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Designing and evaluating an optimal budget plan for parallel network systems through DEA methodology

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Abstract. For an optimal system design (OSD), data envelopment analysis (DEA) treats companies as black boxes disregarding their internal processes. Considering the effect of these processes into account companies can upgrade their internal mechanisms of optimal budgeting allocation. The internal processes can be defined as series and parallel networks or a combination of them. In the literature, DEA is utilized as an approach for OSD in order to determine the optimal budget for a company's activities in a system of series network production; but it is shown that this model is not suitable for budgeting parallel systems. To fill this gap, a new model is presented in this paper to evaluate a company's optimal budgeting which has a parallel network system. The presented parallel network OSD via DEA models, allocates the optimal budget to each of the internal processes of the decision-making units (DMU) based on the efficiency of the parallel internal processes. The model, with a limited budget, also is able to identify the amount of budget deficit and congestion. In this regard, two real cases are studied using the suggested model, which is presented in the parallel network OSD via DEA and Forest production in Taiwan. The optimal budget, budget deficit and congestion, and also the advantages of the proposed model are discussed in these examples as well.

Keywords: Network data envelopment analysis, budgeting, parallel system, DEA.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

Data envelopment analysis (DEA) treats companies as black boxes; indeed, evaluating the black-box production processes, allows one to address only a part of the overall inefficiency that is associated with the production system's exogenous inputs and final outputs. The reason is that the intermediate measures connecting the sub-processes are completely disregarded. To fill these gaps, models are presented to

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evaluate a company's inefficiencies by a network system. The first parallel processes is due to Charnes et al. [5] and the first comprehensive study on network DEA models is done by Fare and Grosskopf [10] and Fare and Grosskopf for the first time used the term "network" DEA model and subsequently improved by several scholars [11] (See the extensive literature on network DEA published in three review articles ([6,16,20]). Emrouznejad and Yang, have surveyed and analyzed articles published in DEA applications from 1978 to 2016 [8]. For several application areas (see, e.g., [1, 11, 12, 22–27, 30]).

The purpose of constructing a network production system has not only addressed the overall inefficiency of a production process but also evaluated the inefficiencies associated with its sub-processes, being linked through intermediate measures. There are broadly two types of network structures of a system: series and parallel, along with their hybrid structures, which are represented by either a parallel system of series components or a series system of parallel components [2]. In a series system, the sub-processes are arranged in a sequence so that the outputs of one sub-process become the inputs of the next sub-process. However, in a parallel system, all the sub-processes appear in parallel so that each sub-process operates independently from one other and has no effect on the other ones. As regards the efficiency evaluation, it should be reminded that the series network structure assumes a priori resource allocation of a firm to be given, and then evaluates the efficiency of its production process along with the sub-processes comprising it. In this way, the efficiency of a production process is decomposed into process-specific efficiencies. This decomposition approach is challenged if the resource allocation of the series network production process is not optimal. As Zeleny has pointed out, in many practical application areas, there is a need to design an optimal production process rather than optimizing the given production system [31]. Following this, Wei and Chang in [28] proposed an optimal system design (OSD) model for the series network via DEA for a production firm to help designing its optimal portfolios of inputs and outputs through maximizing its profit, given its total available budget. Besides determining the optimal budget, their model also helps the firm to identify a situation when budget congestion (an increase in budget yielding lower profits) occurs. Then, Wei and Chang in [29] indicated that the presented model in [28] was not appropriate for the network OSD via DEA models; since this model neither yields correct results in deriving the optimal budget nor in verifying budget congestion. In this regard, Wei and Chang proposed a new method that can help a firm to determine its optimal budget and identify budget congestion [29]. Also, Fang developed a new approach to derive DMUs corresponding optimal budgets also; it checks the existence of budget congestion for the OSD series network DEA models [9]. Note that their proposed method is not applicable to parallel network OSD via DEA models; since their sub-processes are independent and have no intermediate process.

Several researches have also been conducted to determine the efficiency of parallel and hybrid networks. Among these investigations, Gong et al. proposed a new parallel DEA model to determine the performance of an organization where each input/output of the system is not the total of all its components [15]. Such a situation arises from a need for auditing firms or enterprises in Chinese manufacturing industries. The presented model by Liu et al. [18], makes two main contributions: it advances improvements to the methods used in the DEA technique and provides governments with a practical and easy-to-adopt perspective to aid administrators in the process of decision-making. In their research, Bi et al. [4], treat the production unit as a black box; given this, it is not clear, how to arrange the production at the production unit level. Their paper serves to generate resource allocation and target-setting plans for each production unit by opening the black box. The latter proposed model exploits the production information of production lines in generating production plans. Finally, the real data of a production system extracted from extant literature are used to demonstrate the proposed method by Li et al. [17].

A new model for allocating the optimal budget to a company with a dynamic network system is presented by Fakharzadeh Jahromi and Mojtabaei [13]. Moreover, Fakharzadeh Jahromi et al. did optimal budgeting by using a generalized optimal system design DEA model [14]. The presented model by them can meet several secondary goals including optimal budgeting and maximum revenue of the whole organization when the optimal budget is determined.

A survey of the literature reveals that vast amount of research is carried out on the efficiency of parallel networks using the DEA technique. There are also numerous studies on optimal budgeting of the series networks, even though there is no research conducted on their budgeting plans. Considering the importance of budgeting in such parallel networks developing a model for budgeting via DEA can be worthwhile. Accordingly, in our contributed paper, we have presented an OSD-DEA model for a firm in a parallel network system (POSD DEA) so as to determine its optimal budget and identify its budget congestion. The strengths and weaknesses of our proposed method are then discussed and compared with those of conventional DEA methods and series DEA methods. The rest of the paper proceeds as follows: Section 2 presents the parallel network OSD via the DEA model. Section 3 discusses how the optimal budgeting and budget congestion for DMUs operating in the parallel network production system. Finally, two real examples are considered to illustrate the applicability of our proposed method in Section 4. Some concluding remarks in Section 5, is ended this research work.

