Journal of Mathematical Modeling
Vol. 11, No. 3, 2023, pp. 573–585. Research Article

JMM

Picard iterative approach for $\psi-$Hilfer fractional differential problem

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Abstract. In present work, we discuss local existance and uniqueness of solution to the $\psi-$Hilfer fractional differential problem. By using the Picard successive approximations, we construct a computable iterative scheme uniformly approximating solution. Two illustrative examples are given to support our findings.

Keywords: Fractional calculus, $\psi-$Hilfer fractional derivative, Picard's iterative scheme, convergence.
AMS Subject Classification 2010: 26A33, 26D10, 34A08, 40A30.

1 Introduction

Fractional Calculus (FC) has a glorious history of more than three decades and has been evolving in almost all branches of science and engineering. It has emerged and spread its offshoot as a new field of applied mathematics research in twenty first century due to its applicability in many real world applications, for instance see [4, 9, 13, 14, 16, 22] and references therein. During the theoretical development of FC, many fractional differential and integral operators are emerged with specific motive and used by contemporary researchers. Starting with Grunwald-Letnikov, Wyel, Riesz, Liouville-Caputo, Riemann-Liouville, Hadamard, generalized through Hilfer, Katugampola and $\psi-$Hilfer came to be known in material physics and mechanics, signal and image processing, biochemical and electrical engineering, economics and mathematical modelling to name few [1, 3–5, 8, 10–12, 14, 17, 18, 20, 21, 23, 25, 26]. In details of on theory and application of FC, see [14, 20] and their recent citations.

In 2006, Kilbas et al. [14] introduced the concept of fractional differentiation of a function with respect to another function in the Riemann-Liouville sense. They further defined suitable weighted spaces

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Received: 31 June 2023/ Revised: 28 July 2023/ Accepted: 1 August 2023
DOI: 10.22124/JMM.2023.24626.2201

and studied some of its properties by using corresponding fractional integral. Using this idea for Caputo fractional derivative, Almaida [3] proposed a new concept of fractional derivative of a function with respect to anther function called ψ-Caputo derivative. This ψ- Caputo fractional derivative has been widely used by many researchers and studied for its various qualitative properties. Recently in 2018, Sousa and Oliviera [24] proposed interpolator of ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfers [13] sense of definition, and named ψ-Hilfer fractional derivative. This new operator used for generalization of the Gronwall inequality and the data dependence of Cauchy-type problem studied in suitable weighted space [25], also see [2, 15, 19, 26]. Vanterler et al. [25] in 2019 discussed about existence and uniqueness of solution to ψ-Hilfer Cauchy-type problem using Banach contraction mapping principle. Motivated by these results [25], in this paper, we study the initial value problem (IVP) for fractional differential equation (FDE) involving $\psi-$ Hilfer fractional derivative

$$\begin{cases} \mathfrak{D}_{a^+}^{\delta,\rho;\psi}\zeta(s) = \lambda(s,\zeta(s)), & 0<\delta<1, 0\leq\rho\leq 1, s>a>0, \\ \lim\limits_{s\to a}(\psi(s)-\psi(a))^{1-\eta}\zeta(s) = \zeta_0, & \zeta_0\in\mathfrak{R}, \quad \eta=\delta+\rho-\delta\rho, \end{cases} \tag{1}$$

where $\mathfrak{D}_{a^+}^{\delta,\rho;\psi}$ is the ψ-Hilfer fractional derivative, $\lambda : (a,T]\times\mathfrak{R}\longrightarrow\mathfrak{R}$ is given nonlinear function.

We prove the local existence and uniqueness result for IVP (1) using the method presented in Yang et al. [27], Dhaigude et al. [11] and Bhairat [6, 7]. The iterative scheme and uniform convergence criterion for the solution will be discussed.

2 Preliminaries

The following definitions, lemmas, properties of fractional operators will be used in the development of main results.

Definition 1. *[24] Let (a,T) be a finite or infinite interval of $\mathfrak{R}$ and $\delta>0$. Also $\psi(s)$ be an increasing and positive monotone function on $(a,T]$ having a continuous derivative $\psi'(s)$ on (a,T). The (left-sided) fractional integral of λ with respect to another function ψ on $[a,T]$ is defined by*

$$\mathfrak{I}_{a+}^{\delta;\psi}\lambda(s) = \frac{1}{\Gamma(\delta)}\int_a^s \psi'(t)(\psi(s)-\psi(t))^{\delta-1}\lambda(t)dt.$$

Definition 2. *[24] Let $\psi'(s)\neq 0$, $(-\infty\leq a<b\leq\infty)$ and $\delta>0$, $n\in\mathfrak{N}$. The (left-sided) Riemann-Liouville derivative of function λ with respect to ψ of order $\delta>0$ is defined by*

$$\mathfrak{D}_{a+}^{\delta;\psi}\lambda(s) = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n \mathfrak{I}_{a+}^{n-\delta;\psi}\lambda(s).$$

Definition 3. *[24] Let $\delta>0$, $n\in\mathfrak{N}$, $I=[a,T]$ is the interval $(-\infty\leq a<T\leq\infty)$ and λ, $\psi\in C^n([a,b],\mathfrak{R})$ two functions such that ψ is increasing function and $\psi'(s)\neq 0$ for all $s\in I$. The Caputo fractional derivative of λ of order δ is given by*

$${}^c\mathfrak{D}_{a+}^{\delta;\psi}\lambda(s) = \mathfrak{I}_{a+}^{n-\delta;\psi}\left(\frac{1}{\psi'}\frac{d}{ds}\right)^n\lambda(s),$$

where $n=[\delta]+1$ for $\delta\notin\mathfrak{N}$, and $n=\delta$ for $\delta\in\mathfrak{N}$.

