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Synchronization of the chaotic fractional-order multi-agent systems under partial contraction theory

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Abstract. In this paper, a new synchronization criterion for leader-follower fractional-order chaotic systems using partial contraction theory under an undirected fixed graph is presented. Without analyzing the stability of the error system, first the condition of partial contraction theory for the synchronization of fractional systems is explained, and then the input control vector is designed to apply the condition. An important feature of this control method is the rapid convergence of all agents into a common state. Finally, numerical examples with corresponding simulations are presented to demonstrate the efficiency and performance of the stated method in controlling fractional-order systems. The simulation results show the appropriate design of the proposed control input.

Keywords: Synchronization, fractional systems, multi-agent systems, contraction theory, graph theory, partial contraction theory.
AMS Subject Classification 2010: 94B50, 34H10, 37N35.

1 Introduction

The synchronization of multi agent systems during the last decades has been studied in several scientific branches such as mechanics, biology, physics and recently control theories [23]. Multi-agent systems are used in solving complex control problems and have been widely used in sensor networks, unmanned vehicles (UAVs), mobile robots, satellites, and traffic flow control [4, 9, 16]. Generally speaking, in multi-agent systems, synchronization means that agents with desired initial conditions are aligned to the desired goal (phase, position, speed,...) after exchanging information with their neighboring agents. In these systems, graph and matrix theories are often used to achieve synchronization goals. Furthermore

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regarding classification, it is divided into two categories: synchronization with leader and synchronization without leader, the latter being more challenging than the former in terms of stability and connection problems [14]. On the other hand, today it has been proven that the exact model of many multi-agent systems is chaotic. Until recently, due to the inherent features of chaotic systems, the public believed that these systems are uncontrollable.

Gradually, by applying the problem of chaos control and synchronizing this category of systems in scientific and industrial problems such as physics [8, 12], secure communication [5, 24], environmental systems [2] and many others, several methods have been used to achieve these goals. Among others, we can mention the work of Pecora and Carroll [18] in 1990 as the beginning of the synchronization problems of chaotic systems. In the coming years, many successful attempts where used for the control of such systems. However, until 2008, most of the articles have only dealt with integer-order systems, but several phenomena cannot be explained in the integer-order dynamical structure. For example, the search for food by microbes and the group movements of bacteria in greasy environments full of microorganisms, or the movement of ground vehicles on muddy, grassy, or sandy roads [3], and the relationship between temperature and heat flow in the heat release of a semi-solid [19]. Also, compared to integer-order systems, fractional-order systems can well show the inherent features of the system in different processes. In addition, using fractional-order calculus provides new parameters to the designer. These parameters, as a tool in the hand of the designer, lead to better synchronization results in chaotic dynamical systems. The chaotic systems that are sensitive to initial conditions and minor changes in system parameters. In order to stabilize and converge these systems, various control methods have been presented in the literature such as, adaptive control methods, sliding mode control, resistive control, and contraction method. The contraction method is a generalization of Krasovskii's classical theorem. Unlike other control methods, we do not need to have information about the movement or the equilibrium point of the system. In other words, the stability and final behavior of the system are independent of the initial conditions or temporary disturbances. Its performance in a general view is to create a virtual volume from the dynamic equation of the given system instead of finding a decreasing scalar function like Lyapunov's method. A contracting volume can show that the system is contracting and therefore stable. It is shown that the contraction method is more straightforward than other controller methods. In [10], the authors designed a global exponential stable controller for fractional systems using contraction theory. Also, in [6], using the contraction theory, they investigated the synchronization of multi-agent systems and showed that a non-linear control method that causes the stability of the spaceship could stabilize an arbitrarily large ring of connected spaceships. In [7], a variable nonlinear control scheme based on contraction for laser beam stabilizer (LBS) is proposed, which guarantees the convergence of the closed loop system, and it is shown that based on the contraction-based framework, the performance criteria of linear controller can be expanded for nonlinear behavior of the closed loop. Using a comprehensive definition of differential length, we generalize Lyapunov equation and analyze eigenvalues to fulfill the necessary and sufficient conditions for the exponential convergence of fractional-order systems.

