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A covering-based algorithm for resolution of linear programming problems with max-product bipolar fuzzy relation equation constraints

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Abstract. The linear programming problem provided to bipolar fuzzy relation equation constraints is considered in this paper. The structure of bipolar fuzzy relation equation system is studied with the max-product composition. Two new concepts, called covering and irredundant covering, are introduced in the bipolar fuzzy relation equation system. A covering-based sufficient condition is proposed to check its consistency. The relation between two concepts is discussed. Some sufficient conditions are presented to specify one of its optimal solutions or some its optimal components based on the concepts. Also, some covering-based sufficient conditions are given for uniqueness of its optimal solution. These conditions enable us to design some procedures for simplification and reduction of the problem. Moreover, a matrix-based branch-and-bound method is presented to solve the reduced problem. The sufficient conditions and algorithm are illustrated by some numerical examples. The algorithm is compared to existing methods.

Keywords: Bipolar fuzzy relation equations, covering, irredundant covering, linear programming, max-product composition.
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1 Introduction

Let $A^+ = (a^+_{ij})$ and $A^- = (a^-_{ij})$ be two $m \times n$ fuzzy relation matrices with $0 \leq a^+_{ij}, a^-_{ij} \leq 1$ for each $i \in I = \{1,2,\dots,m\}$ and $j \in J = \{1,2,\dots,n\}$. Also, assume that $b = (b_1,\dots,b_m)^T \in [0,1]^m$ and $c = (c_1,\dots,c_n) \in R^{+n}$. In this paper, the linear programming problem subject to a system of max-product

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Bipolar Fuzzy Relation Equations ($BFREs$) is formulated as follows

$$\min \quad Z(x) = \sum_{j=1}^{n} c_j x_j, \tag{1}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{2}$$

$$x \in [0,1]^n. \tag{3}$$

where the notation of $\neg x$ denotes the negation of vector x as $\neg x = (1-x_1, \ldots, 1-x_n)^T$. The notation of "$\circ$" is the max-product operator. The vector $x = (x_1, ..., x_n)^T \in [0,1]^n$ is the vector of decision variables to be determined. The system (2)-(3) consists of finding a set of solution vectors $x \in [0,1]^n$ such that for each $i \in I$, we have

$$\max_{j \in J} \max \left\{ a_{ij}^{+}.x_j, a_{ij}^{-}.(1-x_j) \right\} = b_i. \tag{4}$$

The solution set of system (2)-(3) is denoted by $S(A^+, A^-, b)$. The problem (1)-(3) along with the max-min operator has firstly been studied in a special case where A^- is a zero matrix. This problem is called a linear programming problem provided to a system of fuzzy relation equations. For the first time, the system of $FREs$ was studied by Sanchez [32] in 1976. Thereafter, many researchers have extended and investigated its practical and theoretical aspects. Its applications were considered in different areas such as fuzzy control, fuzzy pattern recognition, models of reasoning processes [24], chemistry engineering problems [25], wireless communication [43], analysis of data transmission mechanism in BitTorrent-like peer-to-peer file sharing system [44], description of the price requirements in the supply chain system [45], the urban sewage treatment system [30], and textile engineering [27]. A comprehensive review of applications of $FREs$ can be seen as their use in estimating the peak hours in transport system, predicting the behaviour to a motor-drive system, medical diagnosis, data compression, and treat analysis in Ref. [33]. Its theoretical aspects focus on the types of fuzzy relation systems with the different composition operators, solvability criteria, and algorithms for finding their solution set. The fuzzy relation systems and their related points have been introduced in [27] and a universal algorithm was given to solve the max-min FRE system in [26]. An algorithm was developed to find the solution set of max-T FRE system where T is a class of pseudo-t-norms [4]. The relationship between the lexicographic solution and the minimal solutions was studied for the max-min Fuzzy Relation Inequalities ($FRIs$) in [42]. The convexity of the solution set and number of minimal solutions were discussed for the addition-min $FRIs$ in [44]. The structure of solution set of arbitrary-term-absent max-product $FRIs$ was investigated as a new kind of max-product $FRIs$ in [29] and an algorithm was proposed to find its lexicographic minimum solutions in [46]. The leximax minimum solution was defined for system of addition-min $FRIs$ and an algorithm was developed to find the solution in [47]. The version of two-sided $FRIs$ was studied with the addition-min composition in [39]. The other version of fuzzy relation programming is to find specific solutions from $FREs$ or $FRIs$ such that they optimize several functions [41] or a certain function in (1) the weighted min-max programming based on discrete functions [40] and developed dichotomy algorithm [20] or (2) posynomial geometric programming [48]. A comprehensive study about the types of $FREs$ and their solvability has been given in [14].

The linear programming problem with the max-min FRE constraints is the result of modeling an applied problem in the textile industry. Fang and Li [7] equivalently converted it to an 0-1 integer linear programming problem and designed a branch-and-bound method to solve it. The problem was considered by the max-product operator and extended Fang and Li's method to solve this problem in [21]. Li and

Fang [13] showed that the problem with Sup-T operator, where T is a continuous triangular norm, can be reduced into a 0-1 integer programming problem in a polynomial time. Wu et al. [36] proposed an upper bound for its optimal objective value and presented a more efficient algorithm to solve the problem with the max-min operator. This algorithm examines fewer nodes to find the optimal solution than Fang and Li's method. A necessary condition was also presented to find one of the optimal solutions of the problem [35]. Three rules were given to simplify the process of finding the optimal solution based on the mentioned necessary condition. Guu and Wu [11] obtained a necessary condition for an optimal solution of the problem with the max-product operator in terms of the maximum solution of $FREs$. This condition was extended to the problem with max-strict-t-norm composition [34]. However, the method in [34] cannot find all the optimal solutions when their number is more than one. To overcome the difficulty, an algorithm was designed based on some necessary conditions for the optimal solutions and some reduction rules in [31]. To decrease the complexity, the size of problem was cut down as much as possible and an algorithm was proposed based on the concept of chained-set suite in [17]. An algorithm was given to solve the problem with the max-product FRI constraints based on the maximum solution and minimal solutions of FRI system in [28]. To decrease the computation rate, some procedures were given to reduce the dimensions of the original problem in [3]. To avoid computing the minimal solutions, an approach was presented to find the optimal solution without computing all minimal solutions of the max-min FRI system in [12]. A new condition was provided to remove unnecessary paths to find the optimal solution in [23]. Then, an algorithm was designed to solve the problem with the addition-min composition in [37] which may require to the resolution of many linear programming problems. Guu and Wu [10] reduced it to a single linear programming problem. To reduce number of the constraints in [10], a smoothing approach was given to solve the problem in [9]. The study of the problem was also extended to the max-pseudo-t-norms [1] and max-aggregation function composition [22].

The FRE and FRI systems of above problems are ascending with respect to their variables. In this paper, we intend to study the linear objective function optimization on the systems which contain both the decision variable vector and its negation, simultaneously, as problem (1)-(3). Freson et al. [8] firstly considered a linear programming problem with such systems to optimize public awareness for their products. The variables induce a degree of appreciation and disappreciation to their products, respectively. Such systems with bipolar variables are called $BFREs$. Freson et al. [8] firstly obtained the solution set of each equation of $BFREs$ (2)-(3) with the max-min composition. According to this point, they determined the solution set of the system of $BFREs$ by a finite set of pairs of minimal and maximal solutions. Lin and Jin [16] showed that checking the consistency of the system of $BFREs$ is NP-complete using the results in [15]. Yang [38] proposed a bipolar path approach for such systems and designed a path-based algorithm to determine its complete solution set. Freson et al. [8] studied the linear optimization problem provided to the constraints of $BFREs$ with an application in management and designed an algorithm to solve the optimization problem based on the structure of its feasible domain. In this method, generating all the feasible candidate elements in a sublattice and checking their optimality are time-consuming. To present a systematic technique, Li and Liu [18] converted the linear optimization problem subject to $BFRE$ constraints with the max-Lukasiewicz t-norm to a 0-1 integer linear programming problem and solved it by the integer and combinational optimization techniques. However, these techniques involve a high computational complexity. To improve computational efficiency, Liu et al. [19] proved that each component of an optimal solution of the mentioned problem can be the component corresponding to the lower or upper bound vector of its feasible domain. Based on the property, they proposed a simple value matrix and some rules to simplify the problem. To further reduce computation, Aliannezhadi et al. [5]

presented some rules to determine a number of optimal variables of the problem with the maximum-parametric hamacher operator and presented an algorithm to find an upper bound for its optimal objective value. They proposed an efficient algorithm based on the rules and the algorithm to solve the problem. Also, Aliannezhadi and Abbasi Molai [6] detected and removed the redundancy constraints in the problem and proposed the sufficient optimality conditions for a feasible solution. An efficient algorithm was designed to solve the problem by a value matrix and the sufficient conditions.