Definition 4. *[24] Let $\lambda,\ \psi \in C^n([a,T],\mathfrak{R})$ be two functions such that $\psi(s)$ is increasing function and $\psi'(s) \neq 0$ for all $s \in I$. The (left-sided) ψ fractional derivative of function of λ of order $n-1<\delta<n$ and type $0 \leq \rho \leq 1$ is defined by*

$$\mathfrak{D}_{a+}^{\delta,\rho;\psi}\lambda(s) = \mathfrak{I}_{a+}^{\rho(n-\delta);\psi}\left(\frac{1}{\psi'}\frac{d}{ds}\right)^n \mathfrak{I}_{a+}^{(1-\rho)(n-\delta);\psi}\lambda(s).$$

The (left-sided) $\psi-$ Hilfer fractional derivative is also defined in the following notational form:

$$\mathfrak{D}_{a+}^{\delta,\rho;\psi}\lambda(s) = \mathfrak{I}_{a+}^{\eta\rho;\psi}\mathfrak{D}_{a+}^{\eta;\psi}\lambda(s),$$

with $\eta = \delta+\rho(n-\delta)$; and $\mathfrak{I}^{\eta-\delta;\psi}$, $\mathfrak{D}^{\eta;\psi}$ are as defined respectively by Definition 1 and Definition 2.

Lemma 1. *[24] Let $\delta, \rho > 0$ then the following semi-group property holds:*

$$\mathfrak{I}_{a+}^{\delta;\psi}\mathfrak{I}_{a+}^{\rho;\psi}\lambda(s) = \mathfrak{I}_{a+}^{\delta+\rho;\psi}\lambda(s).$$

Lemma 2. *[24] Let $\delta>0$, $\tau>0$. If $\lambda(x) = (\psi(x)-\psi(a))^{\tau-1}$, then*

$$\mathfrak{I}_{a+}^{\delta;\psi}\lambda(s) = \frac{\Gamma(\tau)}{\Gamma(\delta+\tau)}(\psi(s)-\psi(a))^{\delta+\rho-1}.$$

Lemma 3. *[24] Let $\delta,\tau>0$. If $\lambda(s) = (\psi(s)-\psi(a))^{\tau-1}$, then*

$$\mathfrak{D}_{a+}^{\delta;\psi}\lambda(s) = \frac{\Gamma(\tau)}{\Gamma(\delta-\tau)}(\psi(s)-\psi(a))^{\delta-\rho-1}.$$

Lemma 4. *[18] For $s>0$,*

$$\Gamma(s) = \lim_{\mu\to\infty}\frac{(\mu)^s\mu!}{s(s+1)(s+2)...(s+\mu)}.$$

Lemma 5. *[24] Let $\lambda \in C^1[a,T]$, $\delta>0$, and $0\leq\rho\leq 1$, we have $\mathfrak{D}^{\delta,\rho;\psi}\mathfrak{I}_{a+}^{\delta;\psi}\lambda(s) = \lambda(s)$.*

Lemma 6. *[24] Let $s>a$. If $n-1<\tau<n$, then $\mathfrak{D}^{\tau,\rho;\psi}(\psi(t)-\psi(a))^{\tau-1} = 0$.*

A function $\zeta(s)$ is said to be a solution of IVP (1) if $\exists\, l>0$ such that $\zeta \in C^0(I)$ satisfies the differential equation $\mathfrak{D}_{a+}^{\delta,\rho;\psi}\zeta(s) = \lambda(s,\zeta(s))$ a.e. on I along with the condition $\lim_{s\to a}(\psi(s)-\psi(a))^{1-\eta} = \zeta_0$.

We denote $\omega = [a,a+\sigma]$, $\omega_\sigma = (a,a+\sigma]$, $I = (a,a+l]$, $J = [a,a+l]$, for $\sigma>0$. Moreover, define $\varepsilon = \{\zeta : |(\psi(t)-\psi(a))^{1-\eta}\zeta(s)-\zeta_0| \leq T\}$. Further

$$l = \min\left\{\sigma, \left(\frac{T}{\aleph}\frac{\Gamma(\delta+k+1)}{\Gamma(k+1)}\right)^{\frac{1}{\nu+k}}\right\} \qquad \text{for } \nu = 1-\rho(1-\delta).$$

The generalized weighted spaces suitable for problem at hand are defined as follows:

$$C_{1-\eta,\nu}^{\delta,\rho}[a,T] = C_{\eta,\psi} = \{\zeta \in C_{1-\eta,\psi}[a,T] | ^H D_{a+}^{\delta,\rho;\psi} \in C_{\nu,\psi}[a,T]\},\ \text{for } 0\leq\nu<1,\ \eta = \delta+\rho(1-\delta),$$

where $C_{\eta,\psi}[a,T] = \{\kappa : (a,T] \to \mathfrak{R} | (\psi(s)-\psi(a))^\eta\kappa \in C[a,T]\}$, $0\leq\eta<1$.

To prove the main result we consider the following hypotheses.