2 Parallel network optimal system design in DEA models

To present our proposed model, let us assume a production system consists of n decision-making units (DMUs) where each $DMU_j(j=1,2,...,n)$ with $X_j=(x_{1j},x_{2j},...,x_{mj})^t$ input and output $Y_j=(y_{1j},y_{2j},...,y_{sj})^t$, has a parallel system comprising the number of D_j subdecision making units, denoted by $subDMU_{jd}(d=1,2,...,D_j)$, which are operated independently. Figure 1 depicts such a case in which the total inputs consumed and outputs produced of $DMU_j(j=1,2,...,n)$, are $x_{ij}=\sum_{d=1}^{D_j}x_{ij}^d$ and $y_{rj}=\sum_{d=1}^{D_j}y_{rj}^d$ respectively, where $X_j^d=(x_{1j}^d,x_{2j}^d,...,x_{mj}^d)^t$ and $Y_j^d=(y_{1j}^d,y_{2j}^d,...,y_{sj}^d)^t$ represent the independent input and output of the $subDMU_{jd}$ of DMU_j respectively (if the number of inputs or outputs of the sub-units is not equal, we make them equal by considering the zero component). Without loss of generality, in the proposed model (PNOSD-DEA), the number of sub-processes related to each DMU_j is considered as $D=max\{D_j, j=1,2,...,n\}$; since, if the number of DMU_j sub-processes is less than D, $D-D_j$ number of virtual parallel sub-processes with zero inputs and zero outputs (or it is possible to have zero weights) are added. Note that the parallel systems do not have intermediate products for connecting different processes. In such a case, let the vector $\lambda=(\lambda^1,\lambda^2,...,\lambda^D)$ be $nD\times 1$ dimension where each vector $\lambda^d=(\lambda_1^d,\lambda_2^d,...,\lambda_n^d)$ is of dimension $n\times 1$ and matrixes $\widehat{X}$ and $\widehat{Y}$ with $m\times nD$ and $s\times nD$ dimensions are the input and output matrixes, respectively such that $\widehat{X_t}=(X_1^t,X_2^t,...,X_n^t)$ and $\widehat{Y_t}=(Y_1^t,Y_2^t,...,Y_n^t)(t=1,2,...,D)$. In this regard, the production possibility set is:

$$T=\{(\overline{X},\overline{Y})|\overline{X}\geq\widehat{X}\lambda,\overline{Y}\leq\widehat{Y}\lambda,\overline{X}\in R^m,\overline{Y}\in R^s\},\tag{1}$$

where it is a semi-positive vector in R^n. By substituting vector quantities $(\widehat{X_t}\widehat{Y_t})$ we have:

$$\begin{aligned}T=\{(\overline{X},\overline{Y})|\overline{X}\geq\textstyle\sum_{j=1}^n\sum_{d=1}^D\lambda_j^dX_j^d,\ \overline{Y}\leq\sum_{j=1}^n\sum_{d=1}^D\lambda_j^dY_j^d,\\ \overline{X}\in R^m,\ \overline{Y}\in R^s,\ \lambda_j^d\geq 0,\ d=1,2,...,D,\ \ j=1,2,...,n\}.\end{aligned}\tag{2}$$

Some studies (e.g., [3, 12] investigated that the best possible way for allocating inputs of a DMU_j to its different $sub-DMU_s$, is to optimize the system.

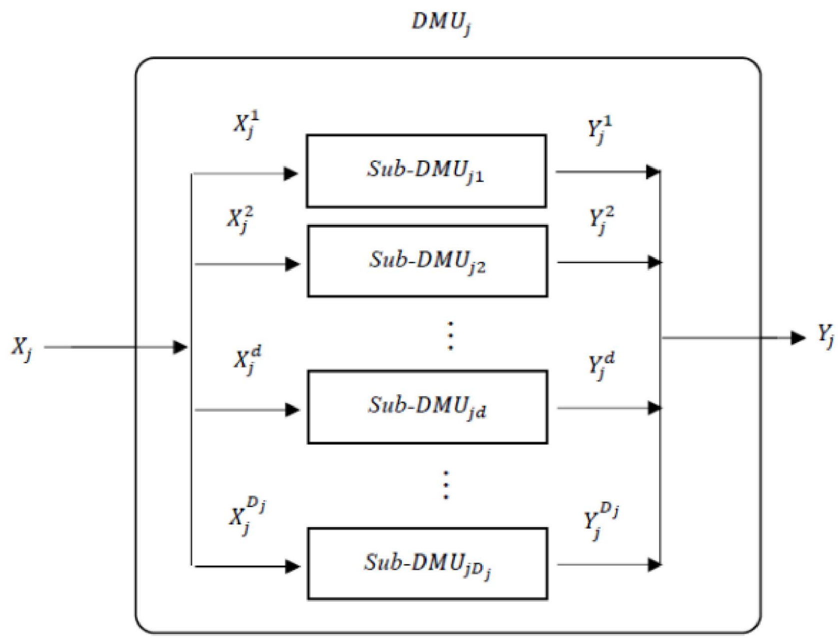

Figure 1: An example of a parallel system of DMU_j.

Let $C = (c_1, c_2, ..., c_m)^t > 0$ and $P = (p_1, p_2, ..., p_m)^t > 0$ be, respectively, the unit price vector of inputs and outputs of DMU_j. Also, the vector $C^d = (c_1^d, c_2^d, ..., c_m^d)(d = 1, 2, ..., D)$ and $P^d = (p_1^d, p_2^d, ..., p_s^d)$ $(d = 1, 2, ..., D)$ are the price vector of inputs and the price vector of outputs of $sub - DMU_j$ respectively, such that $\sum_{d=1}^{D} (C^d)^t X_j^d = C^t X_j > 0$ and $\sum_{d=1}^{D} (P^d)^t Y_j^d = P^t Y_j > 0$ for $j = 1, 2, ..., n$. According to the above definitions, if $D = 1$, the presented OSD-DEA model by Wei and Chang [29] is as following:

$$\begin{array}{ll} f(B) = \max & \sum_{j=1}^{n} \sum_{r=1}^{s} p_r y_{rj} \lambda_j \\ S.t. & \sum_{j=1}^{n} \lambda_j \leq 1; \\ & \sum_{j=1}^{n} \sum_{i=1}^{m} c_i x_{ij} \lambda_j \leq B; \\ & \lambda_j \geq 0, \quad j = 1, 2, \ldots, n, \end{array} \tag{3}$$

where the DMUs total available budget by B, which is known.