A generalization of contraction theory is partial contraction theory, which includes stability and convergence to certain properties. This theory is beneficial for investigating the synchronization behavior of multi-agent systems and was proposed for the first time in [22]. In this article, we study the synchronization of fractional-order multi-agent systems, and by using partial contraction theory and graph theory, we will design a controller so that synchronization takes place; in other words, the dynamic behavior of all agents with different initial conditions will coincide with each other after the passage of time.

The rest of this paper is organized as follows. In the Section 2, we will first describe graph theory and equivalent matrices. Then, after introducing the concepts of Riemann-Lioville fractional operator, we will explain the theory of contraction and partial contraction in fractional systems and state some necessary theorems and lemmas. Then, in Section 3, we refer to the design and proof of a control method using partial contraction theory for the synchronization of chaotic fractional-order multi-agent systems of order $0 < \alpha < 1$. In Section 4, to emphasize the effectiveness of the proposed method, numerical examples are given along with simulation.

2 Preliminaries

In this section, first, some essential concepts of algebraic graph theory and Riemann-Lioville fractional operator are explained. Then, we present the theory of contraction and partial contraction, some definitions, and lemma.

2.1 Graph theory

Using a graph is the most effective and, at the same time, the simplest method for analyzing and modeling the exchange of information between agents in multi-agent systems. Each directed graph is a pair of $G = (V, \varepsilon)$ that $V = (v_0, v_1, v_2, \ldots, v_N)$ which is a set of non- empty nods and finite, $\varepsilon = \{(v_i, v_j), v_i \neq v_j\} \subseteq V \times V$ is the set of edges of the graph that connect the vertices. Each of these edges is represented as an ordered pair $e_{ij} = (v_i, v_j)$ and it indicates that agent j is the receiver of information from agent i, but the opposite is not necessarily true. Therefore, in a directed graph, the order of nodes in each edge is important, and edges are drawn with arrows from the first node to the last node. The set of neighbors of the input node v_i is defined as $N_i = \{v_j \in V : (v_j, v_i) \in \varepsilon\}$, which means the set of nodes from which the edge enters the node v_i, and the number of members of N_i is called the entry degree of the node v_i. Also a sequence of nodes $\{v_0, v_1, \ldots, v_r\}$ in a directed graph G is called a directed path of the graph when

$$(v_i, v_{i+1}) \in \varepsilon, \quad i \in \{0, 1, \ldots, r-1\}.$$

Therefore, node v_i is connected to node v_j whenever there is a directed path from v_i to v_j. Edges of graphs can be weighted or unweighted. A graph whose edges have weights is called a weighted graph. Weights may represent time, cost, relocation or any other factors. In the weighted graph, a weight w_{ij} is considered for each edge $(v_i, v_j) \in \varepsilon$. It is assumed that these coefficients are non-negative so that $w_{ii} = 0$ and $w_{ij} > 0$ if $(v_i, v_j) \in \varepsilon$, otherwise $w_{ij} = 0$ and $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ is called the adjacency matrix of the graph. Also, in a graph, if $e_{ij} \in \varepsilon$ implies $e_{ji} \in \varepsilon$, then it is called an undirected graph such that $\forall i, j \; w_{ij} = w_{ji}$.

Remark 1. *In this article, we use undirected and leader-follower graph, in such a way that one of the vertices is considered the leader agent and is denoted by v_0. The rest of the vertices that are coordinated with the leader agent are called follower agents.*

Among other matrices that play an essential role in designing the control of a multi-agent system is the Laplacian matrix, denoted by $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ and defined as follows:

$$l_{ii} = \sum_{j \neq i} w_{ij}, \qquad \forall i \neq j \quad l_{ij} = -w_{ij}.$$

Also, the degree matrix of G is $D = diag(d_1, \dots, d_N) \in \mathbb{R}^{N\times N}$ where elements on the main diameter $d_i > 0$ if the agent i is the neighbor of the leader agent and $d_i = 0$, otherwise.
In the following, we have an important lemma for the leader-follower graph, which is used for the synchronization of the leader-follower multi-agent system.

