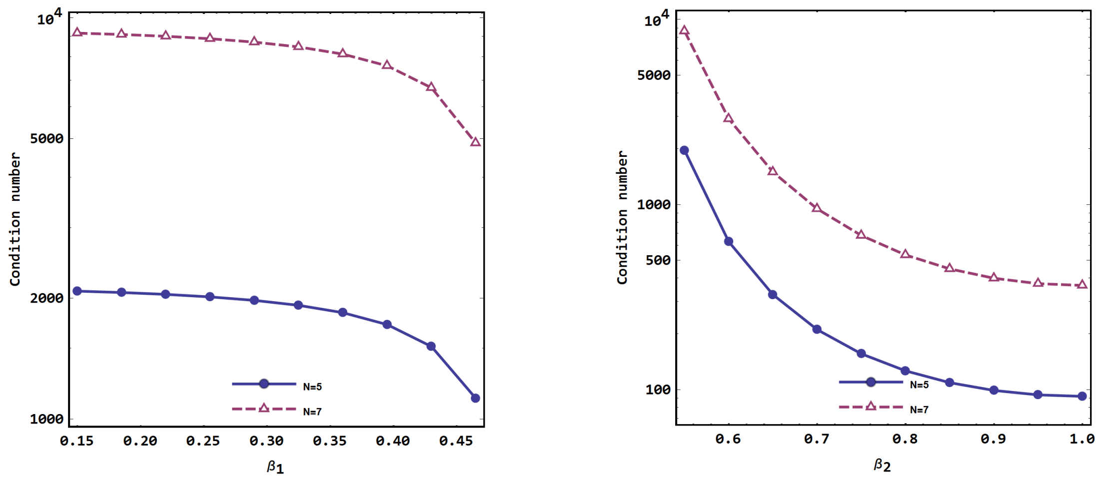


Figure 1: Condition numbers of the stiffness matrix obtained for the space partitions $N = 5$ and $N = 7$ with the various values $\beta_1$ (left panel) and $\beta_2$ (right panel) for Example 1.

$x)^2$ and the diffusion coefficients are given as $K_1 = K_2 = 1$ and the source term is

$$q(x,t) = 200x^2(1-x)^2\Big(\frac{t^{2-\alpha_0}}{\Gamma(3-\alpha_0)} + \frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)}\Big) + \frac{100K_1(t^2+1)}{\cos(\beta_1\pi)}\Big(\frac{\big(x^{2-2\beta_1}+(1-x)^{2-2\beta_1}\big)}{\Gamma(3-2\beta_1)} - \frac{6\big(x^{3-2\beta_1}+(1-x)^{3-2\beta_1}\big)}{\Gamma(4-2\beta_1)} + \frac{12\big(x^{4-2\beta_1}+12(1-x)^{4-2\beta_1}\big)}{\Gamma(5-2\beta_1)}\Big) + \frac{100K_2(t^2+1)}{\cos(\beta_2\pi)}\Big(\frac{\big(x^{2-2\beta_2}+(1-x)^{2-2\beta_2}\big)}{\Gamma(3-2\beta_2)} - \frac{6\big(x^{3-2\beta_2}+(1-x)^{3-2\beta_2}\big)}{\Gamma(4-2\beta_2)} + \frac{12\big(x^{4-2\beta_2}+(1-x)^{4-2\beta_2}\big)}{\Gamma(5-2\beta_2)}\Big).$$

The computed results of this model are represented in Table 3, that the maximal norm and $E_2$-norm errors have been provided with parameters $\beta_1 = 0.45, \beta_2 = 0.7$, and the convergence rate has been found that it is equal to $\mathscr{O}(3-\max(\alpha_i)), i = 1,2$ for all temporal discretization levels. From this table, we see that the convergence rate is 2.5 when $\alpha_1 = \alpha_2 = 0.5$ but it is 2.35 for $\alpha_1 = 0.2, \alpha_2 = 0.65$. Then the largest numbers of the parameters $\alpha_i$ are required to determine the convergence rate. Comparing the error and convergence order of the method [2] with the new method is represented in Table 4 with the parameters $\alpha_0 = 0.7, \alpha_1 = 0.4, \beta_0 = 0.3$ and $\beta_1 = 0.85$. From this table, we observe that the proposed method is much more efficient than method of [2]. Moreover, the error of the new method is much less compared to the paper of [2]. Figure 3 displays the absolute error with parameters $N = 5$, $\alpha_1 = 0.3, \alpha_2 = 0.6, \beta_1 = 0.4$ and $\beta_2 = 0.8$ with different values $M$ which will decrease the absolute error by increasing the number of discretization points.