Wei and Chang expanded model (3) to determine optimal budget for firms operating in a series network system. But this model is, however, not suitable for the firms operating in a parallel network system [29]. Therefore, we develop the OSD parallel network production model to determine optimal budgets. Although, many network DEA models are developed in literature to deal with parallel network systems, such as [21], but no model is provided for a firm in a parallel network system so as to determine its optimal budget and identify its budget congestion.

Based on model (3) and with regarding the parallel network structure, if the DMU's total available budget is known to be B, one can then set up the following parallel network system OSD via DEA model by maximizing the revenue:

$$\begin{array}{lll} f(B)=\max & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s}p_r^d y_{rj}^d\lambda_j^d \\ S.t. & \sum_{j=1}^{n}\lambda_j^d\le 1, \quad d=1,2,\ldots,D; \\ & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{i=1}^{m}c_i^d x_{ij}^d\lambda_j^d\le B; \\ & \lambda_j^d\ge 0, \quad j=1,2,\ldots,n, \quad d=1,2,\ldots,D. \end{array} \tag{4}$$

Since $0\le e\lambda\le 1$ $(e=(1,1,\ldots,1)^t)$, model (4) is NIRS based on [7].

Definition 1. *If the optimal solution of (4) is λ^*, the system associated with optimal design $(\widehat{X}\lambda^*,\widehat{X}\lambda^*)$ given total available budget B is referred to as the optimal system.*

3 Determining the optimal budget

In the previous section, the system efficiency was determined by assuming the total available budget (B) is given. However, in case that the decision-maker wants to determine the optimal budget while evaluating the system efficiency, it is necessary for him/her to know the way of optimizing values of revenue and profit. The conventional OSD-DEA model (4) is a parametric linear programming model taking B as a parameter; thus the solution methods of parametric linear programming can be applied to determine the optimal budget for the target DMU.

3.1 Validating the optimal budget

Let us denote the objective function of (4) as $Z=f(B)$, interpreted as revenue. Since revenue depends on the optimal values of the budget, we express Z as a function of B, i.e., $Z=f(B)$. Over the domain, $\Omega=\{B|B\ge 0, B\in R\}$, $f(B)$ inherits exactly the shape of the production function, as seen from Theorem 1 below.

Theorem 1. *Over the domain Ω, $f(B)$ is a monotonically increasing or a constant concave function of B. Moreover, $f(0)=0$.*

Proof. It is obvious that $f(B)$ is a monotonically increasing or a constant function and $f(0)=0$. The only concept that to be proved is the concavity of $f(B)$. Consider the dual of model (4) as below:

$$\begin{array}{lll} f(B)=\min & (uB+\sum_{d=1}^{D}v_d) \\ S.t. & u(\sum_{i=1}^{m}c_i^d x_{ij}^d)+v_d\ge\sum_{r=1}^{s}p_r^d y_{rj}^d, \quad j=1,2,\ldots,n, \quad d=1,2\ldots,D; \\ & u\ge 0, \quad v_d\ge 0, \quad d=1,2,\ldots,D, \end{array} \tag{5}$$

where $u\in R$ and $v_d\in R$ for all d. For any $B_1\ge 0$, $B_2\ge 0$ and $\alpha\in[0,1]$,

$$\begin{aligned} f(\alpha B_1+(1-\alpha)B_2) &= \min[u(\alpha B_1+(1-\alpha)B_2)+\alpha\sum_{d=1}^{D}v_d+(1-\alpha)\sum_{d=1}^{D}v_d] \\ &= \min[\alpha(uB_1+\sum_{d=1}^{D}v_d)+(1-\alpha)(uB_2+\sum_{d=1}^{D}v_d)] \\ &\ge \min[\alpha(uB_1+\sum_{d=1}^{D}v_d)]+\min[(1-\alpha)(uB_2+\sum_{d=1}^{D}v_d)] \\ &= \alpha f(B_1)+(1-\alpha)f(B_2). \end{aligned} \tag{6}$$

Therefore, the concavity of $f(B)$ is proved. □

Definition 2. *Suppose B^* is the optimal solution of (4), then we call $B^{**} = \min\{B^*\}$ the optimal budget.*

3.2 Optimal budget derivation

The following theorems and the lemma are presented to derive the optimal budget. The proofs are similar to those in [29].

Theorem 2. *Assume that $\max_{1\le j\le n} P^tY_j = P^tY_k$ and $B^* = C^tX_k$. Model (4) is then expressed as:*

$$\begin{array}{ll} f(B^*) = \max & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s} p_r^d y_{rj}^d \lambda_j^d \\ S.to: & \sum_{j=1}^{n} \lambda_j^d \le 1, \quad d = 1,2,...,D; \\ & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{i=1}^{m} c_i^d x_{ij}^d \lambda_j^d \le B^*; \\ & \lambda_j^d \ge 0, \quad j = 1,2,...,n, \quad d = 1,2,...,D. \end{array} \tag{7}$$

Then,
(i) $f(B) = f(B^*)$ for any $B \ge B^*$;
(ii) $f(B) \le f(B^*)$ for any $0 \le B \le B^*$.

Proof. The method of parametric linear programming with a parametric right-hand side is utilized here in order to simplify the proof. Problem (8) which is equivalent to (4) (where $B \ge B^*$) is considered as:

$$\begin{array}{ll} f(B) = \max & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s} (p_r^d y_{rj}^d) \lambda_j^d + 0S_1 + 0S_2 + ... + 0S_D + 0S_{D+1} \\ S.to: & \sum_{j=1}^{n} \lambda_j^d + S_d = 1, \quad d = 1,2,...,D; \\ & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{i=1}^{m} c_i^d x_{ij}^d \lambda_j^d + S_{D+1} = B; \\ & \lambda_j^d \ge 0, \quad j = 1,2,...,n, \quad d = 1,2,...,D, \quad S_d \ge 0, \quad d = 1,2,...,D+1. \end{array} \tag{8}$$