Lemma 1. *[25] If there is at least one path from the leader agent v_0 to all n other follower agents v_i, $i = 1,2,\dots,n$, then for each leader-follower graph G, the matrix $H = L+D$ is positive definite. Since the matrix H is always symmetric and positive definite according to the Lemma 1, therefore the matrix H is uniformly positive definite, namely there is a $\beta > 0$ such that*

$$\frac{1}{2}(H+H^T) \geq \beta I > 0.$$

2.2 Riemann-Lioville fractional derivative

Differential and integral equations of fractional order are generalized equations of integer order. This generalization is not just a curiosity in mathematics. However, in the past decades, with the progress in the fields of chaos and the close connection of fractals with various sciences, it has been increasingly used. Several definitions have been proposed for the fractional derivative. The most famous of these definitions are Riemann-Lioville, Caputo, and Grunwald-Letnikov fractional derivatives. In this article, we use the definition of Riemann-Liouville fractional derivative to present the desired content and objectives. The Riemann-Lioville fractional derivative of $f(t)$ with order α is defined as follows [11]

$$^{RL}_{a}D^{\alpha}_{t} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\tau.$$

where $n-1 < \alpha < n$, $n \in \mathbf{Z}^+$, and $\Gamma(.)$ is the Gamma function

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,\mathrm{d}t.$$

For convenience, the Riemann-Liouville derivative $^{RL}_{a}D^{\alpha}_{t}$ is denoted by ${D_t}^{\alpha}$ in this paper.

2.3 Contraction theory

Contraction theory is based on a new method of stability whose main idea is taken from fluid mechanics. The stability of the equilibrium point is related to the way the solution moves near it. In the Lyapunov linearization method, the local stability of the nonlinear system can be analyzed using its differential approximation. However, what is new and different in the contraction method from other methods is that it has more robust stability, or in other words, exponential and global stability. In this method, there is no need to know the equilibrium point, if we can create a virtual volume from the given dynamics, and if this volume is contracting, it shows that the system is contracting and, therefore, stable and convergent. Now the question is how to calculate the contraction rate of the virtual volume formed by the dynamic system? This is possible by determining the contraction rate of two adjacent paths in the flux field. In this case, the convergence of all paths depends on changes in the contraction rate.

As a result, instead of finding an integral of motion in Lyapanov theory, the systems stability can be checked with the help of this contraction rate. For this purpose, consider the following dynamic, as system

$$\dot{x} = f(x(t)), \qquad x(0) = x_0, \tag{1}$$

where f is a nonlinear vector field and $x(t)$ is an n-dimensional state vector. The dynamic as system (1) can be written as the following fractional-order differential equation: $\dot{x} = D_t^1 x = f(x(t))$.

Definition 1. *The virtual dynamics of fractional order α for system (1) is defined as follows [20]*

$$\delta^\alpha \dot{x} = \delta^\alpha D_t^1 x = \delta^\alpha f(x(t)) = D_x^\alpha f(x(t)) \left(D_x^\alpha x\right)^{-1} \delta^\alpha x, \tag{2}$$

where $\delta^\alpha x$ and $\delta^\alpha \dot{x}$ are virtual relocation and virtual speed, respectively.

Now consider the following fractional order system of order α

$$D_t^\alpha x(t) = f(x(t)). \tag{3}$$

By taking the derivative of fractional order $1-\alpha$ from both sides of the dynamic system (3) we set

$$D_t^{1-\alpha}\left(D_t^\alpha x(t)\right) = D_t^1 x(t) = \dot{x}(t) = D_t^{1-\alpha} f(x(t)).$$

It should be noted that in the definition of Riemann-Liouville fractional derivative, we have $D_t^{1-\alpha} D_t^\alpha = D_t^1$. Hence, according to the relation (2) we have

$$\delta^\alpha D_t^1 x = D_x^\alpha D_t^{1-\alpha} f(x(t)) \left(D_x^\alpha x\right)^{-1} \delta^\alpha x. \tag{4}$$

Theorem 1. *If the matrix $D_x^\alpha D_t^{1-\alpha} f(x(t)) \left(D_x^\alpha x\right)^{-1}$ is uniformly negative definite, all trajectories of system (3) converge exponentially to a single trajectory under any arbitrary initial condition [1].*

Definition 2. *[20] The region $\Omega \subseteq \mathbb{R}^n$ of the state space of fractional-order system (3) is called contraction (semicontraction), if the matrix $D_x^\alpha D_t^{1-\alpha} f(x(t)) \left(D_x^\alpha x\right)^{-1}$ in that region is uniformly negative definite (negative semidefinite), in that case, $f(x(t))$ and system (3) are called contracting function and contracting system, respectively.*

2.4 Partial contraction theory

Partial contraction theory extends the concepts and applications of contraction theory to a set of systems or multi-agent systems. This theory was proposed for the first time in the study of network integration [22]. After that, its usability and flexibility have been proven in many scientific and practical fields. Here, we use the partial contraction theory for the synchronization of leader-follower fractional-order systems, which is the main basis of this article.