(H_1) $(s,\zeta) \to \lambda(s,(\psi(s)-\psi(a))^{\eta-1}\zeta(s))$ is defined on $\omega_\sigma \times \varepsilon$ and satisfies:

(i) $\zeta \to \lambda(s,(\psi(s)-\psi(a))^{\eta-1}\zeta(s))$ is continuous on ε, $\forall\, s \in \omega_\sigma$, $s \to \lambda(s,(\psi(s)-\psi(a))^{\eta-1}\zeta(s))$ is measurable on ω_σ $\forall\, \zeta \in \varepsilon$.

(ii) $\exists\, k > \rho(1-\delta)-1$ and $\aleph \geq 0$ such that

$$\left|\zeta(s,(\psi(s)-\psi(a))^{\eta-1}\zeta(s))\right| \leq \aleph(\psi(s)-\psi(a))^k$$

holds $\forall s \in \omega_\sigma$.

(H_2) $\exists\, \theta > 0$ and $\zeta_1,\, \zeta_2 \in \varepsilon$ such that

$$\left|\lambda(s,(\psi(s)-\psi(a))^{\eta-1}\zeta_1(s)) - \lambda(s,(\psi(s)-\psi(a))^{\eta-1}\zeta_2(s))\right| \leq \theta(\psi(s)-\psi(a))^k\left|\zeta_1-\zeta_2\right|,\ \forall\, s \in I.$$

3 Main result

In this section, we state and prove the existence and uniqueness results for IVP (1).

Lemma 7. *Suppose that (H_1) holds. Then $\zeta : J \longrightarrow \mathfrak{R}$ is the IVP (1) if and only if $\zeta : I \longrightarrow \mathfrak{R}$ is the solution of the Volterra integral equation of second kind*

$$\zeta(s) = \zeta_0(\psi(s)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\zeta(p))dp. \tag{2}$$

Proof. Suppose $\zeta : I \longrightarrow \mathfrak{R}$ is solution of IVP ((1)). Then $|(\psi(s)-\psi(a))^{1-\eta}\zeta(s)-\zeta_0| \leq T$, for all $s \in I$. Since (H_1) holds, $\exists\, k > (\rho(1-\delta)-1)$ and $\aleph \geq 0$ such that

$$\begin{aligned}\left|\lambda(s,\zeta(s))\right| &= \left|\lambda(s,(\psi(s)-\psi(a))^{\eta-1}(\psi(s)-\psi(a))^{1-\eta}\zeta(s)\right| \\ &\leq \aleph(\psi(s)-\psi(a))^k, \qquad \text{for all } s \in I.\end{aligned}$$

We have

$$\begin{aligned}&\left|\tfrac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\zeta(p,\zeta(p))dp\right| \\ &\leq \frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\left|\zeta(p,\zeta(p))\right|dp \\ &\leq \frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\aleph(\psi(p)-\psi(a))^k dp \\ &\leq \frac{\aleph}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}(\psi(p)-\psi(a))^k dp \\ &\leq \aleph\mathfrak{I}_a^{\delta;\psi}(\psi(s)-\psi(a))^k \\ &\leq \frac{\aleph\Gamma(k+1)}{\Gamma(\delta+k+1)}(\psi(s)-\psi(a))^{\delta+k} \\ &= \aleph(\psi(s)-\psi(a))^{\delta+k}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}.\end{aligned}$$

Next

$$\lim_{s\to a}(\psi(s)-\psi(a))^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}\zeta(p,\zeta(p))dp=\lim_{s\to a}\aleph(\psi(s)-\psi(a))^{\delta+k+1-\eta}\times\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}=0.$$

It follows that

$$\zeta(s)=\zeta_0(\psi(s)-\psi(a))^{\delta-1}+\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\zeta(p))dp. \tag{3}$$

Since $k>(\rho(1-\delta)-1)$, we see that $\zeta\in C^0(I)$ is solution of Volterra integral equation(2).

Conversely, $\zeta: I\longrightarrow\mathfrak{R}$ is solution of Volterra integral equation (2). Now, applying ${}^{H}\mathfrak{D}^{\delta,\rho;\psi}$ on both sides of (2)

$$\begin{aligned}
{}^{H}\mathfrak{D}^{\delta,\rho;\psi}\zeta(s) &= {}^{H}\mathfrak{D}^{\delta,\rho;\psi}[\zeta_0(\psi(s)-\psi(a))^{\eta-1}]+\frac{1}{\Gamma(\delta)}{}^{H}\mathfrak{D}^{\delta,\rho;\psi}\mathfrak{J}^{\delta;\psi}\lambda(t,\zeta(t)),\\
{}^{H}\mathfrak{D}^{\delta,\rho;\psi}\zeta(s) &= \lambda(s,\zeta(s)),\\
\lim_{s\to a}(\psi(s)-\psi(a))^{1-\eta} &= \zeta_0,
\end{aligned}$$

which completes the proof. □

Now we present the iterative scheme for approximating the unique solution with following Picard's function.

$$\chi_0(s)=\zeta_0(\psi(s)-\psi(a))^{\eta-1}=\chi_0(s)=\zeta_0\Lambda^{\eta-1}\qquad s\in I, \tag{4}$$

$$\chi_n(s)=\chi_0(s)+\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\chi_{n-1}(p))dp;\ s\in I,\ n=1,2,\cdots. \tag{5}$$