Model (8) has $n\times D$ variable $\lambda_j^d, j = 1,2,...,n, d = 1,2,...,D$ and $D+1$ slack variable S_d, $d = 1,2,...,D+1$. This model has $D+1$ constraints in which D is the number of the parallel constraint, and another one is related to the budget constraint. The initial feasible simplex table of the problem has $(n\times D)+D+1$ variables and $D+1$ constraint. In order to solve (8), the dual to simplex method, is utilized, where $P^tY_j > 0$, $j = 1,2,...,n$. In this regard, the number of basic variables in the possible initial solution is $D+1$ which are slack variables. Suppose λ_k^d is chosen as the entering variable and S_d as the leaving one, then, by noting that $\max_{1\le j\le n} P^tY_j = P^tY_k$ and $B^* = C^tX_k$, the inequality $P^tY_j - P^tY_k \le 0$, $j = 1,2,...,n$ will be obtained, also by paying attention to $B \ge C^tX_k = B^*$, the optimal solution of problem (8) will be

$$\begin{aligned} \lambda_k^d &= 1, \ \ S_d = 0, \ \ d = 1,2,..,D, \\ S_{D+1} &= B - B^* = B - C^tX_k, \\ \lambda_j^d &= 0, \ \ j \ne k, \ \ d = 1,2,...,D. \end{aligned} \tag{9}$$

Hence,

$$\begin{aligned} f(B) &= \max \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s} (p_r^d y_{rj}^d)\lambda_j^d = \sum_{d=1}^{D}\sum_{r=1}^{s} (p_r^d y_{rk}^d) \\ &= \sum_{d=1}^{D} (P^d)^t Y_k^d = P^tY_k = f(B^*); \end{aligned} \tag{10}$$

i.e., (i) will be true. On the other hand, if $B \geq B^*$, then $f(B) \geq f(B^*)$ (according to Theorem 1); i.e., (ii) will be true. □

Theorem 3. *If $K = \{k | \max_{1\leq j\leq n} P^tY_j = P^tY_k\}$ and $B \geq B^*$, then the set of optimal solutions for model (4) will be $\lambda^* = \{\lambda_k^d | k \in K,\ \sum_{k\in K} \lambda_k^d = 1,\ \lambda_k^d \geq 0,\ d = 1,2,\dots,D\}$, and $B^* = \sum_{k\in K}\sum_{d=1}^{D}\sum_{i=1}^{m} c_i^d x_{ik}^d \lambda_k^d$.*

Proof. For $B \geq B^*$, the equation $f(B) = f(B^*)$ holds according to Theorem 2. It is clear that λ^* is a feasible solution of (4), also with regarding the constraints of (4), for $j \notin K$, $\lambda_j^d = 0$. For all $k \in K$, $\sum_{d=1}^{D}\sum_{r=1}^{s}(p_r^d y_{rk}^d) = f(B^*)$ and $\sum_{k\in K}\lambda_k^d = 1$, also $\max_{1\leq j\leq n} P^tY_j = P^tY_k (k \in K)$, then

$$\sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s}(p_r^d y_{rj}^d)\lambda_j^d \leq \sum_{k\in K}\lambda_k^d(\sum_{d=1}^{D}\sum_{r=1}^{s}(p_r^d y_{rk}^d)). \quad (11)$$

Consequently

$$f(B) = \max \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s}(p_r^d y_{rj}^d)\lambda_j^d = f(B^*). \quad (12)$$

According to the constraints of problem (4), it is clear that $B^* = \sum_{k\in K}\sum_{d=1}^{D}\sum_{i=1}^{m} c_i^d x_{ik}^d \lambda_k^d$. □

Theorem 4. *If $K = \{k | \max_{1\leq j\leq n} P^tY_j = P^tY_k\}$ and $\min_{k\in K} C^tX_k = C^tX_{k^0\in K}$, then $B^{**} = C^tX_{k^0\in K}$ (note that B^{**} is the optimal budget).*

Proof. Since, $B^{**} = \min\{B^* | f(B) = f(B^*)\}$ and according to Theorem 3, $B^* = \sum_{k\in K}\sum_{d=1}^{D}(C^d)^tX_k^d\lambda_k^d$, therefore

$$\begin{aligned} B^{**} &= \min\{\sum_{k\in K}\sum_{d=1}^{D}(C^d)^tX_k^d\lambda_k^d | \sum_{k\in K}\lambda_k^d = 1, \lambda_k^d \geq 0, d = 1,2,\dots,D\}. \\ &= \min_{k\in K}\sum_{d=1}^{D}(C^d)^tX_k^d \\ &= \sum_{d=1}^{D}(C^d)^tX_{k^0}^d = C^tX_{k^0\in K} = B^{**}. \end{aligned} \quad (13)$$

According to the Definition 2 which presents the definition of the optimal budget, B^{**} will be the precise amount of the optimal budget. □

3.3 Budget inefficiency of resulted from congestion

In the preceding discussions, the budget constraint $\sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^tX_j^d\lambda_j^d \leq B$ was made by the suggested POSD-DEA models. If the condition $\sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^tX_j^d\lambda_j^d = B$ is considered, it means that the budget B has to be used in full. In this regard, the following linear programming problem is obtained:

$$\begin{aligned} \widehat{f}(B) = \max \quad & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{r=1}^{s} p_r^d y_{rj}^d \lambda_j^d \\ S.to: \quad & \sum_{j=1}^{n}\lambda_j^d \leq 1, \quad d = 1,2,\dots,D; \\ & \sum_{j=1}^{n}\sum_{d=1}^{D}\sum_{i=1}^{m} c_i^d x_{ij}^d \lambda_j^d = B; \\ & \lambda_j^d \geq 0, \quad j = 1,2,\dots,n, \quad d = 1,2,\dots,D. \end{aligned} \quad (14)$$

Suppose that $\overline{\lambda}$ is the optimal solution to (4) and there is a possible solution for (14) with $B \geq 0$. Also a possible scenario in which $B^* = \sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^tX_j^d\overline{\lambda}_j^d < B$ is considered. Then, $\widehat{f}(B) \leq f(B)$, i.e., the amount of the optimal objective function of (14) is lower than that of (4). This phenomenon which seems not to be natural is actually deduced from congestion which means that when the condition $B^* = \sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^t X_j^d \overline{\lambda}_j^d < B$ holds, congestion will happen. This fact is demonstrated in Theorem 5:

Theorem 5. *If the optimal solution of problem (4) is denoted by $\overline{\lambda}$ and $B^* = \sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^t X_j^d \overline{\lambda}_j^d < B$, then $\widehat{f}(B) \leq f(B) = f(B^*)$.*

Proof. $\widehat{f}(B) \leq f(B)$ because $\{\lambda \,|\, \sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^t X_j^d \lambda_j^d = B\} \subset \{\lambda \,|\, \sum_{j=1}^{n}\sum_{d=1}^{D}(C^d)^t X_j^d \lambda_j^d \leq B\}$. Moreover, from Theorem 2, $f(B) = f(B^*)$. Therefore, $\widehat{f}(B) \leq f(B) = f(B^*)$. □

4 Empirical illustration

In this section, two real numerical examples from [20] and [19] are taken to illustrate the applicability of both the conventional OSD model (3) and OSD parallel network DEA model (4). To execute the computations, we have used Lingo 11 software on a PC with Intel Core $i5-7200U$ CPU, 12GB RAM, and a double precision format.