Theorem 2. *[20] Consider a fractional-order system*

$$D_t^\alpha x = f(x, x, t), \tag{5}$$

and its auxiliary system

$$D_t^\alpha y = f(y, x, t), \tag{6}$$

which is contracting with respect to y. Then, in any special feature where the solution trajectory of the auxiliary system y is applied, the solution trajectories of the main system x will also apply to that feature.

Definition 3. *[20] The fractional-order system (5), which has an auxiliary contracting system like (6), is said to be partially contracting.*

Corollary 1. *[20] A convex composition of the system $D_t^\alpha x = f_i(x,t)$, $i = 1,2,\dots,n$, that is contracting and has a common solution path $x_0(t)$ is contracting.*

Remark 2. *The concepts of an auxiliary contracting systems are also used in control problems. For example, consider a fractional-order nonlinear control system $D_t^\alpha x = f(x,x,u,t)$, and suppose we want the system to reach the desired state $x_d(t)$ by entering control $u(x,x_d,t)$ such that $D_t^\alpha x_d = f(x_d,x,u,t)$. It is enough to have an auxiliary contracting system as $D_t^\alpha y = f(y,x,u,t)$ with particular solutions $x(t)$ and $x_d(t)$ so that x converges to x_d.*

3 Main results

In this section, the problem of leader-follower synchronization of fractional-order multi-agent system using partial contraction theory under a fixed communication graph is discussed. First, we consider the fractional-order system of one leader and one follower, whose dynamic equation is as follows

$$D^\alpha x_0 = f(x_0), \tag{7}$$
$$D^\alpha x_1 = f(x_1) + u(x_0) - u(x_1), \tag{8}$$

where D^α means Riemann-Liouville fractional derivative of order α, $0 < \alpha < 1$ and $x_0, x_1 \in R$. System (7) is called leader and system (8) is called follower.

Theorem 3. *[20] If the control vector u is such that the function $(f-u)$ is contracting, then two systems (7) and (8) will be synchronized.*

If we extend this problem to N agent systems with one leader, we will have the following.

Theorem 4. *Consider the leader-follower multi-agent systems consisting of a leader and N agents following*

$$\begin{aligned}
D^\alpha x_0 &= F(x_0,t),\\
D^\alpha x_1 &= F(x_1,t) + U(x_0) - U(x_1),\\
&\vdots\\
D^\alpha x_{N-1} &= F(x_{N-1},t) + U(x_{N-2}) - U(x_{N-1}),\\
D^\alpha x_N &= F(x_N,t) + U(x_{N-1}) - U(x_N),
\end{aligned}$$

where $x_0 = (x_{01}, x_{02}, \dots, x_{0N}) \in \mathbb{R}^N$ and $x_i = (x_{i1}, x_{i2}, \dots, x_{iN}) \in \mathbb{R}^N$ represent the state of the leader, the state of i^{th} agent, respectively. Hence, if the control vector $U = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$ is such that the function $(F-U)$ is contracting, then all the agents follow the leader with any arbitrary initial condition.

Proof. From the coupled system, we have the following

$$\begin{aligned}
g_0(x,t) &= D^\alpha x_0 - F(x_0,t) = D^\alpha x_1 - F(x_1,t) - U(x_0) + U(x_1),\\
g_1(x,t) &= D^\alpha x_1 - F(x_1,t) = D^\alpha x_2 - F(x_2,t) - U(x_1) + U(x_2) + U(x_0) - U(x_1)\\
&= D^\alpha x_2 - F(x_2,t) + U(x_2) - 2U(x_1) + U(x_0),\\
&\vdots\\
g_{N-2}(x,t) &= D^\alpha x_{N-2} - F(x_{N-2},t) = D^\alpha x_{N-1} - F(x_{N-1},t) + U(x_{N-1}) - 2U(x_{N-2}) + U(x_{N-3}) - U(x_1),\\
g_{N-1}(x,t) &= D^\alpha x_{N-1} - F(x_{N-1},t) = D^\alpha x_N - F(x_N,t) + U(x_N) - 2U(x_{N-1}) + U(x_{N-2}).
\end{aligned}$$