The resolution of linear programming problem with *BFRE* constraints has a high computational complexity. Hence, the role of simplification procedures will be noticeable and important to reduce the rate of computations. In this paper, we firstly define a new concept, called covering, in the system of *BFREs* (2)-(3). The irredundant covering is introduced based on the covering. First of all, some sufficient conditions are suggested to check the consistency of system (2)-(3) according to the new concepts. Then, the sufficient conditions for the existence of an optimal solution for problem (1)-(3) are expressed. Furthermore, some sufficient conditions are presented to determine one of its optimal solutions or some of its optimal components. These sufficient conditions enables us to design some simplification procedures for problem (1)-(3). Finally, an algorithm is designed to find one of its optimal solutions based on the procedures and the branch-and-bound method. The sufficient conditions based on covering are completely different with the conditions based on generating all elements of a sublattice and checking their feasibility and optimality [8], the integer optimization techniques [18], some rules on a simple value matrix in [19], some simplification rules and upper bound on the optimal objective function in [5], and detecting the redundancy constraints and removing them [6]. To show its efficiency, the proposed algorithm is compared to the existing algorithms.

The rest of this paper is organized as follows. Section 2 investigates the structure of feasible domain of the problem (1)-(3). Section 3 is divided to two subsections. The first subsection introduces the covering and irredundant covering concepts. Some sufficient conditions are also given to check the consistency of system (2)-(3). The second subsection presents some sufficient conditions to determine one of the optimal solutions of the problem or some of its optimal components without its resolution. In Section 4, a new algorithm is designed to solve the problem using the sufficient conditions and branch-and-bound approach. In Section 5, a numerical example is given to illustrate the algorithm. Section 6 compares the algorithm to the existing methods. Finally, conclusions and future research directions are expressed in Section 7.

2 The structure of the feasible domain of problem (1)-(3)

A system of *BFREs* (2)-(3) is called consistent if $S(A^+,A^-,b) \neq \emptyset$. Otherwise, it is inconsistent. We now focus on the characterizations of system (2)-(3).

Lemma 1 ([6]). *A vector $x \in [0,1]^n$ is a solution for the system of bipolar max-product fuzzy relation equations (4) if and only if* $\max\left\{a^+_{ij}.x_j, a^-_{ij}.(1-x_j)\right\} \leq b_i$ *for all $i \in I$ and $j \in J$, and for each $i \in I$ there exists an index $j_i \in J$ such that* $\max\left\{a^+_{ij_i}.x_{j_i}, a^-_{ij_i}.(1-x_{j_i})\right\} = b_i$.

We assume that if $a^-_{ij} = 0$, then $\max\{1-\frac{b_i}{a^-_{ij}}, 0\} = 0$ is defined. Also, if $a^+_{ij} = 0$, then $\min\{\frac{b_i}{a^+_{ij}}, 1\} = 1$ is defined. The lower and upper bound on the solution set of system (2)-(3) are introduced in the following lemma.

Lemma 2 ([2]). *The vector of $\check{x} = (\check{x}_1, \dots, \check{x}_n)^T$ is the lower bound on the solution set of system (2)-(3) where $\check{x}_j = \max_{i \in I}\{1 - \frac{b_i}{a_{ij}^-} \mid a_{ij}^- > b_i\}$, for each $j \in J$. Also, the vector of $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)^T$ is the upper bound on the solution set of system (2)-(3) where $\hat{x}_j = \min_{i \in I}\{\frac{b_i}{a_{ij}^+} \mid a_{ij}^+ > b_i\}$, for each $j \in J$. Furthermore, it is assumed that max $\emptyset = 0$ and min $\emptyset = 1$.*

Now, we express two special cases for system (2)-(3) in the following lemma.

Lemma 3 ([2]). *Assume that $S(A^+, A^-, b) \neq \emptyset$ and two vectors $\check{x}$ and $\hat{x}$ be its lower and upper bound, respectively.*

1. *If there exists $j_0 \in J$ such that $\check{x}_{j_0} = \hat{x}_{j_0}$, then $x_{j_0} = \check{x}_{j_0} = \hat{x}_{j_0}$ for all $x \in S(A^+, A^-, b)$. Also, the solution set of system (2)-(3) is the same to the following system*

$$\begin{cases} \max\limits_{j \in J - \{j_0\}} \max\left\{a_{ij}^+ . x_j, a_{ij}^- . (1 - x_j)\right\} = b_i, & \forall i \in I - \bar{I}, \\ \qquad\qquad \check{x}_j \leq x_j \leq \hat{x}_j, & \forall j \in J - \{j_0\}; \quad x_{j_0} = \check{x}_{j_0} = \hat{x}_{j_0}, \end{cases}$$

where $\bar{I} = \left\{i \in I \,|\, \max\{a_{ij_0}^+ . x_{j_0}, a_{ij_0}^- . (1 - x_{j_0})\} = b_i\right\}$ and $\bar{I} \neq \emptyset$.

2. *The solution set of system (2)-(3) is the same to the following system*

$$\begin{cases} \max\limits_{j \in J} \max\{a_{ij}^+ . x_j, a_{ij}^- . (1 - x_j)\} = b_i, & \forall i \in I - I_0, \\ \qquad \check{x}_j \leq x_j \leq \hat{x}_j, & \forall j \in J, \end{cases}$$

where $I_0 = \{i \in I | b_i = 0\}$. Also, $\check{x}_j$ and $\hat{x}_j$, for all $j \in J$, are defined on the basis of system (2)-(3).

We will assume that $\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$, without loss of generality. The characteristic matrices of Q^+ and Q^- are defined below.

Definition 1 ([6]). *Define two characteristic matrices $Q^+ = (q_{ij}^+)_{m \times n}$ and $Q^- = (q_{ij}^-)_{m \times n}$ such that for each $i \in I$ and $j \in J$, we have*

$$q_{ij}^+ = \begin{cases} 1, & \text{if } a_{ij}^+ . \hat{x}_j = b_i, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad q_{ij}^- = \begin{cases} 1, & \text{if } a_{ij}^- . (1 - \check{x}_j) = b_i, \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, a series of index sets is defined based on the matrices Q^+ and Q^- as follows.

Definition 2 ([2]). ***(i)** For the matrix of Q^+, define $I_j^+(x) = \{i \in I \mid x_j = \hat{x}_j$ and $q_{ij}^+ = 1\}$ and $J_i^+(x) = \{j \in J \mid x_j = \hat{x}_j$ and $q_{ij}^+ = 1\}$. Also, for the matrix of Q^-, define $I_j^-(x) = \{i \in I \mid x_j = \check{x}_j$ and $q_{ij}^- = 1\}$ and $J_i^-(x) = \{j \in J \mid x_j = \check{x}_j$ and $q_{ij}^- = 1\}$, for each $i \in I$ and $j \in J$. Furthermore, let $I_j(x) = I_j^+(x) \cup I_j^-(x)$, for each $j \in J$. **(ii)** Let $I_j^+ = I_j^+(\hat{x})$, $J_i^+ = J_i^+(\hat{x})$, $I_j^- = I_j^-(\check{x})$, and $J_i^- = J_i^-(\check{x})$, for each $i \in I$ and $j \in J$.*

The vectors $\check{x}$ and $\hat{x}$ are not necessarily feasible solutions for the system (2)-(3) and $S(A^+, A^-, b) \subseteq \{x | \check{x} \leq x \leq \hat{x}\}$. But their components can satisfy the equations in system (2)-(3). This fact is expressed in the following theorem.