4.1 Example 1

Table 1 exhibits the data set for a parallel system comprising five DMUs with each having three processes. For the computation of optimal budgets in both models, we have assumed the unit price vectors for inputs and outputs as $C^t = (1,1,1,1)$ and $P^t = (1,1,1,1)$, respectively, and the total available budget (B) as 60. The OSD model (3) is solved to compute the conventional budgets, and Table 3 exhibits the results where one finds three DMUs E, C, and D exhibiting the optimal budgets. The optimal budgeting results based on the OSD parallel network DEA model (4) are presented in Table 2 where one finds no DMUs exhibiting optimal budget. However, it can still rank from best to worst, as can be seen by the first $(\lambda_4^3, \lambda_5^1, \lambda_5^2)$, second $(\lambda_2^2, \lambda_3^1, \lambda_5^3)$ and third $(\lambda_3^2, \lambda_3^3, \lambda_4^1)$ rows of Table 4. The three inefficient processes (the first process of DMU A, the third process of DMU B, and the second process of DMU D) identified by the parallel network model (4) are still evaluated as the worst by the conventional model (3). Indeed, Table 4 presents the comparative results of optimal budgeting by the parallel and conventional models.

We now present below a list of advantages of the parallel network budgeting model over its conventional counterpart:

1. The most important advantage lies in its ability to distribute the available budget at the DMU levels.

2. As can be observed from Table 4, in the parallel network DEA model, in the first budget allocation, only the first and second processes of DMU E are prioritized, and instead of allocating the budget to its third process, it is allocated to the third process of DMU D. In the conventional model, however, the full budget is allocated to the whole DMU E itself (with regarding its processes).

3. In the parallel network model, in the case of a budget deficit, it is quite possible to select the most efficient processes, which is, however, not possible in the conventional model. This can be easily verified from the example of the budget allocation of DMU E. With the optimal budget of 12, unlike in the case of the conventional model where the full budget is allocated to the DMU E itself,

Table 1: Data for the parallel system [20].

	Process 1		Process 2				Process 3	
DMU	X_1^1	Y_1^1	X_1^2	X_2^2	Y_1^2	Y_2^2	X_1^3	Y_1^3
A	3	2	1	2	2	3	2	2
B	2	3	1	4	2	4	2	1
C	4	3	3	2	2	4	3	2
D	3	3	3	3	3	2	3	3
E	3	4	4	3	4	3	2	3

in the parallel network model, however, the budget is allocated to the first and the second processes of DMU E and the third process of DMU D. This example reveals the priority of budget allocation of DMU E by shifting the budget from its third process to the third process of DMU D.

4. When the DMU's internal processes are more detailed, the parallel network model becomes more important and efficient by the ability to distribute the limited available budgets effectively among various processes of the DMUs.

5. While the parallel network model enables one to identify various processes of the DMUs that are not profitable, the conventional model is unable to do so and declares rather the whole DMU as unprofitable. Table 5 exhibits such comparative optimal budgeting results in which one finds that the first three priorities are profitable and are, hence, chosen for budget allocation; in this manner, no budget is allocated to the fourth and fifth priorities as they are incurring losses.

6. Contrary to the previous situation, due to the structure of the conventional model (which not considers the sub-processes) no budget is allocated to a DMU which is not profitable. The parallel network model, however, properly identifies the unprofitable sub-processes of the above DMU and then, allocates budgets to other sub-processes which are more efficient in comparison with those of other DMUs.

4.2 Example 2 (Forest production in Taiwan)

For better efficiency, some organizations allocate their production to several independent units and forest production is one of them in the nation of Taiwan. In this regard, geographical separation is very important. Taiwan is an island with a 36000 sq. km area, half of that is covered by forest. This forestland is divided into eight districts, and each is further divided into four or five sub-districts called Working Circles (WCs). Altogether, there are 34 WCs in this country, which are the basic unit in forest management. The forest production system, indeed, is a common parallel production system kind, where each district consists of several subordinate divisions that work independently. In organizational terms, each district is an independent unit whereas each WC is not, since it has no administrator. Rather, Taiwan Forestry Bureau (which presides over Taiwan's national forests) is interested in the efficiency of all districts in the efficiency of subordinated WCs as well. The administrator of each distinct can reallocate the resources to working circles for improving productivity. The parallel network model developed by Kao [19] discusses

Table 2: The optimal budgeting results with OSD parallel network DEA model.

Row	λ^*	$f(B)$	B^{**}	$B-B^{**}$
1	$\lambda_4^3, \lambda_5^1, \lambda_5^2$	14	13	47
2	$\lambda_2^2, \lambda_3^1, \lambda_5^3$	12	11	36
3	$\lambda_3^2, \lambda_3^3, \lambda_4^1$	11	11	25
4	$\lambda_1^2, \lambda_1^3, \lambda_2^1$	10	7	18
5	$\lambda_1^1, \lambda_2^3, \lambda_4^2$	8	10	8

Table 3: The optimal budgeting results with conventional OSD DEA model

Row	λ^*	*DMU*	$f(B)$	B^{**}	$B-B^{**}$
1	λ_5	$DMU5$	14	12	48
2	λ_3	$DMU3$	11	12	36
3	λ_4	$DMU4$	11	12	24
4	λ_1	$DMU1$	10	9	15
5	λ_2	$DMU2$	10	9	6

Table 4: Comparing results on optimal budgeting.