By combining equations $g_{N-2}(x,t)$ and $g_{N-1}(x,t)$, we have

$$g_{N-2}(x,t) = D^\alpha x_N - F(x_N,t) + U(x_N) - U(x_{N-1}) - U(x_{N-2}) + U(x_{N-3}).$$

Furthermore, in the same way, it is obtained from the combination of the above recursive relations

$$g_0(x,t) = D^\alpha x_N - F(x_N,t) + U(x_N) - U(x_{N-1}).$$

Now we consider the following auxiliary system

$$D^\alpha y = F(y,t) + g_0(x,t) + U(x_{N-1}) - U(y),$$

which has two specific solutions $y = x_{N-1}(t)$ and $y = x_N(t)$. Since $F - U$ is a contracting function, the auxiliary system is contracting, so $y = x_{N-1}(t)$ and $y = x_N(t)$ will converge to each other as two solutions of the auxiliary system. □

Theorem 5. *Consider the following multi-agent system consisting of a leader and N followers. Also assume that there is at least one path from the leader to all the followers*

$$\begin{aligned}
D^\alpha x_0 &= F(x_0) = Ax_0 + G(x_0),\\
D^\alpha x_1 &= F(x_1) + U(x_0) - U(x_1) = Ax_1 + G(x_1) + U(x_0) - U(x_1),\\
&\vdots\\
D^\alpha x_N &= F(x_N) + U(x_{N-1}) - U(x_N) = Ax_N + G(x_N) + U(x_{N-1}) - U(x_N),
\end{aligned}$$

where $U = (u_1, u_2, \dots u_N) \in \mathbb{R}^N$, $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ and $G(x) = (g_1(x), g_2(x), \dots, g_N(x))^T$ are the control vector, matrix of system coefficients and a chaos factor in the leader-follower systems, respectively. Let the control vector $U(x)$ be as follows

$$U(x) = Ax + G(x) + Hx, \tag{9}$$

where $H = L + D$. Then, synchronization of all follower agents with the leader is established at each initial condition.

Proof. According to Theorem 4, if $F - U$ becomes a contracting function, the statement of theorem holds. So we have

$$F(x) - U(x) = Ax + G(x) - Ax - G(x) - Hx = -Hx.$$

Moreover, since according to Lemma 1, H is uniformly positive definite, $-H$ is uniformly negative definite, and the statement holds. □

4 Numerical example

In this section, we discuss some illustrative examples to show the convergence of the proposed method in the synchronization problems of chaotic fractional-order multi-agent systems. For numerical simulation, Adams-bashforth-Moulton method is used [13, 17].

Example 1. Consider a chaotic fractional-order system with a leader and three followers Newton-Leipnik [21] under the graph of Figure 1. Also, assuming that throughout this section $v_0 = (x_0, y_0, z_0)$ and $v_i = (x_i, y_i, z_i)$, $i = 1, 2, \ldots, N$ are leader and follower factors respectively. Let $\alpha = 0.99$ and the leader agent with the initial condition $(2, -3, 1)$ and the initial conditions of the follower agents as $(2, -1, 0), (0, 1, 3), (0, -1, 2)$. The leader and follower systems defied as follows

$$
\begin{aligned}
&D^{\alpha} x_0 = -a x_0 + y_0 + 10 y_0 z_0,\\
&D^{\alpha} y_0 = -x_0 - 0.4 y_0 + 5 x_0 z_0,\\
&D^{\alpha} z_0 = -5 x_0 y_0 + b z_0,\\
&D^{\alpha} x_i = -a x_i + y_i + 10 y_i z_i + u(x_{i-1}, y_{i-1}, z_{i-1}) - u(x_i, y_i, z_i),\\
&D^{\alpha} y_i = -x_i - 0.4 y_i + 5 x_i z_i + u(x_{i-1}, y_{i-1}, z_{i-1}) - u(x_i, y_i, z_i),\\
&D^{\alpha} z_i = -5 x_i y_i + b z_i + u(x_{i-1}, y_{i-1}, z_{i-1}) - u(x_i, y_i, z_i), \qquad i = 1, 2, 3.
\end{aligned}
$$

where $a = 0.4$ and $b = 0.175$.