Theorem 1 ([6]). *A vector $x \in [0,1]^n$ is a solution of system (2)-(3) if and only if $\check{x} \leq x \leq \hat{x}$ and $\bigcup_{j\in J} I_j(x) = I$.*

The following lemma expresses an important property of the optimal solution of problem (1)-(3).

Lemma 4 ([6]). *Suppose that problem (1)-(3) is feasible. Then there exists an optimal solution $x^* = (x_1^*, \ldots, x_n^*)^T$ such that for each $j \in J$ either $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$.*

For each $j \in J$, label the values of $\hat{x}_j$ and $\check{x}_j$ with boolean variables of y_j and $\neg y_j$, respectively. The following theorem presents a criteria to check the consistency of system (2)-(3). With regard to Definitions 1 and 2, we can express a similar to [15, Theorem 2.5] as follows.

Theorem 2. *The system (2)-(3) is consistent if and only if its characteristic boolean formula $C = \bigwedge_{i\in I} C_i$ is well-defined and satisfiable, where $C_i = \bigvee_{j\in J_i^+} y_j \vee \bigvee_{j\in J_i^-} \neg y_j$.*

Now, we are ready to present the sufficient conditions for consistency of system (2)-(3) and detect optimal variables based on the covering concept.

3 Covering-based sufficient conditions for consistency and simplification of problem (1)-(3)

This section is divided to two subsections. The first subsection presents the definition of covering and its related points. Then, sufficient conditions are proposed to check the consistency of system (2)-(3). The second subsection provides some sufficient conditions to determine some of the optimal variables or one of the optimal solutions for problem (1)-(3) based on the covering concept.

3.1 The sufficient conditions for consistency of system (2)-(3)

In this subsection, some sufficient conditions are proposed for consistency of system (2)-(3) based on the covering concept. First of all, two index sets are defined as follows

$$I_1 = \bigcup_{j\in J} I_j^- \qquad \text{and} \qquad I_2 = I \setminus I_1. \tag{5}$$

Obviously, if $I_2 = \emptyset$, then $x^* = \check{x}$ will be an optimal solution of problem (1)-(3). We are now ready to present the following definition.

Definition 3. *(i) The matrix of $Q^{R+} = (q_{ij}^{R+})_{|I_2|\times n}$ is the matrix of Q^+ with the removed row(s) $i \in I_1$.*
(ii) A set $C \subseteq J$ is a covering of Q^{R+} if $\bigcup_{j\in C} I_j^+ \supseteq I_2$. A covering C is irredundant if each proper subset of C is not a covering of Q^{R+}. Let $CS(Q^{R+})$ be the set of all irredundant coverings of Q^{R+}.
(iii) A covering set $C \in CS(Q^{R+})$ is a feasible covering of Q^{R+} if $(\bigcup_{j\in C} I_j^+) \cup (\bigcup_{j\in J\setminus C} I_j^-) = I$. Also, $FCS(Q^{R+})$ is called the set of all feasible irredundant coverings of Q^{R+}.

It is noticeable that we may have $S(A^+, A^-, b) \neq \emptyset$ when $FCS(Q^{R+}) = \emptyset$. The following example explains this point.

Example 1. Consider the following system of bipolar max-T_p *FREs* $A^+ \circ x \vee A^- \circ \neg x = b$, where

$$A^+ = \begin{pmatrix} 0.4 & 0.05 & 0.02 & 0.09 & 0.07 \\ 1 & 0.24 & 0.12 & 0.19 & 0.22 \\ 0.58 & 1 & 0.42 & 0.31 & 0.26 \\ 0.35 & 0.25 & 0.18 & 0.54 & 0.72 \\ 0.17 & 0.2 & 0.45 & 0.33 & 0.48 \end{pmatrix}, \quad A^- = \begin{pmatrix} 0.01 & 0.05 & 0.08 & 0.03 & 0.2 \\ 0.2 & 0.19 & 0.23 & 1 & 0.16 \\ 0.51 & 0.7 & 0.9 & 0.6 & 0.42 \\ 0.46 & 0.3 & 0.29 & 0.23 & 0.53 \\ 0.14 & 0.1 & 0.16 & 0.3 & 0.31 \end{pmatrix},$$

and $b = (0.1, 0.25, 0.63, 0.54, 0.36)^T$. Considering Lemma 2, the lower and upper bound of $\check{x}$ and $\hat{x}$ can be computed as follows: $\check{x} = (0, 0.1, 0.3, 0.75, 0.5)^T$ and $\hat{x} = (0.25, 0.63, 0.8, 1, 0.75)^T$. With regard to Definition 1, two matrices Q^+ and Q^- are obtained as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad Q^- = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

In this example, we have $I_1 = \{1,2,3\}$, $I_2 = \{4,5\}$, $CS(Q^{R+}) = \Big\{\{3,4\},\{5\}\Big\}$, and $FCS(Q^{R+}) = \emptyset$, where

$$Q^{R+} = \begin{array}{c} \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Consider $x = (\hat{x}_1, \check{x}_2, \check{x}_3, \check{x}_4, \hat{x}_5)$. Since $\check{x} \le x \le \hat{x}$ and $(\bigcup_{j\in\{1,5\}} I_j^+) \cup (\bigcup_{j\in\{2,3,4\}} I_j^-) = I$, then $x \in S(A^+, A^-, b)$ and $S(A^+, A^-, b) \neq \emptyset$. Hence, the bipolar system is not necessarily inconsistent if $FCS(Q^{R+}) = \emptyset$.

We now present some sufficient conditions for consistency of system (2)-(3) in the following theorem.

Theorem 3. *If there exists a $C \in CS(Q^{R+})$ such that $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$, then the solution set of system (2)-(3) is not empty.*

Proof. Assume that there exists an $C \in CS(Q^{R+})$ such that $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$. Then, we have: $I \supseteq (\bigcup_{j\in C} I_j^+) \cup (\bigcup_{j\in J-C} I_j^-) \supseteq (\bigcup_{j\in C} I_j^+) \cup (\bigcup_{j\in C} I_j^-) \cup (\bigcup_{j\in J-C} I_j^-) = (\bigcup_{j\in C} I_j^+) \cup (\bigcup_{j\in J} I_j^-) \supseteq I_2 \cup I_1 = I$. The recent relations implies that $(\bigcup_{j\in C} I_j^+) \cup (\bigcup_{j\in J-C} I_j^-) = I$. Hence, $C \in FCS(Q^{R+})$ and C is a feasible irredundant covering. Now, define vector $x = [x_j]_{j\in J}$ as follows:

$$x_j = \begin{cases} \hat{x}_j, & \text{if } j \in C, \\ \check{x}_j, & \text{otherwise}, \end{cases} \qquad \forall j \in J.$$

Obviously, we have $x \in S(A^+, A^-, b)$. Hence, $S(A^+, A^-, b) \neq \emptyset$. □

The following lemma provides some sufficient conditions to satisfy $FCS(Q^{R+}) \supseteq CS(Q^{R+})$. Hence, it implies that $FCS(Q^{R+}) = CS(Q^{R+})$.

Lemma 5. *If for each $C \in CS(Q^{R+})$, $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$, then $FCS(Q^{R+}) = CS(Q^{R+})$.*

Proof. It is obvious that $FCS(Q^{R+}) \subseteq CS(Q^{R+})$ with regard to Definition 3. Now, it is enough to show that $CS(Q^{R+}) \subseteq FCS(Q^{R+})$. For each $C \in CS(Q^{R+})$, we have: $I \supseteq (\bigcup_{j\in C} I_j^+)\bigcup(\bigcup_{j\in J-C} I_j^-) \supseteq (\bigcup_{j\in C} I_j^+)\bigcup(\bigcup_{j\in C} I_j^-)\bigcup(\bigcup_{j\in J-C} I_j^-) = (\bigcup_{j\in C} I_j^+)\bigcup(\bigcup_{j\in J} I_j^-) \supseteq I_2 \bigcup I_1 = I$. The above relations imply that $(\bigcup_{j\in C} I_j^+)\bigcup(\bigcup_{j\in J-C} I_j^-) = I$. Hence, $C \in FCS(Q^{R+})$. Therefore, $CS(Q^{R+}) \subseteq FCS(Q^{R+})$. □

Now, we are ready to present some sufficient conditions to simplify the problem (1)-(3).