Lemma 8. *Suppose that (H_1) holds. Then χ_n is continuous on I and satisfies $\left|\Lambda^{1-\eta}\chi_n(s)-\zeta_0\right|\le T$*

Proof. From (H_1), clearly we obtain $\left|\lambda(s,\Lambda^{\eta-1})\zeta(s)\right|\le\aleph\Lambda^k$, for all $s\in\omega_\sigma$ and $\left|\Lambda^{1-\eta}\chi_n(s)-\zeta_0\right|\le T$. For $n=1$, we have

$$\chi_1(t)=\zeta_0\Lambda^{\eta-1}+\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\chi_0(p))dp. \tag{6}$$

Then

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\chi_0(p))dp\right|\\
&\quad\le\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}(\psi(p)-\psi(a))^{\eta-1}\times(\psi(p)-\psi(a))^{1-\eta}\lambda(p,\chi_0(p))dp\\
&\quad\le\frac{\aleph}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}(\psi(p)-\psi(a))^{k}dp\le\aleph\Lambda^{\delta+k}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}.
\end{aligned}$$

This implies that $\chi_1\in C^0(I)$, we get

$$\Lambda^{1-\eta}\chi_1(s)-\zeta_0=\Lambda^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s\psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\chi_0(p))dp.$$

Therefore

$$\left|\Lambda^{1-\eta}\chi_1(S)-\zeta_0\right| \leq \Lambda^{1-\eta}\aleph\Lambda^{\delta+k}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)} \leq \aleph\Lambda^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)} \leq \aleph l^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}.$$

Now by mathematical induction hypothesis, for $n=m$, suppose $\chi_m \in C^0(I)$ for all $s \in J$

$$\left|\Lambda^{1-\eta}\chi_m(t)-\zeta_0\right| \leq T.$$

We have

$$\chi_{m+1}(s) = \zeta_0\Lambda^{\eta-1} + \frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\chi_m(p))dp.$$

From the above discussion, we obtain $\chi_{m+1} \in C^0[I]$ for all $s \in J$,

$$\begin{aligned}\left|\Lambda^{1-\eta}\chi_{m+1}(s)-\zeta_0\right| &\leq \Lambda^{1-\eta}\aleph\Lambda^{\delta+k}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)} \leq \aleph\Lambda^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\\ &\leq \aleph l^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)} \leq T.\end{aligned}$$

Thus, the result is true for $n=m+1$. Hence by mathematical induction principle the result is true for all n. The proof is complete. □

Theorem 1. *Suppose that (H_1) and (H_2) hold. Consider the Picard's function χ_n, then the sequence $\{\Lambda^{1-\eta}\chi_n(s)\}$ is uniformly convergent on J.*

Proof. Consider the series:

$$\Lambda^{1-\eta}\chi_0(s)+\Lambda^{1-\eta}[\chi_1(s)-\chi_0(s)]+\cdots+\Lambda^{1-\eta}[\chi_n(s)-\chi_{n-1}(s)]+\cdots, \quad s \in J$$

$$\Lambda^{1-\eta}|\chi_1(s)-\chi_0(s)| \leq \aleph\Lambda^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}; \quad s \in J,$$

and

$$\begin{aligned}&\Lambda^{1-\eta}\left|\chi_1(s)-\chi_0(s)\right|\\ &\quad\leq \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\left|\lambda(s,\chi_1(s))-\lambda(p,\chi_0(p))\right|dp\\ &\quad\leq \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\times\left|\lambda(p,\Lambda^{1-\eta}\Lambda^{\eta-1}\chi_1(p))-\lambda(p,\Lambda^{1-\eta}\Lambda^{\eta-1}\chi_0(s))\right|dp\\ &\quad\leq \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\theta(\psi(p)-\psi(a))^k\times(\psi(p)-\psi(a))^{1-\eta}\left|\chi_1(p)-\chi_0(p)\right|dp\\ &\quad\leq \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\theta\aleph\Lambda^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}dp\\ &\quad\leq \theta\aleph\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}(\psi(p)-\psi(a))^{\delta+2k+1-\eta}dp\\ &\quad\leq \theta\aleph\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\frac{\Gamma(\delta+2k+2-\eta)}{\Gamma(2\delta+2k+2-\eta)}\Lambda^{2(\delta+k+1-\eta)}.\end{aligned}$$

Similarly, we get

$$\Lambda^{1-\eta}\left|\chi_2(s)-\chi_1(s)\right| \le \theta^2 \aleph \frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\frac{\Gamma(\delta+2k+(1-\eta)+1)}{\Gamma(2(\delta+k)+(1-\eta)+1)}\Lambda^{3(\delta+k+1-\eta)}.$$

Suppose $n=m$. Then, we obtain

$$\Lambda^{1-\eta}\left|\chi_{m+1}(s)-\chi_m(s)\right| \le \theta^m \aleph \Lambda^{(m+1)(\delta+k+1-\eta)}\Pi_{i=0}^{m}\frac{\Gamma[(i+1)k+i(\delta+1-\eta)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}.$$

Now

$$\begin{aligned}
&\Lambda^{1-\eta}\left|\chi_{m+1}(s)-\chi_m(s)\right| \\
&\le \Lambda^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\left|\lambda(p,\chi_{m+1}(p))-\lambda(p,\chi_m(p))\right|dp \\
&\le \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\left|\lambda(p,\Lambda^{1-\eta}\Lambda^{\eta-1}\chi_{m+1}(p))-\lambda(p,\Lambda^{1-\eta}\Lambda^{\eta-1}\chi_m(p))\right|dp \\
&\le \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(s)(\psi(s)-\psi(p))^{\delta-1}\theta[(\psi(p)-\psi(a))^{1-\eta}|\chi_{m+1}(p)-\chi_m(p)|]dp \\
&\le \theta^{m+1}\aleph\Lambda^{(m+2)(\delta+k+1-\eta)}\Pi_{i=0}^{m+1}\frac{\Gamma[(i+1)k+i(\delta+1-\eta)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}.
\end{aligned}$$

Thus the result is true for $n=m+1$. Hence, by mathematical induction, the result is true for all n.