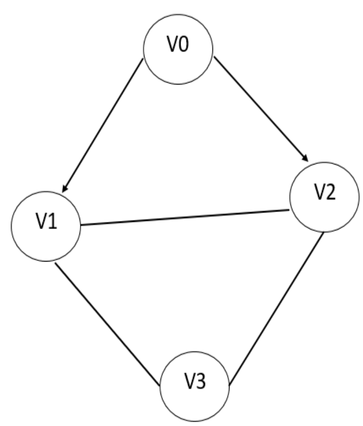

Figure 1: The topology of the leader-follower system under the undirected graph in Example 1.

For convenience, let $w_{ij} = 1 (d_i = 1)$ if $w_{ij} > 0 (d_i > 0)$ and $w_{ij} = 0 (d_i = 0)$, otherwise. Thus, the degree matrix D and Laplacian matrix L of the graph are as follows

$$
D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{10}
$$

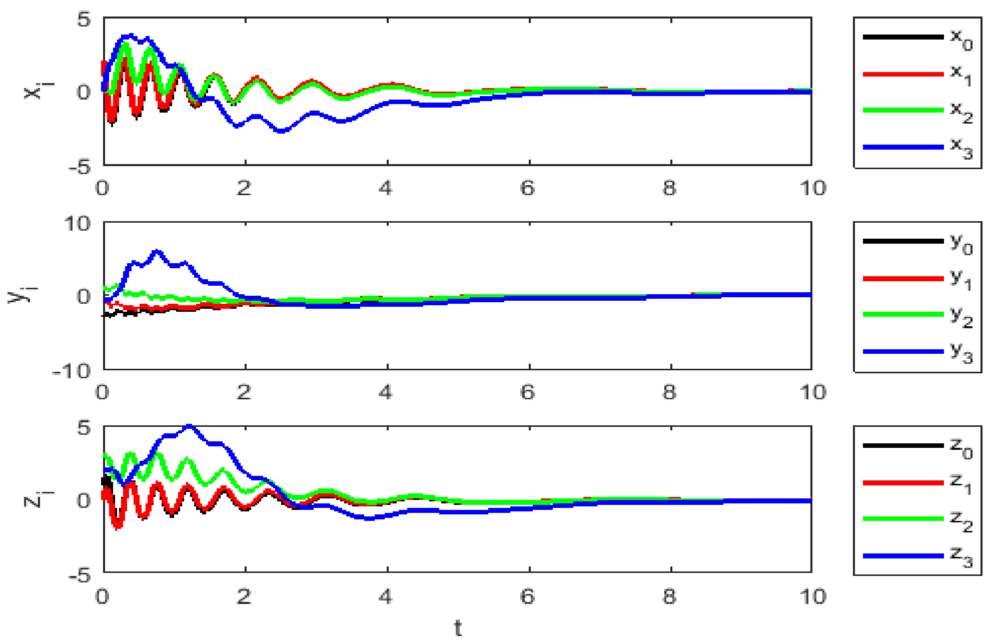

Figure 2: The state trajectories of $x_i, y_i, z_i, i = 0,1,2,3$ in Example 1.