3.2 Some sufficient conditions for simplification of problem (1)-(3)

This section provides some sufficient conditions under which, one of the optimal solutions or some of the optimal variables of problem (1)-(3) are directly determined.

Theorem 4. *If there exists a cover $C' \in FCS(Q^{R+})$ such that*

$$\sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j) \le \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j),\ \forall C \in CS(Q^{R+}), \tag{6}$$

then there exists an optimal solution $x^* = (x_j^*)_{j\in J}$ for problem (1)-(2) as follows:

$$x_j^* = \begin{cases} \hat{x}_j & \text{if } j \in C', \\ \\ \check{x}_j & \text{otherwise}, \end{cases} \qquad \forall j \in J. \tag{7}$$

Proof. It is enough to show that *(i)* $x^* \in S(A^+, A^-, b)$ and *(ii)* $Z(x^*) \le Z(x)$, for each $x \in S(A^+, A^-, b)$. Since $C' \in FCS(Q^{R+})$, then $x^* \in S(A^+, A^-, b)$ with regard to Definition 3 *(iii)*. On the other hand, for any $x \in S(A^+, A^-, b)$, there exists $C \in CS(Q^{R+})$ such that $x_j = \hat{x}_j$, for each $j \in C$ and

$$Z(x) \ge \sum_{j\in J\setminus C} c_j\check{x}_j + \sum_{j\in C} c_j\hat{x}_j = \sum_{j\in J} c_j\check{x}_j + \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j).$$

Since

$$\sum_{j\in C} c_j(\hat{x}_j - \check{x}_j) \ge \sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j),$$

for each $C \in CS(Q^{R+})$, then $Z(x) \ge \sum_{j\in J} c_j\check{x}_j + \sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j) = Z(x^*)$, for any $x \in S(A^+, A^-, b)$. □

It is necessary to remind the following point about Theorem 4.

Remark 1. *Consider relation (6) in Theorem 4. This inequality should be held for any $C \in CS(Q^{R+})$. If this inequality holds for any $C \in FCS(Q^{R+})$, then Theorem 4 is not necessarily true. More precisely, there may exist $C^1 \in CS(Q^{R+}) \setminus FCS(Q^{R+})$ and $K \subseteq J \setminus C^1$ such that $(\bigcup_{j\in C^2} I_j^+) \cup (\bigcup_{j\in J\setminus C^2} I_j^-) = I$ and $\sum_{j\in C^2} c_j(\hat{x}_j - \check{x}_j) < \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j)$, for any $C \in CS(Q^{R+})$ and $C \neq C^1$, where $C^2 = C^1 \cup K$. This point is explained in Example 2.*

Example 2. Consider the following optimization problem:

$$\begin{aligned} \min \quad & x_1 + 3x_2 + 2x_3 + 5x_4 + 8x_5 + 7x_6, \\ \text{s.t.} \quad & A^+ \circ x \vee A^- \circ \neg x = b, \\ & x \in [0,1]^6, \end{aligned}$$

where

$$A^+ = \begin{pmatrix} 0.4 & 0.05 & 0.21 & 0.19 & 0.09 & 0.05 \\ 0.32 & 0.12 & 0.13 & 0.08 & 0.04 & 0.48 \\ 0.05 & 0.03 & 0.01 & 0.1 & 0.01 & 0.04 \\ 0.11 & 0.3 & 0.13 & 0.2 & 0.09 & 0.02 \\ 0.06 & 0.2 & 0.02 & 0.02 & 0.15 & 0.24 \\ 0.26 & 0.48 & 0.6 & 0.24 & 0.75 & 0.34 \end{pmatrix}, \ A^- = \begin{pmatrix} 0.3 & 0.4 & 0.13 & 0.17 & 0.09 & 0.02 \\ 0.21 & 0.32 & 0.8 & 0.06 & 0.13 & 0.01 \\ 0.1 & 0.05 & 0.04 & 0.18 & 0.04 & 0.05 \\ 0.13 & 0.02 & 0.12 & 0.03 & 0.3 & 0.2 \\ 0.08 & 0.01 & 0.02 & 0.02 & 0.08 & 0.02 \\ 0.13 & 0.26 & 0.23 & 0.24 & 0.22 & 0.14 \end{pmatrix},$$

$b = (0.3, 0.24, 0.09, 0.18, 0.12, 0.6)^T$, and $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$.

In this example, the lower and upper bound of $\check{x}$ and $\hat{x}$ are as follows: $\check{x} = (0.1, 0.25, 0.7, 0.5, 0.4, 0.1)^T$ and $\hat{x} = (0.75, 0.6, 1, 0.9, 0.8, 0.5)^T$. Applying Definition 1, the matrices of Q^+ and Q^- are obtained as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \text{ and } Q^- = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Also, the index sets of I_j^+ and I_j^-, for all $j \in J$, can be computed as follows: $I_1^+ = \{1,2\}$, $I_2^+ = \{4,5\}$, $I_3^+ = \{6\}$, $I_4^+ = \{3,4\}$, $I_5^+ = \{5,6\}$, $I_6^+ = \{2,5\}$, $I_1^- = \{3\}$, $I_2^- = \{1,2\}$, $I_3^- = \{2\}$, $I_4^- = \{3\}$, $I_5^- = \{4\}$, and $I_6^- = \{4\}$.

Moreover, we can compute the index sets of J_i^+ and J_i^-, for all $i \in I$, as follows: $J_1^+ = \{1\}$, $J_2^+ = \{1,6\}$, $J_3^+ = \{4\}$, $J_4^+ = \{2,4\}$, $J_5^+ = \{2,5,6\}$, $J_6^+ = \{3,5\}$, $J_1^- = \{2\}$, $J_2^- = \{2,3\}$, $J_3^- = \{1,4\}$, $J_4^- = \{5,6\}$, and $J_5^- = J_6^- = \emptyset$.

The system of the bipolar max-product $FREs$ of $A^+ \circ x \vee A^- \circ \neg x = b$ is consistent. Also, two index sets I_1 and I_2 are as follows: $I_1 = \{1,2,3,4\}$ and $I_2 = \{5,6\}$. The matrix of Q^{R+} is obtained as follows:

$$Q^{R+} = \begin{array}{c} \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Also, two sets $CS(Q^{R+})$ and $FCS(Q^{R+})$ are computed as follows:

$$CS(Q^{R+}) = \{\{2,3\}, \{5\}, \{3,6\}\} \ \text{ and } \ FCS(Q^{R+}) = \{\{5\}, \{3,6\}\}.$$

We now check the conditions of Theorem 4. Set $C' = \{5\} \in FCS(Q^{R+})$. Although inequality $c_5(\hat{x}_5 - \check{x}_5) = 3.2 \leq \sum_{j \in C} c_j(\hat{x}_j - \check{x}_j)$ holds for any $C \in FCS(Q^{R+})$, it does not hold for any $C \in CS(Q^{R+})$ and

Theorem 4 cannot be used. It is easy to see that there exists $C^1 = \{2,3\} \in CS(Q^{R+}) \setminus FCS(Q^{R+})$ and $K = \{1\}$ such that $(\bigcup_{j\in\{1,2,3\}} I_j^+) \cup (\bigcup_{j\in\{4,5,6\}} I_j^-) = I$ and $\sum_{j\in\{1,2,3\}} c_j(\hat{x}_j - \check{x}_j) = 2.3 < \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j)$, for each $C \in CS(Q^{R+})$ and $C \neq \{2,3\}$. In this problem, the optimal objective value is 10.95 and the optimal solution is $x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*)^T = (0.75, 0.6, 1, 0.5, 0.4, 0.1)^T$. In fact, the optimal solution x^* has been obtained from the cover $C^1 = \{2,3\}$.

Now, we are ready to discuss on uniqueness of the obtained optimal solution by Theorem 4. Under the assumptions of Theorem 4, the optimal solution is not necessarily unique. The important point is the cost coefficients of problem (1)-(3). Since $c_j \geq 0$, for each $j \in J$, there may exist $k \in J \setminus \bigcup_{C\in CS(Q^{R+})} \{j \mid j \in C\}$ such that $c_k = 0$ and $I_k^+ = I_k^- = \emptyset$. In this case, each point in the closed interval $[\check{x}_k, \hat{x}_k]$ can be selected as the k^{th} component of an optimal solution. Under some conditions expressed in Theorem 5, the optimal solution introduced in Theorem 4 is unique.