	Parallel network with B = 60				Conventional model with B = 60			
ROW	*Solution*	$f(B)$	B^{**}	$B-B^{**}$	*Solution*	$f(B)$	B^{**}	$B-B^{**}$
1	$\lambda_4^3, \lambda_5^1, \lambda_5^2$	14	13	47	λ_5	14	12	48
2	$\lambda_2^2, \lambda_3^1, \lambda_5^3$	12	11	36	λ_3	11	12	36
3	$\lambda_3^2, \lambda_3^3, \lambda_4^1$	11	11	25	λ_4	11	12	24
4	$\lambda_1^2, \lambda_1^3, \lambda_2^1$	10	7	18	λ_1	10	9	15
5	$\lambda_1^1, \lambda_2^3, \lambda_4^2$	8	10	8	λ_2	10	9	6
		55	52			56	54	

Table 5: Comparing results on optimal budgeting based on Profit.

	Parallel network with B = 60				Conventional model with B = 60			
ROW	*Solution*	$f(B)$	B^{**}	$B-B^{**}$	*Solution*	$f(B)$	B^{**}	$B-B^{**}$
1	$\lambda_1^2, \lambda_5^1, \lambda_5^3$	4	8	52	λ_5	2	12	48
2	λ_2^1, λ_2^9	2	7	45	λ_1	1	8	40
3	λ_3^2	1	5	40	λ_2	1	9	31
		7	20	40		4	29	31

efficiency determination without considering the allocated budget; however, no model has yet been offered for the determination of parallel network budgets. In this paper, the parallel DEA model (4) is used to measure the optimal budget both for the eight forest districts and the 34 WCs. Tables 6 and 7 display the input and output data of Taiwan forests taken from [19] respectively; the mentioned four inputs are as follow:

- Land: area in thousand hectares;

- Labor: number of employees in person;
- Expenditures: money spent each year in ten thousand New Taiwan dollars (1000 NTD $\simeq$ 30 USD);
- Initial stocks: volume of forest stock before the period of evaluation in 10,000 m^3.

The considered outputs of the model are also as follows:

- Timber production: timber harvested each year in cubic meters;
- Soil conservation: volume of forest stock in 10,000 m^3, as higher stock level leads to less soil erosion;
- Recreation: visitors serviced by forests every year in thousands of visits.

For each input/output, the amount of a district is the sum of its subordinated WCs. For the computation of optimal budgets in both models, we have assumed the unit price vectors for inputs and outputs as $C^d = (1,1,1,1)$ and $P^d = (1,1,1,1),(d = 1,2,3,4,5)$ respectively, and the total available budget (B) as 50000. The OSD model (3) is solved to compute the conventional budgets, and Table 8 section (a) exhibits such results where three DMUs 2, 4, and 7 are found to exhibit the optimal budgets. The results of optimal budgeting based on the OSD parallel network DEA model (4) are indicated in section (b) of Table 8. The first row indicates the priority in optimal budgeting distribution in which the WCs 9, 16, 24, 27, and 5 should get the optimal budget as predicted. Then these WCs are omitted and the amount of budget is reduced to 43780.21 and again model (4) is applied to get the second-row results. In this regard, we had done 8 iterations to reach the final results. The amounts in the last row of the Table 8 show the total revenue, the consumed optimal budget, and the remaining budget respectively in each section. Table 8 also displays the results of comparing the parallel network budgeting model and the conventional model. The following presents the advantages of the results of the parallel network budgeting model:

1. As can be observed from Table 8, in the parallel network DEA model, in the first apportioning, the budget is allocated to working circles Ho-ping, Ta-hu, Pu-li, Chao-chou, and Kuan-shan from forest districts of Lotung , Hsinchu, Nantou, Pingtung, and Taitung in the mentioned respective order. The requisite optimal budget is 33786.04 which has a revenue of 33786.4. However, in the conventional model, Hsinchu District is entirely selected in the priority, which by allocating the optimal budget of 3939 has a revenue portion of 19736. As is revealed from the results indicated in Table 8, using a parallel network model which considers the budget effect within the network causes the budget to be allocated exclusively to the working circles with higher efficiencies; while in the conventional model, the budget is allocated to the entirety of a district.

2. In the parallel network model, in case of a budget deficit, the facility of selecting more efficient processes is provided, whereas it is not provided in the conventional model. This is readily confirmed by the example of allocating 33786.04 to working circles Ho-ping, Ta-hu, Pu-li, Chao-chou, and Kuan-shan from forest districts of Lotung , Hsinchu, Nantou , Pingtung, and Taitung in the first priority. This is totally unlike the conventional model which has allocated the entire budget to the first order, Hsinchu districts.

Table 6: Input data of Taiwan forests.

Labor(1000ha)		Inputs			
		land	*labor* (*person*)	*Expenditures* (10000*NT*)	*Initialstocks* ($10000m^3$)
	Lotung Distric	175.73	248.33	1581.60	1604.38
1	Taipei	18.23	45.33	608.32	125.46
2	Tai-ping-shan	55.49	98.00	336.33	584.85
3	Chao-chi	31.44	51.00	263.99	147.76
4	Nan-au	28.94	27.33	166.78	263.02
5	Ho-ping	41.63	26.67	206.18	483.29
Hsinchu	District	162.81	316.67	850.05	2609.79
6	Guay-shan	41.48	86.33	158.49	386.03
7	Ta-chi	29.72	58.00	260.02	638.87
8	Chu-tung	59.28	77.67	220.97	1218.07
9	Ta-hu	32.33	94.67	210.57	366.82
Tungshi	District	138.42	310.34	864.42	2348.03
10	Shan-chi	10.40	50.67	218.55	103.86
11	An-ma-shan	33.64	111.33	153.07	731.43
12	Li-yang	38.01	97.67	272.32	421.41
13	Li-shan	56.37	50.67	220.48	1091.33
Nantou	District	211.82	287.32	1835.20	2352.10
14	Tai-chung	10.57	64.33	319.51	39.12
15	Tan-ta	52.69	49.00	340.54	688.60
16	Pu-li	77.22	68.33	652.53	966.44
17	Shui-li	54.29	59.33	348.33	602.24
18	Chu-shan	17.05	46.33	174.29	55.70
Chiayi	District	139.65	203.00	215.77	1316.48
19	A-li-shan	42.81	69.33	62.51	527.44
20	Fan-chi-hu	19.28	35.33	54.71	96.00
21	Ta-pu	32.86	44.67	60.41	196.30
22	Tai-nan	44.70	53.67	38.14	496.74
Pingtung	District	196.06	250.33	1230.56	1588.02
23	Chih-shan	35.64	61.33	37.92	150.90
24	Chao-chou	70.19	62.00	188.12	624.80
25	Liu-guay	70.96	55.67	461.42	722.46
26	Heng-chun	19.27	71.33	543.10	89.86
Taitung	District	226.54	141.67	755.20	2679.98
27	Kuan-shan	113.42	54.67	272.35	1607.90
28	Chi-ben	44.54	41.00	184.65	552.13
29	Ta-wu	44.03	20.33	100.70	394.03
30	Chan-kong	24.55	25.67	197.50	125.92
Hualien	District	320.43	284.00	1092.92	401.21
31	Shin-chan	85.95	64.00	314.71	1074.86
32	Nan-hua	51.60	76.00	228.40	886.07
33	Wan-yong	59.53	74.00	282.01	829.11
34	Yu-li	123.35	70.00	267.80	1611.17