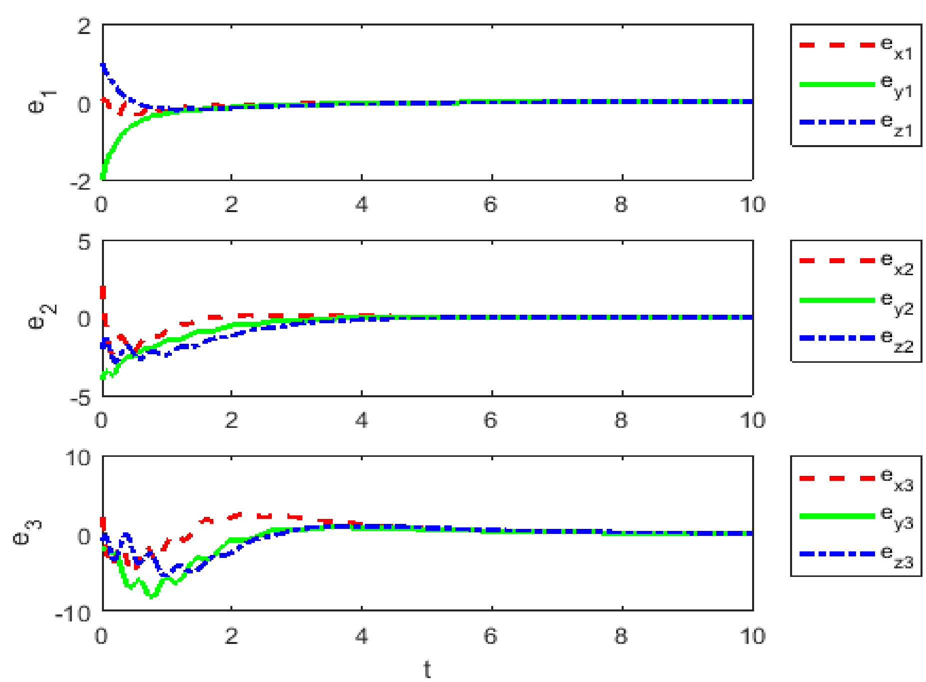

Figure 3: The error trajectories of leader and agents within Example 1.

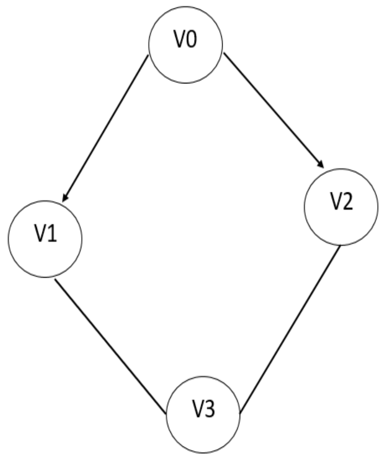

Figure 4: The topology of leader-follower system under the undirected graph in Example 2.

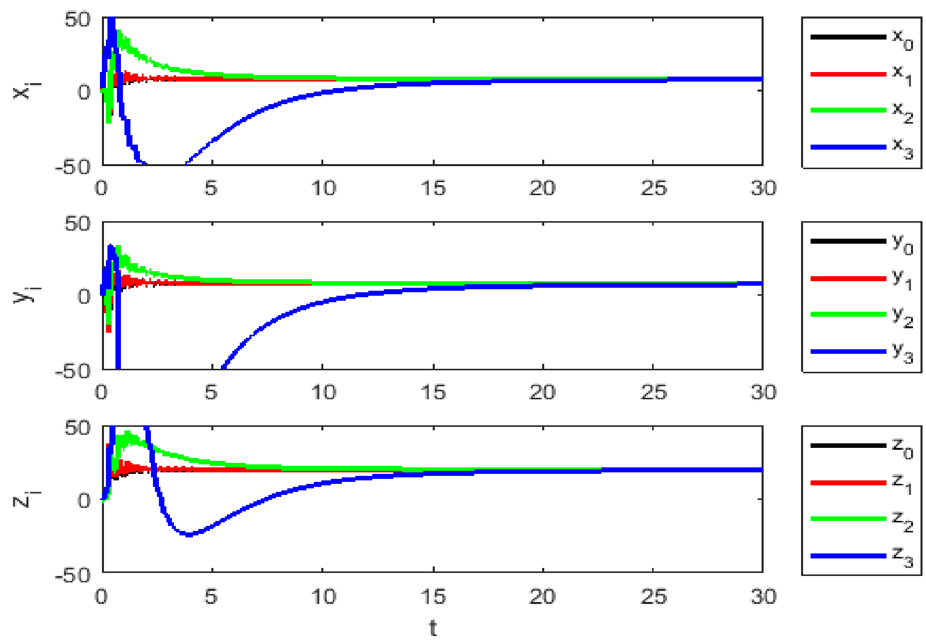

Figure 5: The state trajectories of x_i, y_i, z_i, $i = 0, 1, 2, 3$ in Example 2.