Theorem 5. *If the cover $C' \in FCS(Q^{R+})$ satisfies the following conditions,*

1. $c_j > 0$, $\forall j \in J \setminus C'$,
2. $\sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j) < \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j)$, $\forall C \in CS(Q^{R+})$ *and* $C \neq C'$,

then the optimization problem of (1)-(3) has a unique optimal solution of $x^* = (x_j^*)_{j\in J}$ as relation (7).

Proof. Considering proof of Theorem 4, the proof can be completed by showing that $Z(x^*) < Z(x)$, for each $x \in S(A^+, A^-, b)$ with $x \neq x^*$. It is obvious that for any $x \in S(A^+, A^-, b)$, there exists $C \in CS(Q^{R+})$ such that $x_j = \hat{x}_j$, for each $j \in C$ and $Z(x) \geq \sum_{j\in J} c_j\check{x}_j + \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j)$. If $C \neq C'$ for $x \in S(A^+, A^-, b)$ with $x \neq x^*$, then inequality $Z(x) > Z(x^*)$ can be easily obtained with regard to condition 2. Otherwise, we have $C = C'$ for $x \in S(A^+, A^-, b)$ with $x \neq x^*$, i.e., $x_j = \hat{x}_j$, for each $j \in C'$. In this case, there exists $k \in J \setminus C'$ such that $x_k = \hat{x}_k$ due to $x \neq x^*$. Thus, we conclude that

$$Z(x) \geq \sum_{j\in J} c_j\check{x}_j + \sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j) + c_k(\hat{x}_k - \check{x}_k). \tag{8}$$

It can be easily seen that $c_k(\hat{x}_k - \check{x}_k) > 0$ with regard to $\check{x}_k < \hat{x}_k$ and condition 1. Applying inequality (8) and $c_k(\hat{x}_k - \check{x}_k) > 0$, we have $Z(x) > Z(x^*)$ and the proof is completed. □

The following theorem provides covering-based sufficient conditions to determine one of the optimal solutions of problem (1)-(3).

Theorem 6. *If for each $C \in CS(Q^{R+})$, we have $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$, then there exists a $C^* \in CS(Q^{R+})$ such that problem (1)-(3) has an optimal solution $x^* = [x_j^*]_{j\in J}$ as follows*

$$x_j^* = \begin{cases} \hat{x}_j, & \textit{if } j \in C^*, \\ \check{x}_j, & \textit{otherwise}, \end{cases} \qquad \forall j \in J. \tag{9}$$

Proof. According to Lemma 5, it is concluded that $FCS(Q^{R+}) = CS(Q^{R+})$. Since $\mid J \mid$ and the dimensions of matrix Q^{R+} are finite, $\mid FCS(Q^{R+}) \mid$ is finite. Hence, there exists a covering $C^* \in FCS(Q^{R+})$ such that $\sum_{j\in C^*} c_j(\hat{x}_j - \check{x}_j) \le \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j)$, $\forall C \in CS(Q^{R+})$. According to Theorem 4, there exists an optimal solution $x^* = [x_j^*]_{j\in J}$ as follows

$$x_j^* = \begin{cases} \hat{x}_j, & \text{if } j \in C^*, \\ \check{x}_j, & \text{otherwise}, \end{cases} \qquad \forall j \in J.$$

□

Two following theorems present some sufficient conditions to detect some optimal components of problem (1)-(3). These theorems can be used to simply or reduce the dimensions of the original problem.

Theorem 7. *Suppose that: (1) for each $C \in CS(Q^{R+})$, we have $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$, and (2) $\bigcap_{C\in CS(Q^{R+})} C \neq \emptyset$. Then there exists an optimal solution $x^* = [x_j^*]_{j\in J}$ for problem (1)-(3) where for each $j \in \bigcap_{C\in CS(Q^{R+})} C$, $x_j^* = \hat{x}_j$.*

Proof. According to assumption (1) and Theorem 3, the feasible domain of problem (1)-(3) is non-empty. Since its feasible domain is a closed and bounded set, problem (1)-(3) has at least one optimal solution as $x^* = [x_j^*]_{j\in J}$. With regard to assumption (1), it is concluded that $CS(Q^{R+}) = FCS(Q^{R+})$ according to Lemma 5. Since $\mid FCS(Q^{R+}) \mid$ is finite, there exists one $C^1 \in FCS(Q^{R+})$ such that $\sum_{j\in C^1} c_j(\hat{x}_j - \check{x}_j) \le \sum_{j\in C} c_j(\hat{x}_j - \check{x}_j)$, $\forall C \in CS(Q^{R+})$. Hence, the assumptions of Theorem 4 are satisfied. So, there exists an optimal solution $x^* = [x_j^*]_{j\in J}$ such that $x_j^* = \hat{x}_j$, for each $j \in C^1$. Since $\bigcap_{C\in CS(Q^{R+})} C \subseteq C^1$, it is concluded that $x_j^* = \hat{x}_j$, for each $j \in \bigcap_{C\in CS(Q^{R+})} C$. □

Theorem 8. *Assume that for each $C \in CS(Q^{R+})$, the relation $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$ holds. If there exists a covering $C' \in FCS(Q^{R+})$ such that (1) for each $C \in CS(Q^{R+}) - \{C'\}$, we have $C \bigcap C' = \emptyset$ and (2) for one $C'' \in CS(Q^{R+}) - \{C'\}$, we have $\sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j) > \sum_{j\in C''} c_j(\hat{x}_j - \check{x}_j)$. Then there exists an optimal solution $x^* = [x_j^*]_{j\in J}$ for problem (1)-(3) such that $x_j^* = \check{x}_j$, for each $j \in C'$.*

Proof. The assumption of $\bigcup_{j\in C} I_j^- \subseteq \bigcup_{j\in J-C} I_j^-$, for each $C \in CS(Q^{R+})$, implies that the feasible domain of problem (1)-(3) is a non-empty, according to Theorem 3. The feasible domain is also a closed, and bounded set. Hence, problem (1)-(3) has at least one optimal solution as $x^* = [x_j^*]_{j\in J}$ with $C^* \in FCS(Q^{R+})$. On the other hand, the assumption (2) is concluded that

$$\sum_{j\in C'} c_j(\hat{x}_j - \check{x}_j) > \sum_{j\in C''} c_j(\hat{x}_j - \check{x}_j) \ge \sum_{j\in C^*} c_j(\hat{x}_j - \check{x}_j).$$

Obviously, $C' \neq C^*$. Since $C' \cap C^* = \emptyset$, Theorem 4 implies that $x_j^* = \check{x}_j$, for each $j \in C'$. □

The lemmas, theorems, and corollaries of this section are firstly applied to reduce the dimensions of the original problem using the following remark and the original problem is reduced to a problem with smaller dimensions. Since the resolution of the original problem is a NP-hard problem, these simplifications can reduce the computational complexity, noticeably.

Remark 2. *Assume that some components of an optimal solution x^* are determined under some sufficient conditions in the resolution process of problem (1)-(3). If value $\check{x}_k$ is assigned the k^{th} component of an optimal solution x^*, then row(s) $i \in I_k^-$ and column k can be removed from matrices Q^+ and Q^-. Also, if value $\hat{x}_k$ is assigned to the k^{th} component of an optimal solution x^*, then row(s) $i \in I_k^+$ and column k can be deleted from matrices Q^+ and Q^-.*

If all components of the optimal solution of problem (1)-(3) is not completely determined, then we have to solve the reduced problem. To do this, we will apply the matrix-based branch-and-bound method in the next section.