Further

$$\sum_{n=1}^{\infty}\mathfrak{U}_{n+1} = \sum_{n=1}^{\infty}\aleph\theta^{n+2}l^{(n+3)(\delta+k+1-\eta)}\Pi_{i=0}^{n+2}\frac{\Gamma[(i+1)k+i(\delta+1-\gamma)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}. \tag{7}$$

The following ratio

$$\begin{aligned}
\frac{\mathfrak{U}_{n+1}}{\mathfrak{U}_n} &= \frac{\aleph\theta^{n+2}l^{(n+3)(\delta+k+1-\eta)}\Pi_{i=0}^{n+2}\frac{\Gamma[(i+1)k+i(\delta+1-\gamma)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}}{\aleph\theta^{n+1}l^{(n+2)(\delta+k+1-\eta)}\Pi_{i=0}^{n+1}\frac{\Gamma[(i+1)k+i(\delta+1-\gamma)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}} \\
&= \theta l^{\delta+k+1-\eta}\frac{\Gamma((n+2+1)k+(n+2)(\delta+1-\eta)+1)}{\Gamma((n+2+1)(\delta+k)+(n+2)(1-\eta)+1)},
\end{aligned}$$

in the light of Lemma 4, can be written as

$$\frac{\mathfrak{U}_{n+1}}{\mathfrak{U}_n} = \theta l^{\delta+k+1-\eta}\left\{\frac{\lim\limits_{m\to\infty}\frac{m^{(n+3)k+(n+2)(\delta+1-\eta)+1}m!}{[(n+3)k+(n+2)(\delta+1-\eta)+1][(n+3)k+(n+2)(\delta+1-\eta)+2]\cdots[(n+3)k+(n+2)(\delta+1-\eta)+m+1]}}{\lim\limits_{m\to\infty}\frac{m^{(n+3)(\delta+k)+(n+2)(1-\eta)+1}m!}{[(n+3)(\delta+k)+(n+2)(1-\eta)+1][(n+3)(\delta+k)+(n+2)(1-\eta)+2]\cdots[(n+3)(\delta+k)+(n+2)(1-\eta)+m+1]}}\right\}.$$

Clearly,

$$\frac{[(n+3)(\delta+k)+(n+2)(1-\eta)+1][(n+3)(\delta+k)+(n+2)(1-\eta)+2]\cdots[(n+3)(\delta+k)+(n+2)(1-\eta)+m+1]}{[(n+3)k+(n+2)(\delta+1-\eta)+1][(n+3)k+(n+2)(\delta+1-\eta)+2]\cdots[(n+3)k+(n+2)(\delta+1-\eta)+m+1]}$$

is bounded for all m,n. Thus, $\lim\limits_{n\to\infty}\frac{\mathfrak{U}_{n+1}}{\mathfrak{U}_n}=0$. This implies that $\sum_{n=1}^{\infty}\mathfrak{U}_n$ is convergent. Hence

$$\Lambda^{1-\eta}\chi_0(s)+\Lambda^{1-\eta}[\chi_1(s)-\chi_0(s)]+\cdots+\Lambda^{1-\eta}[\chi_n(s)-\chi_{n-1}(s)]+\cdots$$

is uniformly convergent for $s\in J$; i.e. the sequence $\{\Lambda^{1-\eta}\chi_n(s)\}$ is uniformly convergent on J. □

Theorem 2. *Suppose (H_1) and (H_2) hold. Then $\chi(s) = \Lambda^{\eta-1}\lim_{n\to\infty}\Lambda^{1-\eta}\chi_n(s)$ is unique continuous solution of integral equation (2) on J.*

Proof. Since
$$\chi(s) = \Lambda^{\eta-1}\lim_{n\to\infty}\Lambda^{1-\eta}\chi_n(s)$$
on J. We have
$$\begin{aligned}\Lambda^{1-\eta}\big|\chi(s)-\chi_0(s)\big| &\le T\\ \big|\lambda(s,\chi_n(s))-\lambda(s,\chi(s))\big| &\le \theta\Lambda^k\big|\chi(s)-\chi_0(s)\big|;\ s\in I.\end{aligned}$$
Clearly
$$\Lambda^{1-\eta}\big|\lambda(s,\chi_n(s))-\lambda(s,\chi(s))\big| \le \theta\big|\chi_n(s)-\chi(s)\big| \longrightarrow 0,\text{ uniformly as } n\longrightarrow\infty \text{ on } I.$$
Therefore
$$\begin{aligned}\Lambda^{1-\eta}\chi(s) &= \lim_{n\to\infty}\chi_n(s)\\ &= \zeta_0 + (\psi(s)-\psi(a))^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}(\psi(p)-\psi(a))^k\\ &\quad\times\lim_{n\to\infty}(\psi(p)-\psi(a))^{-k}\lambda(p,\chi_{n-1}(p))dp\\ &= \zeta_0 + (\psi(s)-\psi(a))^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\chi(p))dp.\end{aligned}$$
Hence $\chi(s)$ is continuous solution of integral equation (2) defined on J.

