Journal of Mathematical Modeling
Vol. 11, No. 3, 2023, pp. 425–439. Research Article

JMM

A new approach for solving constrained matrix games with fuzzy constraints and fuzzy payoffs

Sabiha Djebara[†], **Farida Achemine**[‡∗], **Ouiza Zerdani**[†]

[†] *Laboratoire de Recherche Opérationnelle et de Mathématiques de la Décision, Faculté des sciences,Université Mouloud Mammeri de Tizi Ouzou, 15000 Tizi-Ouzou, Algeria*
[‡] *Laboratoire de Mathématiques Pures et Appliquées, Faculté des sciences, Université Mouloud Mammeri de Tizi Ouzou, 15000 Tizi-Ouzou, Algeria*
Email(s): sabiha.djebara@ummto.dz, farida.achemine@ummto.dz, ouizabouarab@gmail.com

Abstract. The main purpose of this study is to construct a new approach for solving a constrained matrix game where the payoffs and the constraints are LR-fuzzy numbers. The method that we propose here is based on chance constraints and on the concept of a comparison of fuzzy numbers. First, we formulate the fuzzy constraints of each player as chance constraints with respect to the possibility measure. According to a ranking function $\mathscr{R}$, a crisp constrained matrix game is obtained. Then, we introduce the concept of $\mathscr{R}$-saddle point equilibrium. Using results on ordering fuzzy numbers, sufficient existence conditions of this concept are provided. The problem of computing this solution is reduced to a pair of primal-dual linear programs. To illustrate the proposed method, an example of the market competition game is given.

Keywords: Chance-constraints, constrained matrix games, fuzzy games, linear programming, saddle point equilibrium.
AMS Subject Classification 2010: 91A10, 91A05

1 Introduction

Game theory is a branch of applied mathematics that provides tools for modelling and analyzing different types of competitive situations. It has several applications in all fields of social sciences, engineering, biological science, and political sciences. The present literature on game theory applications is very large (see for example, Refs. [2–4, 17, 34]).

The simplest model in game theory is a two player finite strategic zero-sum game (matrix game). It is a mathematical representation of a competitive situation which involves two players, in which the players

[∗]Corresponding author

Received: 9 November 2022 / Revised: 10 March 2023 / Accepted: 4 May 2023
DOI: 10.22124/jmm.2023.23207.2072

have a finite number of alternatives to choose from and one player's loss is the other's gain. From the seminal paper of [45], it is known that any matrix game has at least one saddle point in mixed strategies. Dantzing [10] showed that a saddle point equilibrium of a matrix game can be obtained from the optimal solution of a primal-dual pair of linear programs. However, in many matrix games coming from real life situation, the two players often lack full information about some aspects of the problem. To tackle this uncertainty, fuzzy set theory of Zadeh [47] offers powerful techniques and methods. The research on fuzzy matrix games was initiated by Campos [8] and several results have already been obtained in this direction of research. Just to mention a few recent works on some families of fuzzy matrix games, the reader can consult the references [11, 18, 21, 31, 34, 35, 37–41, 44].

Another direction of research related to the study of matrix games is the investigation of constrained matrix games. Indeed, in some real matrix games the choice of strategies is subject to linear inequality constraints rather than being in whole strategy space. Such a matrix game is called a constrained matrix game initially introduced by Charnes [9] and then later in some what more generally by Kawaguchi and Maruyama [20]. Later, Dresher [12] gave a real example of the constrained matrix games. Owen [29] showed that any classical constrained matrix game always has optimal strategies and game value of both players are obtained by solving a pair of primal-dual linear programming problems.

In recent years, many research articles examined fuzzy constrained matrix games. Li and Cheng [22] underlined that there is no method in the literature solving constrained matrix games in which payoffs are represented by fuzzy numbers and studied fuzzy constrained matrix games using multi-objective programming methods. Later, Li and Hong [24], proposed a method to solve constrained matrix games with payoffs of triangular fuzzy numbers. In Li and Hong [23], an α-cut linear programming algorithm for solving fuzzy constrained matrix games is considered. Nan and Li [27] established an effective linear programming technique for solving constrained matrix games with interval payoffs. Mansoori et al. [26] investigated the constrained matrix game with fuzzy payoffs and fuzzy constraints, using the concepts of recurrent neural networks. A literature review of constrained matrix games with fuzzy payoffs is given in the book of Verma and Kumar [44]. The authors discussed several existing methods and showed that some mathematically incorrect assumptions have been considered in these methods. Then, they proposed a novel method for solving fuzzy constrained matrix games. The book offers a comprehensive guide on non-cooperative games with fuzzy payoffs to both students and researchers. In Verma [43], a new method is proposed to find the complete solution of constrained matrix games with fuzzy payoffs.