4 An algorithm for solving problem (1)-(3)

First of all, the objective function of problem (1)-(3) is rewritten as follows:

$$Z(x) = \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} c_j \check{x}_j + \sum_{j=1}^{n} c_j \check{x}_j = \sum_{j=1}^{n} c_j (x_j - \check{x}_j) + \sum_{j=1}^{n} c_j \check{x}_j.$$

Therefore, the problem (1)-(3) can equivalently rewrite as follows:

$$\min \quad \bar{Z}(x) = \sum_{j=1}^{n} c_j (x_j - \check{x}_j), \tag{10}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{11}$$

$$x \in [0,1]^n. \tag{12}$$

Now, we want to solve problem (10)-(12) according to Lemma 4. The problem (10)-(12) gives us more information with respect to problem (1)-(3). For instance, consider two vectors $x' = [x'_j]_{j \in J}$ and $x'' = [x''_j]_{j \in J}$ with $x'_t = \hat{x}_t$ and $x'_j = x_j$, for each $j \in J - \{t\}$, and $x''_k = \hat{x}_k$ and $x''_j = x_j$, for each $j \in J - \{k\}$. If $c_k.\hat{x}_k > c_t.\hat{x}_t$, then we cannot conclude that $Z(x'') > Z(x')$ or $Z(x'') < Z(x')$. But, if $c_k(\hat{x}_k - \check{x}_k) < c_t(\hat{x}_t - \check{x}_t)$, then $\bar{Z}(x'') < \bar{Z}(x')$ is concluded. This useful property is applied to design a matrix-based branch-and-bound approach to solve problem (1)-(3). To do this, all rows $i \in I_2$ are transferred to the top of matrices Q^+ and Q^-. Now, define the following value matrix according to problem (10)-(12).

Definition 4 ([6]). *The value matrix $M = [m_{ij}]_{m \times 2n}$ is defined as follows*

$$m_{i,2j-1} = \begin{cases} c_j.(\hat{x}_j - \check{x}_j), & \text{if } q_{ij}^+ = 1, \\ \infty, & \text{otherwise}, \end{cases} \quad \text{and} \quad m_{i,2j} = \begin{cases} 0, & \text{if } q_{ij}^- = 1, \\ \infty, & \text{otherwise}. \end{cases}$$

for each $i \in I$ and $j \in J$.

The branch-and-bound method is employed on the matrix M with three modifications as follows [2]:

1. If we choose $\hat{x}_j$ (or $\check{x}_j$) to branch from one node to another node, then we never use $\check{x}_j$ (or $\hat{x}_j$) to branch further on the current node.

2. Under the following conditions, we cannot branch further on Node k.

2.1. We have reached to the last row of matrix M.

2.2. The selected variables along Node 0 to Node k together with $\check{x}_j$, for each $j \in J \setminus J_k$, satisfy all the equations where $J_k = \{j \in J | x_j \text{ has been selected along the branches from Node 0 to Node k}\}$.

2.3. We do not have any candidate for satisfying an equation with regard to modification 1.

3. If we cannot branch further on Node k under the conditions 2.1 and 2.2, then we assign $\check{x}_j$ to x_j, for each $j \in J \setminus J_k$.

We now provide an algorithm to solve problem (1)-(3) based on the sufficient conditions and branch-and-bound method.

Algorithm 1 An algorithm for solving problem (1)-(3)

Step 1. Compute the lower and upper bound of $\check{x}$ and $\hat{x}$ by Lemma 2.
Step 2. If $b = 0$ and $\check{x} \leq \hat{x}$, then $S(A^+, A^-, b) = [\check{x}, \hat{x}]$ and $x^* = \check{x}$ is an optimal solution of problem (1)-(3) according to Lemma 3 (2) and stop!
Step 3. If $\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$, then go to Step 4. Otherwise, use Lemma 3.
Step 4. Compute the matrices of Q^+ and Q^-, the index sets of I_j^+ and I_j^-, for each $j \in J$, and the index sets of J_i^+ and J_i^-, for each $i \in I$ using Definitions 1 and 2.
Step 5. Check the consistency of system (2)-(3) by Theorem 2 or Theorem 3. If it is inconsistent, then stop! Otherwise, go to Step 6.
Step 6. Check the following sufficient conditions:

- **6.1.** Find two index sets I_1 and I_2 by relation (5).
- **6.2.** If $I_2 = \emptyset$, then $x^* = \check{x}$ and stop!
- **6.3.** Compute matrix Q^{R+} and two sets $CS(Q^{R+})$ and $FCS(Q^{R+})$ using Definition 3
- **6.4.** If the conditions of Theorem 5 are satisfied, then the unique optimal solution of problem (1)-(3) is obtained by relation (7) and stop!
- **6.5.** If the conditions of Theorem 4 are satisfied then choose a covering $C^* \in CS(Q^{R+})$ such that $\sum_{j \in C^*} c_j(\hat{x}_j - \check{x}_j) \leq \sum_{j \in C} c_j(\hat{x}_j - \check{x}_j)$, for each $C \in CS(Q^{R+})$. Then the optimal solution is as relation (7) and stop!
- **6.6.** If the conditions of Theorem 6 are satisfied, then the problem (1)-(3) has an optimal solution as relation (9) and stop!
- **6.7.** If the conditions of Theorem 7 are satisfied, then for each $j \in \bigcap_{C \in CS(Q^{R+})} C$, we have $x_j^* = \hat{x}_j$. Update Q^+, Q^-, and Q^{R+} using Remark 2.
- **6.8.** If the conditions of Theorem 8 are satisfied, then for each $j \in C'$, we have $x_j^* = \check{x}_j$. Update Q^+, Q^-, and Q^{R+} using Remark 2.

Step 7. If $Q^+ = Q^- = \emptyset$, then assign $\check{x}_j$ to x_j^* for other unknown variables. Then, go to Step 10.
Step 8. Transfer all rows $i \in I_2$ to the top of matrices Q^+ and Q^-. Also, generate the value matrix of M using Definition 4.
Step 9. Apply the modified branch-and-bound method on the matrix of M to solve the problem (1)-(3).
Step 10. Produce the optimal solution and the optimal value of problem (1)-(3). End.

5 Numerical Examples

An example is given to illustrate the algorithm.

Example 3. Consider the following problem:

$$\begin{aligned} \min \quad & 4x_2+3x_3+x_4+5x_5+6x_6+x_7+2x_8, & (13)\\ \text{s.t.} \quad & A^+\circ x \vee A^-\circ \neg x = b, & (14)\\ & x\in[0,1]^8, \end{aligned}$$

where

$$A^+=\begin{pmatrix} 0.3 & 0.35 & 0.25 & 0.6 & 0.45 & 0.35 & 0.4 & 0.32\\ 0.2 & 0.04 & 0.1 & 0.15 & 0.3 & 0.2 & 0.3 & 0.24\\ 0.17 & 0.12 & 0.1 & 0.03 & 0.2 & 0.01 & 0.15 & 0.14\\ 0.14 & 0.25 & 0.3 & 0.5 & 0.1 & 0.03 & 0.3 & 0.25\\ 0.2 & 0.17 & 0.1 & 0.3 & 0.15 & 0.17 & 0.15 & 0.1\\ 0.6 & 0.45 & 0.5 & 0.65 & 0.4 & 0.45 & 0.3 & 0.4\\ 0.9 & 0.63 & 0.6 & 1 & 0.51 & 0.4 & 0.6 & 0.2\\ 0.52 & 0.4 & 0.45 & 0.7 & 0.3 & 0.14 & 0.25 & 0.4\\ 0.23 & 0.15 & 0.2 & 0.3 & 0.12 & 0.1 & 0.2 & 0.03\\ 0.45 & 0.4 & 0.3 & 0.1 & 0.5 & 0.25 & 0.6 & 0.15 \end{pmatrix},$$

$$A^-=\begin{pmatrix} 0.45 & 0.35 & 0.23 & 0.4 & 0.3 & 0.6 & 0.7 & 1\\ 0.25 & 0.3 & 0.28 & 0.07 & 0.2 & 0.3 & 0.45 & 0.8\\ 0.1 & 0.2 & 0.2 & 0.17 & 0.05 & 0.22 & 0.15 & 0.4\\ 0.3 & 0.35 & 0.2 & 0.14 & 0.21 & 0.45 & 0.5 & 1\\ 0.2 & 0.15 & 0.01 & 0.22 & 0.18 & 0.3 & 0.25 & 0.55\\ 0.48 & 0.5 & 0.4 & 0.15 & 0.32 & 0.7 & 0.65 & 0.9\\ 0.7 & 0.6 & 0.65 & 0.3 & 0.43 & 1 & 0.8 & 1\\ 0.5 & 0.4 & 0.3 & 0.6 & 0.35 & 0.55 & 0.72 & 0.85\\ 0.25 & 0.2 & 0.15 & 0.2 & 0.1 & 0.25 & 0.4 & 0.6\\ 0.6 & 0.5 & 0.35 & 0.52 & 0.4 & 0.8 & 0.9 & 0.75 \end{pmatrix},$$

$$b=(0.36,0.24,0.15,0.3,0.18,0.45,0.63,0.42,0.2,0.48)^T,$$

$$x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)^T.$$

Now, we want to solve this example by Algorithm 1.
Step 1. In this example, the lower and upper bound of $\check{x}$ and $\hat{x}$ are as follows:

$$\check{x}=(0.2,0.25,0.25,0.3,0,0.4,0.5,0.7)^T \text{ and } \hat{x}=(0.7,1,0.9,0.6,0.75,1,0.8,1)^T.$$

Step 2. Since $b>0$, we go to Step 3.
Step 3. Since $\check{x}_j<\hat{x}_j$, for each $j\in J$ and $b_i>0$, for each $i\in I$, we go to Step 4.