To prove the uniqueness, suppose that $\xi(s)$ is solution of integral equation (2), which implies for all $s\in I$, $\Lambda^{1-\eta}|\xi(s)|\le T$, and
$$\xi(s) = \zeta_0(\psi(s)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\xi(p))dp. \tag{8}$$
We prove, $\chi(s)=\xi(s)$ on I. From (H_1), $\exists\, k > (\rho(1-\delta)-1)$, $m\ge 0$, such that
$$\begin{aligned}\big|\lambda(s,\xi(s))\big| &= \big|\lambda(s,\Lambda^{\eta-1}\Lambda^{1-\eta}\xi(s))\big|\\ \big|\lambda(s,\xi(s))\big| &\le \aleph\Lambda^k\ ;\forall\, s\in I.\end{aligned}$$
Therefore
$$\begin{aligned}\Lambda^{1-\eta}\Big|\chi_0(s)-\xi(s)\Big| &= \Lambda^{1-\eta}\Big|\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\lambda(p,\xi(p))dp\Big|\\ &\le \Lambda^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\aleph(\psi(p)-\psi(a))^k dp\\ \Lambda^{1-\eta}\Big|\chi_1(s)-\zeta_0\Big| &\le \Lambda^{1-\eta}\aleph\Lambda^{\delta+k}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\\ &\le \aleph\Lambda^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\\ &\le \aleph l^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}.\end{aligned}$$

Moreover

$$\begin{aligned}
\Lambda^{1-\eta}\left|\chi_1(s)-\xi(s)\right| &\le \Lambda^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\left|\lambda(p,\chi_0(p))-\lambda(p,\xi(p))\right|dp\\
&\le \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\theta(\psi(p)-\psi(a))^k(\psi(p)-\psi(a))^{1-\eta}\left|\chi_0(p)-\xi(p)\right|dp\\
&\le \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\theta\aleph\Lambda^{\delta+k+1-\eta}\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}dp\\
&\le \theta\aleph\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}(\psi(p)-\psi(a))^{\delta+2k+1-\eta}dp\\
&\le \theta\aleph\frac{\Gamma(k+1)}{\Gamma(\delta+k+1)}\frac{\Gamma(\delta+2k+2-\eta)}{\Gamma(2\delta+2k+2-\eta)}\Lambda^{2(\delta+k+1-\eta)}.
\end{aligned}$$

Suppose

$$\Lambda^{1-\eta}\left|\chi_m(s)-\xi(s)\right| \le \theta^m\aleph\Lambda^{(m+1)(\delta+k+1-\eta)}\Pi_{i=0}^{m}\frac{\eta[(i+1)k+i(\delta+1-\eta)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]},$$

then

$$\begin{aligned}
\Lambda^{1-\eta}\left|\chi_{m+1}(s)-\xi(s)\right| &\le \Lambda^{1-\eta}\frac{1}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\left|\lambda(p,\chi_{m+1}(p))-\lambda(p,\xi(p))\right|dp\\
&\le \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}|\lambda(p,(\psi(p)-\psi(a))^{1-\eta}\Lambda^{\eta-1}\chi_{m+1}(p))\\
&\quad -\lambda(p,(\psi(p)-\psi(a))^{1-\eta}(\psi(p)-\psi(a))^{\eta-1}\xi(p))|dp\\
&\le \frac{\Lambda^{1-\eta}}{\Gamma(\delta)}\int_a^s \psi'(p)(\psi(s)-\psi(p))^{\delta-1}\theta[(\psi(p)-\psi(a))^{1-\eta}|\chi_{m+1}(p)-\xi(p)|]dp\\
&\le \theta^{m+1}\aleph\Lambda^{(m+2)(\delta+k+1-\eta)}\Pi_{i=0}^{m+1}\frac{\Gamma[(i+1)k+i(\delta+1-\eta)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}\\
&\le \theta^{m+1}\aleph\Lambda^{(m+2)(\delta+k+1-\eta)}\Pi_{i=0}^{m+1}\frac{\Gamma[(i+1)k+i(\delta+1-\eta)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]},
\end{aligned}$$

is convergent. Therefore

$$\theta^{m+1}\aleph\Lambda^{(m+2)(\delta+k+1-\eta)}\Pi_{i=0}^{m+1}\frac{\Gamma[(i+1)k+i(\delta+1-\eta)+1]}{\Gamma[(i+1)(\delta+k)+i(1-\eta)+1]}\longrightarrow 0, \text{as}\, n\longrightarrow\infty.$$

Also we observe that $\lim_{n\to\infty}\Lambda^{1-\eta}\psi_n(s)=\Lambda^{1-\eta}\xi(s)$ uniformly on J. Thus $\psi(s)=\xi(s)$ on I. □