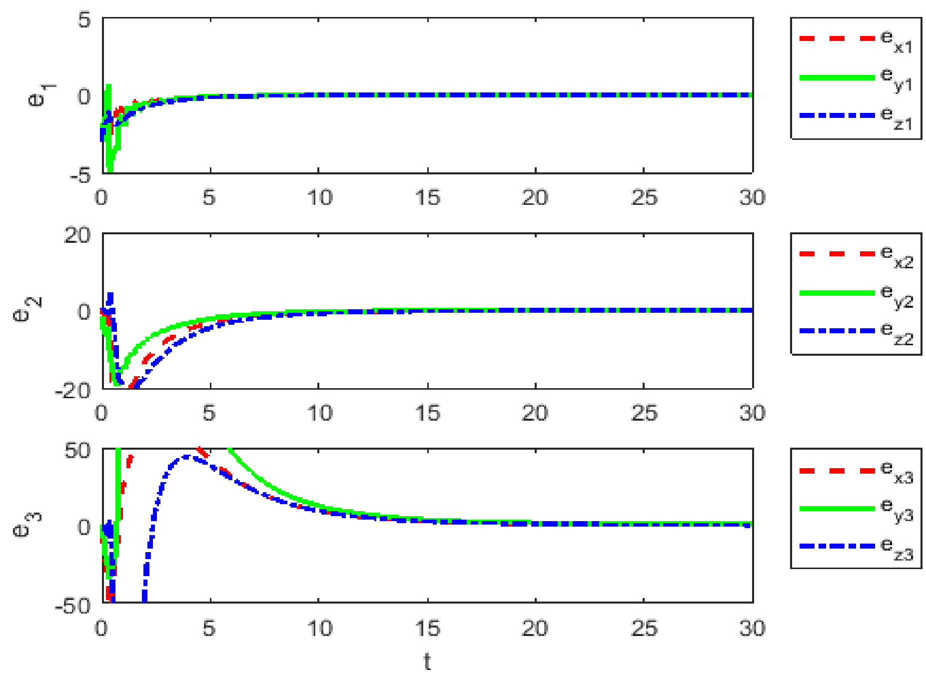

Figure 6: The error trajectories of leader and agents within Example 2

Now, we define the control vector $U(x,y,z)$ under the control law (9) as follows

$$U(x,y,z)=\begin{pmatrix}-0.4 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & 0.175\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}+\begin{pmatrix}10yz\\ 5xz\\ -5xy\end{pmatrix}+\begin{pmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

According to Figure 2, all factors reach a common state. Figure 3 also shows the synchronization error between agents.

Example 2. Consider a chaotic fractional-order system consisting of a leader and three followers Lü [15] under the graph of Figure 4 with fractional order $\alpha = 0.9$ and the initial conditions of leader factor as $(-2,0,-1)$ and the initial conditions of follower agents as $(1,3,2),(-2,1.5,-1),(2,0,-0.5)$. The leader and three followers Lü system are as follows

$$\begin{aligned}
&D^{\alpha}x_0 = a(y_0-x_0),\\
&D^{\alpha}y_0 = -x_0z_0-by_0,\\
&D^{\alpha}z_0 = x_0y_0-cz_0,\\
&D^{\alpha}x_i = a(y_i-x_i)+u(x_{i-1},y_{i-1},z_{i-1})-u(x_i,y_i,z_i),\\
&D^{\alpha}y_i = -x_iz_i-by_i+u(x_{i-1},y_{i-1},z_{i-1})-u(x_i,y_i,z_i),\\
&D^{\alpha}z_i = x_iy_i-cz_i+u(x_{i-1},y_{i-1},z_{i-1})-u(x_i,y_i,z_i), \qquad i=1,2,3.
\end{aligned}$$

where $a=36$, $b=20$ and $c=3$.

According to the connection graph, the degree matrix D and Laplacian matrix L are equal to

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{11}$$

Therefore, by defining the control vector $U(x,y,z)$ with the help of Theorem 5, synchronization occurs in the leader-follower system according to Figure 5. Figure 6 also shows the synchronization error between agents. The designed controller is as follows

$$U(x,y,z) = \begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

5 Conclusions

In this paper, a nonlinear control method is designed for the synchronization of the chaotic fractional-order multi-agent systems of leader-follower. The relationships and results in this article are based on the partial contraction theory, which is a generalization of the contraction theory. Convergence to the equilibrium point with any arbitrary and uncertain initial condition and high resistance to disturbances are the main features of this method. Also, for a better understanding of the content, two numerical examples of chaotic systems are presented to show the importance of the proposed method.

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