**Example 3.** This example is not similar to Examples 1 and 2, in which the exact solution is $v(x,\tau) = \exp(-t)\sin x$ and the diffusion coefficients are given $K_1 = K_2 = 1$ and the source term is achieved with the exact solution. The initial and boundary values are $v(x,0) = \sin x$, $v(0,\tau) = 0$, and $v(1,\tau) = \sin 1$

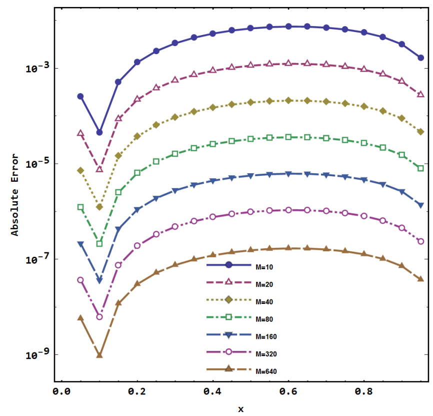


Figure 2: Absolute error for the first example with parameter $N = 5$, $\alpha_1 = 0.5, \alpha_2 = \beta_1 = 0.2$, and $\beta_2 = 0.7$ at $T = 1$.

Table 3: Maximal norm, $E_2$-norm error and convergence rate with $\beta_1 = 0.45, \beta_2 = 0.7$, and $N = 5$ for Example 2 at $T = 1$.

| | with $\alpha_1 = 0.2, \alpha_2 = 0.65$ | | | | with $\alpha_1 = \alpha_2 = 0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\delta\tau$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ |
| $\frac{1}{10}$ | $1.30566E-3$ | – | $4.49206E-7$ | – | $1.52349E-4$ | – | $4.64827E-4$ | – |
| $\frac{1}{20}$ | $2.20088E-5$ | 2.56862 | $2.30350E-6$ | 2.56742 | $2.31367E-5$ | 2.71912 | $7.06426E-5$ | 2.71808 |
| $\frac{1}{40}$ | $3.97720E-6$ | 2.46826 | $1.21600E-5$ | 2.46729 | $3.68642E-6$ | 2.64989 | $1.12627E-5$ | 2.64899 |
| $\frac{1}{80}$ | $7.53043E-7$ | 2.40095 | $6.72457E-5$ | 2.40025 | $6.06790E-7$ | 2.60295 | $1.85482E-6$ | 2.60220 |
| $\frac{1}{160}$ | $1.46802E-7$ | 2.35886 | $3.98597E-4$ | 2.35838 | $1.02142E-7$ | 2.57063 | $3.12350E-7$ | 2.57004 |

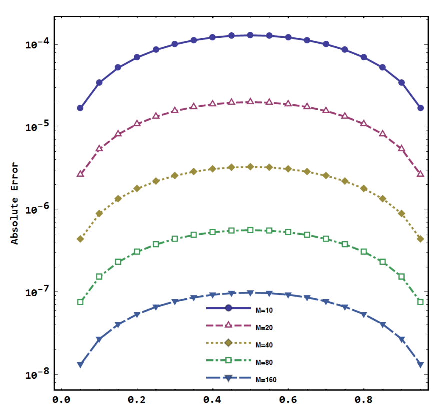


Figure 3: Absolute error with parameters $N = 5$, $\alpha_1 = 0.3, \alpha_2 = 0.6, = \beta_1 = 0.4$, and $\beta_2 = 0.8$ for Example 2 at $T = 1$.

respectively. The computed results of this model are represented in Table 5, that the maximal norm and $E_2$-norm errors have been provided with parameters $\beta_1 = 0.6, \beta_2 = 0.8$, and the convergence rate has