Step 4. The matrices of Q^+ and Q^- can be obtained as follows:

$$Q^+ = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&1&1&0&0&1&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix} \end{array} \text{ and } Q^- = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{pmatrix} 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&1&0\\ 1&0&0&0&0&1&0&0 \end{pmatrix} \end{array}.$$

The index sets of I_j^+ and I_j^-, for all $j \in J$, can be computed as follows: $I_1^+ = \{7\}$, $I_2^+ = \{6,7\}$, $I_3^+ = \{6\}$, $I_4^+ = \{1,4,5,8\}$, $I_5^+ = \{3\}$, $I_6^+ = \{6\}$, $I_7^+ = \{2,10\}$, $I_8^+ = \{2\}$, $I_1^- = \{1,9,10\}$, $I_2^- = \{3\}$, $I_3^- = \{3\}$, $I_4^- = \{8\}$, $I_5^- = \{5\}$, $I_6^- = \{1,5,10\}$, $I_7^- = \{9\}$, and $I_8^- = \{2,4\}$.

Also, we can compute the index sets of J_i^+ and J_i^-, for all $i \in I$, as follows: $J_1^+ = \{4\}$, $J_2^+ = \{7,8\}$, $J_3^+ = \{5\}$, $J_4^+ = \{4\}$, $J_5^+ = \{4\}$, $J_6^+ = \{2,3,6\}$, $J_7^+ = \{1,2\}$, $J_8^+ = \{4\}$, $J_9^+ = \emptyset$, $J_{10}^+ = \{7\}$, $J_1^- = \{1,6\}$, $J_2^- = \{8\}$, $J_3^- = \{2,3\}$, $J_4^- = \{8\}$, $J_5^- = \{5,6\}$, $J_6^- = J_7^- = \emptyset$, $J_8^- = \{4\}$, $J_9^- = \{1,7\}$, and $J_{10}^- = \{1,6\}$.

Step 5. Since the bipolar max-T_p *FREs* of $A^+ \circ x \vee A^- \circ \neg x = b$ is consistent, we go to Step 6.

Step 6. Perform the process of problem reduction as follows:

6.1. Two index sets I_1 and I_2 can be obtained as follows: $I_1 = \{1,2,3,4,5,8,9,10\}$ and $I_2 = \{6,7\}$. It can be easily verified that Substep 6.2 cannot be used. So, we go to Step 6.3.

6.3. The matrix of Q^{R+} is obtained as follows:

$$Q^{R+} = \begin{array}{c} \\ 6\\7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{pmatrix} 0&1&1&0&0&1&0&0\\ 1&1&0&0&0&0&0&0 \end{pmatrix} \end{array}.$$

Also, we have $CS(Q^{R+}) = \{\{2\},\{1,3\},\{1,6\}\}$ and $FCS(Q^{R+}) = \{\{2\},\{1,3\}\}$.

6.4. Since the following conditions are satisfied:

1. $C' = \{1,3\} \in FCS(Q^{R+})$,
2. $c_j > 0$, for each $j \in \{2,4,5,6,7,8\}$,
3. $\sum_{j\in\{1,3\}} c_j(\hat{x}_j - \check{x}_j) = 1.95 < 3 = c_2(\hat{x}_2 - \check{x}_2)$,
4. $\sum_{j\in\{1,3\}} c_j(\hat{x}_j - \check{x}_j) = 1.95 < 3.6 = \sum_{j\in\{1,6\}} c_j(\hat{x}_j - \check{x}_j)$,

then by Theorem 5, the optimization problem of (13)-(14) has a unique optimal solution x^* with the optimal objective value 8.3 as follows: $x^* = (\hat{x}_1, \check{x}_2, \hat{x}_3, \check{x}_4, \check{x}_5, \check{x}_6, \check{x}_7, \check{x}_8)^T = (0.7, 0.25, 0.9, 0.3, 0, 0.4, 0.5, 0.7)^T$.

6 Comparison of the proposed algorithm with the existing methods

This section compares the proposed algorithm with other existing methods to solve problem (1)-(3) with regard to the results in Example 3 of Section 5. As it is considered in Example 3, the proposed algorithm solves the problem without using the branch-and-bound method. Its optimal solution is obtained by Theorem 5 in Substep **6.4**. The proposed algorithm can also be applied for large scale problems because the sufficient conditions are proved in a general case and the modified branch-and-bound method have no limitations on the dimensions of problem. The sufficient conditions in the algorithm are based on some theorems and lemmas which their results are exact. On the other hand, the optimal solution produced by Step 9 is exact. Therefore, the proposed algorithm produces an exact optimal solution. We don't use from approximate methods or solutions in the steps of the algorithm. The authors in [8] investigated problem (1)-(3) with the max-min composition operator. They presented an algorithm to solve the problem based on the structure of its feasible domain. Its computational complexity is as $T_F = O(mn(2m+2)^n)$, where m and n denote the number of rows and columns of matrix A^+ (or A^-), respectively [6]. If we use the method for an instance of the problem with the dimensions $m = 10$ and $n = 8$ in Example 3, its computational cost is as: $O(10 \times 8 \times (2 \times 10 + 2)^8) = O(80 \times 22^8) = 438869882880 \times O(1)$. In fact, $(2 \times 10 + 2)^8 = 54875873536$ elements are created and their feasibility should be checked in m equations of n-variable. Then the optimal solution is obtained by computing the objective function values in the feasible vectors and their comparison. So, this method needs huge computations and it is very time-consuming. Li and Liu [18] studied the problem (1)-(3) using the max-Lukasiewicz t-norm composition. In this method, the problem is directly converted to a 0-1 integer linear programming problem without simplification and reduction procedures. If this method is applied to solve Example 3 with the max-product composition, we should solve the following 0-1 integer programming problem by the branch-and-bound method:

$$
\begin{aligned}
Z = \min\ \ & 6.35 + 3u_2 + 1.95u_3 + 0.3u_4 + 3.75u_5 + 3.6u_6 + 0.3u_7 + 0.6u_8, \\
s.t.\ \ & \begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 1 & 0
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}
\geq
\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \\
& u \in \{0,1\}^8.
\end{aligned}
$$