Theorem 3. *Suppose* (H_1) *and* (H_2) *hold. Then IVP* (1) *has unique continuous solution*

$$\psi(s)=\Lambda^{\eta-1}\lim_{n\to\infty}\Lambda^{1-\eta}\psi_n(s).$$

Proof. Evidently if follows from Lemma 7 and Theorem 2. □

4 Illustrative examples

Example 1. For the choice of $\psi(s) = \log(s)$, $s \in (1,e)$, $\delta = \frac{2}{3}$, $\rho = \frac{3}{4}$, $\eta = \frac{17}{20}$,

$$\begin{cases} \mathfrak{D}_{a^+}^{\frac{2}{5},\frac{3}{4};\log(s)}\zeta(s) = \log(s)^{-\frac{1}{2}}(\sin(\zeta(s)) + \sqrt{2}s),\ s \in [1,e], \\ \lim\limits_{s\to a}(\log(s) - \log(a))^{\frac{3}{4}}\zeta(s) = 2. \end{cases} \tag{9}$$

In this application,

$$\begin{cases} \lambda(s,\zeta(s)) = \log(s)^{-\frac{1}{2}}(\sin(\zeta(s)) + \sqrt{2}s), \quad \text{for} \quad s \in [1,e], \quad \zeta \in \mathfrak{R} \\ \lambda(1,\zeta(1)) = 0, \quad \text{for} \quad \zeta \in \mathfrak{R}. \end{cases}$$

clearly, λ is singular at $s = 1$, and is continuous on $[1,e]$. We consider $\nu = \frac{11}{20}$ and $k = -\frac{1}{5} \geq -\frac{11}{20}$,

$$M = \max_{s\in[1,e],\lambda\in[-2,2]}(\sin\lambda(t) + \sqrt{2}e) \approx 3.8791$$

and

$$l = \min\left\{5, \left(\frac{2.7182818258}{3.8791305282} \times \frac{\Gamma(1.2)}{\Gamma(1.8)}\right)^4\right\} \approx 0.09328,$$

with

$$\begin{cases} \zeta_0(s) = 2\log(s)^{-\frac{3}{20}}, & \text{for } s \in [1,e] \\ \zeta_n(s) = \zeta_0(s) + \dfrac{1}{\Gamma(\frac{1}{2})}\displaystyle\int_0^s \psi'(p)(\psi(s) - \psi(a))^{-\frac{1}{3}}\lambda(p,\zeta_{n-1})(p)dp, & \text{for } n = 1,2,\cdots. \end{cases}$$

Hence, IVP (9) has a unique and continuous solution $\zeta(s) = \log(s)^{\frac{1}{2}}\lim\limits_{s\to\infty}\log(s)^{-\frac{1}{2}}\zeta_n(s)$ on $(0,1]$.

Example 2. For the choice of $\psi(s) = s$, $s \in (0,1)$, $\delta = \frac{2}{3}$, $\rho = \frac{1}{2}$, $\eta = \frac{1}{6}$, consider the following problem

$$\begin{cases} \mathfrak{D}_{a^+}^{\frac{2}{3},\frac{1}{2};s}\zeta(s) = s^{-\frac{1}{4}}(\zeta(s)^{\frac{1}{2}} + |\sin(s)|),\ s \in [0,1], \\ \lim\limits_{s\to a}(\psi(s) - \psi(a))^{-\frac{1}{4}}\zeta(s) = \Gamma(\frac{1}{2}) \approx 1.7725, \end{cases} \tag{10}$$

where

$$\begin{cases} \lambda(s,\zeta(s)) = (s)^{-\frac{1}{4}}(\zeta(s)^{\frac{1}{2}} + |\sin(s)|), \quad \text{for} \quad s \in [0,1], \quad \zeta \in \mathfrak{R}, \\ \lambda(0,\zeta(0)) = 0, \quad \text{for} \quad \zeta \in \mathfrak{R}. \end{cases}$$

Clearly λ is singular at $s = 0$, and is continuous on $[0,1]$. We consider $\nu = \frac{5}{6}$ and $k > -\frac{5}{6}$ which gives $M \approx 1.8268$, $l \approx 0.0395$ with

$$\begin{cases} \zeta_0(s) = \Gamma(\frac{1}{2})s^{\frac{1}{6}}, & \text{for } s \in (0,1) \\ \zeta_n(s) = \zeta_0(s) + \frac{1}{\Gamma(\frac{1}{2})}\int_0^s \psi'(p)(\psi(s) - \psi(a))^{-\frac{1}{4}}\lambda(p,\zeta_{n-1})(p)dp, & \text{for } n = 1,2,\ldots. \end{cases}$$

Hence, IVP (10) has a unique continuous solution $\zeta(s) = s^{-\frac{1}{4}}\lim\limits_{s\to\infty}s^{\frac{1}{4}}\zeta_n(s)$ on $(0,1]$.

Remark.