Table 4: Comparing the error and the CPU time of the method [2] with the new method for example 2 with $\alpha_0 = 0.7, \alpha_1 = 0.4, \beta_0 = 0.3$, and $\beta_1 = 0.85$.

| $\delta\tau$ | Error and the CPU time of paper [2] | | | | Error and the CPU time *of the new method with* $N = 5$ | | |
|---|---|---|---|---|---|---|---|
| | $E_\infty(2\delta\tau, \delta\tau)$ | $CPU(s)$ | | | $E_\infty(\delta h, \delta\tau)$ | $CPU(s)$ | $Rate_\infty$ |
| | | Gauss | CGNR | MG | | *New method* | |
| $\frac{1}{128}$ | $1.9108E-3$ | 0.02 | 0.02 | 0.04 | $1.98616E-5$ | 0.0189 | – |
| $\frac{1}{256}$ | $4.7880E-4$ | 0.24 | 0.38 | 0.19 | $3.36372E-6$ | 0.2154 | 2.56186 |
| $\frac{1}{512}$ | $1.2254E-5$ | 3.19 | 3.14 | 0.84 | $6.06274E-7$ | 1.0259 | 2.47202 |
| $\frac{1}{1024}$ | $2.7918E-5$ | 40.15 | 34.59 | 4.20 | $1.13894E-7$ | 3.2468 | 2.41227 |
| $\frac{1}{2048}$ | $8.5361E-6$ | 698.80 | 693.64 | 22.54 | $2.20280E-8$ | 8.5496 | 2.37028 |

Table 5: Maximal norm, $E_2$-norm error and convergence rate with $\beta_1 = \beta_2 = 0.7$, and $N = 5$ for Example 3 at $T = 1$.

| $\delta\tau$ | with $\alpha_1 = \alpha_2 = 0.4$ | | | | with $\alpha_1 = \alpha_2 = 0.7$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ | $E_\infty(\delta h, \delta\tau)$ | $Rate_\infty$ | $E_2(\delta h, \delta\tau)$ | $Rate_2$ |
| $\frac{1}{20}$ | $3.38983E-3$ | – | $1.48477E-2$ | – | $4.11076E-3$ | – | $1.82796E-2$ | – |
| $\frac{1}{40}$ | $5.49963E-4$ | 2.60200 | $2.44861E-3$ | 2.60020 | $8.33588E-4$ | 2.30200 | $3.71140E-3$ | 2.30020 |
| $\frac{1}{80}$ | $9.05216E-5$ | 2.60300 | $4.03787E-4$ | 2.60030 | $1.68919E-4$ | 2.30300 | $7.53492E-4$ | 2.30030 |
| $\frac{1}{160}$ | $1.48892E-5$ | 2.60399 | $6.65815E-5$ | 2.60040 | $3.42064E-5$ | 2.30400 | $1.52964E-4$ | 2.30040 |
| $\frac{1}{320}$ | $2.44739E-6$ | 2.60495 | $1.09780E-5$ | 2.60050 | $6.92212E-6$ | 2.30498 | $3.10506E-5$ | 2.30050 |

been found that it is equal to $\mathscr{O}(3-\max(\alpha_i)), i = 1,2$ for all temporal discretization levels. From this table, we can see that the convergence rate is 2.6 when $\alpha_1 = \alpha_2 = 0.4$ but it is 2.3 for $\alpha_1 = \alpha_2 = 0.7$. Then the largest numbers of the parameters $\alpha_i$ are required to determine the convergence rate. From this table, we can see that the proposed method is much more efficient and the absolute error will decrease by increasing the number of discretization points.

## 5 Conclusion

In this paper, a square interpolation to discrete of the time-fractional derivative with a high-order and spectral method based on Legendre polynomials is considered to solve MTFADE. For the discretized design of the main equation, a matrix-approximate preconditioner is constructed, and this operator is applied to solve it. Moreover, the stability and convergence of the scheme are discussed, and the semi-discrete scheme is unconditionally stable. Theoretically, the convergence rate of the preconditioned method is also given to be $\mathscr{O}(3-\max(\alpha_i))$, $i = 1,2$. The experimental results demonstrate that the proposed preconditioners are efficient for the discretization of MTFADE. In the future, we will consider the variable coefficients of higher order fractional differential equations and consider developing some fast numerical methods.

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