Table 7: Output data of Taiwan forests.

Labor(1000ha)		Outputs		
		Timer (m^3)	*Soidcons* ($10000m^3$)	*Recreation* ($1000vis$)
	Lotung Distric	746.04	1604.01	207.59
1	Taipei	19.59	125.46	0.00
2	Tai-ping-shan	17.70	584.85	207.59
3	Chao-chi	0.00	147.39	0.00
4	Nan-au	38.00	263.02	0.00
5	Ho-ping	670.75	483.29	0.00
Hsinchu	District	16823.42	2603.99	308.97
6	Guay-shan	26.37	386.03	114.16
7	Ta-chi	42.53	638.87	181.01
8	Chu-tung	1350.65	1214.48	13.80
9	Ta-hu	15403.87	364.61	0.00
Tungshi	District	4778.32	2819.48	264.92
10	Shan-chi	2842.34	165.63	0.00
11	An-ma-shan	0.00	728.19	38.98
12	Li-yang	1935.98	558.17	111.26
13	Li-shan	0.00	1367.49	114.68
Nantou	District	11429.54	2343.86	0.00
14	Tai-chung	3330.16	39.12	0.00
15	Tan-ta	1242.50	688.60	0.00
16	Pu-li	4134.43	966.44	0.00
17	Shui-li	2574.87	602.24	0.00
18	Chu-shan	147.58	47.46	0.00
Chiayi	District	1086.00	1330.10	845.05
19	A-li-shan	0.00	527.40	845.05
20	Fan-chi-hu	1086.00	95.97	0.00
21	Ta-pu	0.00	195.85	0.00
22	Tai-nan	0.00	510.88	0.00
Pingtung	District	7236.45	1588.02	939.69
23	Chih-shan	1405.76	150.90	0.00
24	Chao-chou	1802.85	624.80	0.00
25	Liu-guay	4027.84	722.46	8.08
26	Heng-chun	0.00	89.86	931.61
Taitung	District	8086.47	2679.98	161.38
27	Kuan-shan	7669.57	1607.90	57.87
28	Chi-ben	416.90	552.13	103.51
29	Ta-wu	0.00	394.03	0.00
30	Chan-kong	0.00	125.92	0.00
Hualien	District	2263.01	4410.58	53.19
31	Shin-chan	17.77	1085.88	0.00
32	Nan-hua	110.28	882.20	16.50
33	Wan-yong	339.91	819.16	0.00
34	Yu-li	1795.05	1623.34	36.69

Table 8: Optimal budgeting results with OSD parallel and conventional DEA model.

(a) Parallel model B = 50000

ROW	*Solution*	*WCs*	$f(B)$	B^{**}	$B-B^{**}$
1	$\lambda_7^1,\lambda_6^2,\lambda_4^3,\lambda_2^4,\lambda_1^5$	$\lambda_{27},\lambda_{24},\lambda_{16},\lambda_9,\lambda_5$	33786.04	33786.04	43780.21
2	$\lambda_4^1,\lambda_4^2,\lambda_6^3,\lambda_8^4,\lambda_4^5$	$\lambda_{14},\lambda_{15},\lambda_{25},\lambda_{34},\lambda_{18}$	13708.6	5241.03	38539.18
3	$\lambda_3^1,\lambda_5^2,\lambda_3^3,\lambda_4^4$	$\lambda_{10},\lambda_{20},\lambda_{12},\lambda_{17}$	9972.49	2481.92	36057.26
4	$\lambda_6^1,\lambda_7^2,\lambda_2^3,\lambda_3^4$	$\lambda_{23},\lambda_{28},\lambda_8,\lambda_{13}$	6690.64	4103.16	31954.1
5	$\lambda_5^1,\lambda_8^2,\lambda_8^3,\lambda_6^4$	$\lambda_{19},\lambda_{32},\lambda_{33},\lambda_{26}$	4561.52	3912.28	28041.82
6	$\lambda_8^1,\lambda_2^2,\lambda_7^3,\lambda_5^4$	$\lambda_{31},\lambda_7,\lambda_{29},\lambda_{22}$	2871.32	3718.95	24322.87
7	$\lambda_2^1,\lambda_1^2,\lambda_5^3,\lambda_1^4$	$\lambda_6,\lambda_2,\lambda_{21},\lambda_4$	1834.01	2566.98	21755.89
8	$\lambda_1^1,\lambda_3^2,\lambda_1^3,\lambda_7^4$	$\lambda_1,\lambda_{11},\lambda_3,\lambda_{30}$	1185.48	2694.3	19061.59
	......	.....	74610.1	30938.41	1961.59

(b) Conventional model B =50000.