(1) Taking $\rho \to 1$, the nonsingular differential problem (1) *becomes $\psi-$Caputo fractional differential problem.*

(2) Taking $\rho \to 0$, the nonsingular differential problem (1) *becomes $\psi-$Reimann-Liouville fractional differential problem.*

(3) For $\rho \to 1$ and $\psi(s) = s$, the nonsingular differential problem (1) *becomes Caputo fractional differential problem.*

(4) For $\rho \to 0$ and $\psi(s) = s^p$, $p \geq 1$ the nonsingular differential problem (1) *becomes Katugompola fractional differential problem.*

(5) For $\rho \to 0$ and $\psi(s) = s$ the nonsingular differential problem (1) *reduces to Riemann-Liouville fractional differential problem.*

(6) For $\rho \to 0$ and $\psi(s) = \log(s)$, the nonsingular differential problem (1) *becomes Hadmard fractional differential problem.*

(7) Taking $\rho \to 1$ and $\psi(s) = s$, the nonsingular differential problem (1) *becomes Caputo-Hadamard fractional differential problem.*

5 Concluding remarks

This study focused on establishing the local existence and uniqueness of solutions to the ψ-Hilfer fractional differential problem. By employing Picard's approximations, a computable iterative scheme is developed to uniformly approximate the solution. The validity of the findings is supported by two illustrative examples. This work contributes to a deeper understanding of the singular ψ-Hilfer fractional differential problem and offers a more generalized computational approach for approximating solutions, extending the existing contributions of various researchers.

References

[1] M.S. Abdo, S.K. Panchal, *Fractional integro-differential equations involving $\psi-$ Hilfer fractional derivative,* Adv. Appl. Math. Mech. **11** (2019) 338–359.

[2] M.A. Almalahi, M.S. Abdo, S.K. Panchal *ψ-Hilfer fractional functional differential equation by Picard operator method,* J. Appl. Nonlinear Dynam. **9** (2020) 685–702.

[3] R. Almeida, *A Caputo fractional derivative of a function with respect to another function,* Comm. Nonlinear Sci. Numer. Sinulat., **44** (2017) 460–481.

[4] T.M. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica, *Fractional calculus with applications in mechanics: Vibrations and diffusion processes,* Wiley-ISTE, London, Hoboken, 2014.

[5] S.P. Bhairat, *Existence and continuation of solutions of Hilfer fractional differential equations,* J. Math. Model. **7** (2019) 1–20.

[6] S.P. Bhairat, *On local existence of solution for nonlinear Hilfer fractional differential equation,* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **29** (2022) 363–374.

[7] S.P. Bhairat, *New approach to existence of solution for weighted Cauchy-type problem,* J. Math. Model. **8** (2020) 377–391.

[8] D. Delbosco, L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation,* J. Math. Anal. Appl. **204** (1996) 609–625.

[9] K. Diethelm, *The Analysis of Fractional Differential Equations,* Lecture Notes in Mathematics, Springer–Verlag Berlin Heidelberg, 2010.

[10] D.B. Dhaigude, S.P. Bhairat, *Existence and uniqueness of solution of Cauchy-type problem for Hilfer fractional differential equations,* Comnun. Appl. Anal. **22** (2018) 121–134.

[11] D.B. Dhaigude, S.P. Bhairat, *Local existence and uniqueness of solution of Hilfer fractional differential equations,* Nonlinear Dyn. Syst. Theory. **18** (2018) 144–153.

[12] K.M. Furati, M.D. Kassim,*Existence and uniqueness for a problem involving Hilfer fractional derivative,* Computer Math. Appl. **64** (2012) 1616–1626.

[13] R. Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific, Singapore, 2000.

[14] A.A. Kilbas, H.M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Math Stud, 204 Elsevier, Amsterdam, 2006.

[15] K.D. Kucche, A.D. Mali, J. Vanterler da C. Sousa, *On the nonlinear ψ-Hilfer fractional differential equations,* Comput. Appl. Math., **38** (2019) 73.

[16] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity,* an Introduction to Mathematical Model, Imperical College Press, World Scientific Publishing, London, 2010.

[17] F. Norouzi, G.M. N'Guerekata *A study of $\psi-$Hilfer fractional differential system with application in financial crisis,* Chaos Solitons Fractals **6** (2021) 100056.

[18] I. Podlubny, *Fractional Differential Equations,* Academic Press, 1998.

[19] A. Salim, M. Benchohra, J.R. Graef, J.E. Lazreg, *Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations,* J. Fixed Point Theory Appl. **24** (2022) 1–14.

[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives - Theory and Applications,* Gordon and Breach, 1993.

[21] C. Sousa, E.C. de Oliveira, *Leibniz type rule: ψ-Hilfer fractional operator,* Comnun. Nonlinear Sci. Numer. Sinulat. 77 (2019) 305–311.

[22] H.G. Sun, Y. Zhang, D. Baleanu, W. Chena, Y.Q. Chene, *A new collection of real world applications of fractional calculus in science and engineering,* Comnun. Nonlinear Sci. Numer. Sinulat. **64** (2018) 213–231.

[23] R. Toledo-Henrnandez, V. Rico-Ramirez, G.A. Iglesias-Silva, U.M. Diwekar, *A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: Fractional models for biological reactions,* Chem. Eng. Sci., **117** (2014) 217–228.

[24] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *On the ψHilfer fractional derivative,* Comnun. Nonlinear Sci. Numer. Sinulat. **60** (2018) 72–91.

[25] J. Vanterler da C. Sousa, E. Capelas Oliveira, *A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator,* Diff. Equ. Appl. **11** (2019), 87–106.

[26] N. Vieira, M.M. Rodregues, M. Ferriera, *Fractional gradient methods via $\psi-$Hilfer derivative,* Fractal Fract. **7** (2023) 275.

[27] X. Yang and Y. Liu, *Picard iterative processes for initial value problems of singular fractional differential equations,* Adv. Differ. Equat. **2014** (2014) 102.