We now consider the algorithm in [5] with the max-parametric hamacher composition operators along with some rules to simplify the original problem. The rules are completely different from the sufficient conditions for simplification. If the rules in [5] are checked for Example 3, Rule 1 can be used for this example. Let $K = \{8\} \subseteq J = \{1,2,...,8\}$. Then $\bigcup_{k \in K} I_k^+ = \{2\} \subseteq \bigcup_{k \in K} I_k^- = \{2,4\}$ and the conditions of Rule 1 in [5] are satisfied. Therefore, $x_8^* = \check{x}_8 = 0.7$ and rows 2 and 4 and column 8 from matrices Q^+ and Q^- are removed. Rule 1 cannot again be applied for the simplified problem. Rules 2,3, and 4 are not applicable for the simplified problem. To check Rule 5 in [5] for the simplified problem, we should run Algorithm 1 in [5]. In Step 1, a feasible solution $x = (\check{x}_1, \hat{x}_2, \check{x}_3, \check{x}_4, \check{x}_5, \check{x}_6, \check{x}_7) = (0.2, 1, 0.25, 0.3, 0, 0.4, 0.5)$ is obtained by characteristic boolean formula $C = \bigwedge_{i \in I'} C_i$ where $I' = \{1,3,5,6,7,8,9,10\}$ and $C_1 = y_4 \bigvee \neg y_1 \bigvee \neg y_6$, $C_3 = y_5 \bigvee \neg y_2 \bigvee \neg y_3$, $C_5 = y_4 \bigvee \neg y_5 \bigvee \neg y_6$, $C_6 = y_2 \bigvee y_3 \bigvee y_6$, $C_7 = y_1 \bigvee y_2$, $C_8 = y_4 \bigvee \neg y_4$, $C_9 = \neg y_1 \bigvee \neg y_7$, and $C_{10} = y_7 \bigvee \neg y_1 \bigvee \neg y_6$. The formula is satisfiable for $y_2 = 1$, and $y_1 = y_3 = y_4 = y_5 = y_6 = y_7 = 0$. In Step 2, $J' = \{2\}$. In Step 3, since $J' \neq \emptyset$, we go to Step 4. In Step 4, $c_2(\hat{x}_2 - \check{x}_2) = 4 \times (1 - 0.25) = 3$ is computed. In Step 5, $H = \{h_2 = c_2(\hat{x}_2 - \check{x}_2)\} = \{h_2 = 3\}$. Then Procedure Decreasing Order(H) is run and with $s_1 = 2$, we return to Step 6 of Algorithm 1. The condition of Substep 6.1 is not satisfied. Therefore, we go to Step 7 and $U = 7.95$. For each $k \in \{1, 2, ..., 7\}$, we have $c_k(\hat{x}_k - \check{x}_k) \leq U$. Therefore, Rule 5 isn't applicable for this example. Hence, the modified branch-and-bound should be run on the following matrix:

$$
M = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} & \multicolumn{2}{c}{5} & \multicolumn{2}{c}{6} & \multicolumn{2}{c}{7} \end{array} \\
\begin{pmatrix}
\infty & 0 & \infty & \infty & \infty & \infty & 0.3 & \infty & \infty & \infty & \infty & 0 & \infty & \infty \\
\infty & \infty & \infty & 0 & \infty & 0 & \infty & \infty & 3.75 & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 0.3 & \infty & \infty & 0 & \infty & 0 & \infty & \infty \\
\infty & \infty & 3 & \infty & 1.95 & \infty & \infty & \infty & \infty & \infty & 3.6 & \infty & \infty & \infty \\
0 & \infty & 3 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 0.3 & 0 & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 \\
\infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & 0.3 & \infty
\end{pmatrix}
\end{array}.
$$

We now consider the algorithm given in [19] to solve the problem in Example 3. Rules 1, 2-1, and 2-2 in [19] aren't applicable for this example. Rule 3 in [19] is satisfied for equations 2 and 8 because $J_2(M) \bigcap J_2(\underline{M}) = \{8\}$ and $J_8(M) \bigcap J_8(\underline{M}) = \{4\}$. Therefore, equations 2 and 8 can be removed. Hence, columns 2 and 8 can be removed from matrices Q^+ and Q^-. According to Rule 4 in [19], no equations can be satisfied by $x_8 = \hat{x}_8$. So, $x_8^* = \check{x}_8 = 0.7$. The rows corresponding to $\hat{x}_8$ and $\check{x}_8$ and column 4 are also removed from the matrix V corresponding to this problem in [19]. The reduction matrix V is as follows:

$$
V = \begin{array}{c} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \\ \hat{x}_6 \\ \hat{x}_7 \\ \check{x}_1 \\ \check{x}_2 \\ \check{x}_3 \\ \check{x}_4 \\ \check{x}_5 \\ \check{x}_6 \\ \check{x}_7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 3 & 5 & 6 & 7 & 9 & 10 \end{array} \\
\begin{pmatrix}
\infty & \infty & \infty & \infty & 0 & \infty & \infty \\
\infty & \infty & \infty & 3 & 3 & \infty & \infty \\
\infty & \infty & \infty & 1.95 & \infty & \infty & \infty \\
0.3 & \infty & 0.3 & \infty & \infty & \infty & \infty \\
\infty & 3.75 & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & 3.6 & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 0.3 \\
0 & \infty & \infty & \infty & \infty & 0 & 0 \\
\infty & 1 & \infty & \infty & \infty & \infty & \infty \\
\infty & 0.75 & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & 0 & \infty & \infty & \infty & \infty \\
2.4 & \infty & 2.4 & \infty & \infty & \infty & 2.4 \\
\infty & \infty & \infty & \infty & \infty & 0.5 & \infty
\end{pmatrix}
\end{array}.
$$

Rules 5 and 6 in [19] aren't applicable for this example. To obtain the optimal values of other variables, the 0-1 integer linear programming problem equivalent to matrix V should be written which the problem is solved by the branch-and-bound method. The method has a high computational complexity. We now consider Algorithm 1 in [6] to solve the problem in Example 3. In its Step 1, the lower and upper bound of $\check{x}$ and $\hat{x}$ are computed. Since $b_i \neq 0$, for each $i \in I$, Step 2 is not applicable for this problem. In this problem, $\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$, are satisfied and the conditions of Lemmas 4 and 6 in [6] aren't held for the problem. Matrices Q^+ and Q^- and sets I_j^+, I_j^-, J_i^+, and J_i^-, for each $i \in I$ and $j \in J$, are computed in Step 4. Also, the problem is feasible according to Step 5. The process of problem reduction is done in Step 6. The conditions of Substep 6.2, 6.3, 6.4, and 6.5, aren't applicable for this problem. Only the condition of Substep 6.6 is satisfied for this problem and it causes that rows 2 and 8 in the computation of the minimum objective value are removed. Also, Step 7 isn't applicable for this problem. Hence, we can't find an optimal value of any variable by the reduction or simplification procedures of Algorithm 1 in [6]. So, the reduced problem should be solved by the modified branch-and-bound method on matrix M to find the optimal values of all its variables.

$$M = \begin{array}{c} \\ 6 \\ 7 \\ 1 \\ 3 \\ 4 \\ 5 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} & \multicolumn{2}{c}{5} & \multicolumn{2}{c}{6} & \multicolumn{2}{c}{7} & \multicolumn{2}{c}{8} \end{array} \\ \begin{pmatrix} \infty & \infty & 3 & \infty & 1.95 & \infty & \infty & \infty & \infty & \infty & 3.6 & \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 3 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & 0.3 & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & 0 & \infty & \infty & 3.75 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & 0.3 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty & \infty & 0.3 & \infty & \infty & 0 & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & 0.3 & \infty & \infty & \infty \end{pmatrix} \end{array}.$$

In the proposed algorithm, the main challenge is to compute two sets $CS(Q^{R+})$ and $FCS(Q^{R+})$ with regard to the matrix Q^{R+}. However, for this example, the dimensions of matrix Q^{R+} is 2×8 which is less than the dimensions of above matrices.

7 Conclusions and future research directions

In this paper, a linear programming problem subject to a system of max-product bipolar fuzzy relation equations was studied. The structure of its feasible domain was briefly investigated. The covering and irredundant covering concepts were introduced for the system and a sufficient condition was proposed for its consistency based on the concepts. The resolution of the bipolar fuzzy relation programming is a NP-hard problem. Hence, some sufficient conditions were presented to find one of its optimal solutions or some components of its optimal solution based on the covering concepts. Some sufficient conditions were also suggested for its uniqueness. Finally, an algorithm was designed by the branch-and-bound method and simplification rules based on the covering concepts and compared with other existing methods. In the proposed algorithm, the main challenge is to compute two sets $CS(Q^{R+})$ and $FCS(Q^{R+})$ with regard to the matrix Q^{R+}. Our future research directions can be focused on the following items:
(i) Sufficient conditions to reduce the dimensions of matrix Q^{R+}.
(ii) A systematic method to compute all of the elements of two sets $CS(Q^{R+})$ and $FCS(Q^{R+})$.
(iii) An efficient algorithm for finding all the optimal solutions of problem (1)-(3).

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