ROW	*Solution*	$f(B)$	B^{**}	$B-B^{**}$
1	λ_2	19736	3939	46061
2	λ_4	13773	4686	41375
3	λ_7	10928	3803	37572
4	λ_3	9764	3265	34307
5	λ_6	9764	3265	31042
6	λ_8	6727	2099	28943
7	λ_5	3261	1875	27068
8	λ_1	2558	3610	23458
	.....	76511	26542	23458

5 Concluding remarks

The optimal budgeting models for series networks cannot work effectively for parallel networks which are independent of one another and whose internal mechanisms are essentially different and no model is provided for a firm in a parallel network system to determine its optimal budget and identify its budget congestion. Therefore, in this paper, a new model for optimal budgeting of the parallel networks is presented utilizing the implications of data envelopment analysis. In addition, to provide optimal budgeting within the network, this model can determine both the budget deficit and the budget waste. Two real examples from [20] and Taiwan forests are also presented to illustrate the applicability of the model and its advantages are also discussed in comparison with the conventional model. Developing a new model whose goal is to evaluate the optimal budgets in dynamic systems, can be considered as future research.

References

[1] N.K. Avkiran, *Opening the black box of efficiency analysis: an illustration with UAE banks*, Omega **37** (2009) 930–941.

[2] R.E. Barlow, F. Proschan, *Importance of system components and fault tree events*, Stochastic Process. Appl. **3** (1975) 153–173.

[3] J.E. Beasley, *Determining teaching and research efficiencies*, J. Oper. Res. Soc. **46** (1995) 441–452.

[4] G. Bi, J. Ding, Y. Luo, L. Liang, *Resource allocation and target setting for parallel production system based on DEA*, Appl. Math. Model. **35** (2011) 4270–4280.

[5] A. Charnes, W.W. Cooper, B. Golany, R. Halek, G. Klopp, E. Schmitz, D. Thomas, *Two phase data envelopment analysis approach to policy evaluation and management of army recruiting activities: tradeoffs between joint services and army advertising*, Research Report CCS. **532** (1986).

[6] W.D. Cook, L. Liang, J. Zhu, *Measuring performance of two-stage network structures by DEA: a review and future perspective*, Omega **38** (2010) 423–430.

[7] W.W. Cooper, L.M. Seiford, K. Tone, *Introduction to Data Envelopment Analysis and its Uses: With DEA-Solver Software and References*, New York, Springer, 2007.

[8] A. Emrouznejad, G.I. Yang, *A survey and analysis of the first 40 years of scholarly literature in DEA: 1978-2016*, Socio-Econ. Plan. Sci. **61** (2018) 4–8.

[9] L. Fang, *Optimal budget for system design series network DEA model*, J. Oper. Res. Soc. **65** (2014) 1781–1787.

[10] R. Fare, S. Grosskopf, *Intertemporal production frontiers: with dynamic DEA*, Kluwer Academic Publishers, Boston, 1996.

[11] R. Fare, S. Grosskopf, *Network DEA*, Socio-Econ. Plan. Sci. **34** (2000) 35–49.

[12] R. Fare, S. Grosskopf, *Theory and application of directional distance functions*, J. Product Anal. **13** (2000) 93–103.

[13] A. Fakharzadeh Jahromi, M. Mojtabaei, *An optimal budgeting model for dynamic network systems design via data envelopment analysis*, J. Math. Ext. **14** (2020) 1–28.

[14] A. Fakharzadeh Jahromi, M.H. Mojtabaei, M. Haghguyan Jahromi, *Optimal budgeting by using a generalized optimal system design DEA MODEL and the effect of a new DMU'S budget in the MODEL*, J. Organ. Behav. **4** (2019) ID: 91S2547.

[15] Y. Gong, J. Zhu,Y. Chen, D.C. Wade, *DEA as a tool for auditing: application to Chinese manufacturing industry with parallel network structures*, Ann. Oper. Res. **263** (2018) 247–269.

[16] G.E. Halkos, N. G.Tzeremes, S. Kourtzidis, *A unified classification of two-stage DEA models*, Surv. Oper. Res. Manag. Sci. **19** (2014) 1–16.

[17] H. Li, C. Chen, D.W. Cook, J. Zhang, J. Zhu, *Two-stage Network DEA: Who is the Leader?*, Omega **74** (2018) 15–19.

[18] H. Liu, Y. Zhang, Q. Zhu, J. Chu, *Environmental efficiency of land transportation in China: A parallel slack-based measure for regional and temporal analysis*, J. Clean. Prod. **142** (2017) 867–876.

[19] C. Kao, *Efficiency measurement for parallel production systems*, European J. Oper. Res. **196** (2009) 1107–1112.

[20] C. Kao, *Efficiency decomposition in network data envelopment analysis. In: W. D. Cook and J. Zhu (eds.) Data Envelopment Analysis: A Handbook on the Modeling of Internal Structures and Networks*, Boston, Springer, (2014) 55–77.

[21] C. Kao, *Measurement and decomposition of the Malmquist productivity index for parallel production systems*, Omega **67** (2017) 54–59.

[22] C. Kao, S.N. Hwang, *Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan*, European J. Oper. Res. **185** (2008) 418–429.

[23] A.M. Prieto, J.L. Zofio, *Network DEA efficiency in inputoutput models: With an application to OECD countries*, European J. Oper. Res. **178** (2007) 292–304.

[24] B. K. Sahoo, J. Zhu, K. Tone, B.M. Klemen, *Decomposing technical efficiency and scale elasticity in two-stage network DEA*, European J. Oper. Res. **233** (2014) 584–594.

[25] C. Sheth, K. Triantis, D. Teodorovic, *Performance evaluation of bus routes: A provider and passenger perspective*, Transp. Res. E: Logist. Transp. **43** (2007) 453–478.

[26] K. Tone, B.K. Sahoo, *Scale, indivisibilities and production function in data envelopment analysis*, Int. J. Prod. Econ. **84** (2003) 165-192.

[27] C.B. Vaz, A.S. Camanho, R.C. Guimares, *The assessment of retailing efficiency using network data envelopment analysis*, Ann. Oper. Res. **173** (2010) 5–24.

[28] Q.L. Wei, T.S. Chang, *Optimal system design series-network DEA models*, J. Oper. Res. Soc. **62** (2011) 1109–1119.

[29] Q. Wei, T.S. Chang, *Optimal profit-maximizing system design data envelopment analysis models*, Comput. Ind. Eng. **61** (2011) 1275–1284.

[30] M.M. Yu, E.T. Lin, *Efficiency and effectiveness in railway performance using a multi-activity network DEA model*, Omega **36** (2008) 1005–1017.

[31] M. Zeleny, *Optimizing given systems vs. designing optimal systems: The De Novo programming approach*, Int. J. General Syst. **17** (1999) 295–307.