In the present work, we focus on a constrained matrix game in which the payoffs and the constraint matrices are *LR*-fuzzy numbers. The *LR*-fuzzy number initialized by Dubois and Prade [13], is a commonly used type of fuzzy numbers in various problems over the past few decades (see for example, Refs. [1, 7, 15, 30, 50]). However, to the best of our knowledge, there is no study on constrained matrix games with both payoffs and constraints are *LR*-fuzzy numbers. In this study, we use the chance-constrained method which is one of the major approaches for solving optimization problems under various uncertainties. The proposed approach is based on chance constraints and on the concept of a comparison of fuzzy numbers. First, we formulate the fuzzy constraints of each player as chance constraints with respect to the possibility measure. Second, using ranking function $\mathscr{R}$, a crisp constrained matrix game is obtained. Then, we introduce the concept of $\mathscr{R}$-saddle point equilibrium. Using results on ordering fuzzy numbers, sufficient existence conditions of this concept are provided. The problem of computing this solution is reduced to a pair of primal-dual linear programs. Finally, the proposed method is illustrated with the example of the market share problem.

The remainder of the paper is organized as follows. Section 2 reviews some notations and definitions about constrained matrix games and fuzzy sets theory. In Section 3, we present the fuzzy constrained matrix game and we define the concept of $\mathscr{R}$-saddle point equilibrium. Then, the existence of this concept is proved. In Section 4, we explain the proposed methodology for solving the considered constrained matrix game. An illustrative example is given in Section 5. Conclusion is made in Section 6.

2 Preliminaries

In this section, the basic definitions, involving matrix game, constrained matrix game, fuzzy sets and fuzzy numbers, arithmetic on fuzzy numbers and ranking function are reviewed.

2.1 Constrained matrix games

In this sub-section, we recall some basic definitions and preliminaries used throughout our paper, namely two person zero-sum matrix games and two person zero-sum constrained matrix games. For various notations, terminology and basics related to this subject, the reader is referred to the work of Bector and Chandra [5], and Verma and Kumar [44].

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ be an $(m \times n)$ real matrix and $e^T = (1,1,\dots,1)$ be a vector of ones whose dimension is specified as per the specific context. By a (crisp) two person zero-sum matrix game G we mean the triplet $G = (S^m,\ S^n,\ A)$ where $S^m = \{x \in \mathbb{R}^m_+,\ e^T x = 1\}$ and $S^n = \{y \in \mathbb{R}^n_+,\ e^T y = 1\}$. In the terminology of the matrix game theory, S^m (respectively S^n) is called the strategy space for player I (respectively player II) and A is called the payoff matrix. Then, the elements of S^m (respectively S^n) which is in the form $x = (0,0,\dots,1,\dots,0)^T = e_i$, where 1 is at the i^{th} place (respectively $y = (0,0,\dots,1,\dots,0)^T = e_j$, where 1 is at the j^{th} place) are called pure strategies for player I (respectively player II). If player I chooses i^{th} pure strategy and player II chooses j^{th} pure strategy then a_{ij} is the amount paid by player II to player I. If the game is zero-sum then a_{ij} is the amount paid by player I to player II i.e. the gain of one player is the loss of other player.

The quantity $E(x,y) = x^T Ay$ is called the expected payoff of player I by player II, as elements of S^m (respectively S^n) are set of all probability distribution over $\mathbb{I} = \{1,2,\dots,\ m\}$ (respectively $\mathbb{J} = \{1,2,\dots,n\}$). Also, it is a convention to assume that player I is a maximizing player and player II is a minimizing player. As player I is a maximizing player and player II is a minimizing player, the expected payoff for player I is the expected loss for player II. The triplet $PG = (\mathbb{I},\mathbb{J},A)$ is called the pure form of the game G, whenever G is the mixed extension of the pure game PG.

In the two person finite zero-sum game (matrix game) denoted by the triplet $G = (S^m,S^n,A)$, S^m refers to the (mixed) strategy space of player I, S^n refers to the (mixed) strategy space of player II, and A refers to the payoff matrix which introduces the function $E : S^m \times S^n \to \mathbb{R}$ given by

$$E(x,\ y) = x^T Ay = \sum_{j=1}^{n} (\sum_{i=1}^{m} x_i a_{ij}) y_j,$$

called the expected payoff function.

Definition 1. *([28]). (Solution of the game G) Assume that there exist mixed strategies $x^* \in S^m$ and $y^* \in S^n$ such that*

$$x^{*T} Ay^* = \max_x \min_y (x^T Ay) = \min_y \max_x (x^T Ay).$$

Then, $(x^*,\ y^*)$ *and* $v^* = {x^*}^T A y^*$ *are called a saddle point and a value of the matrix game G, respectively.*

An equivalence between two person zero-sum matrix game $G = (S^m, S^n, A)$ and a pair of primal-dual linear programming problems is established in [28]. This equivalence besides being interesting mathematically, offers a very useful way to solve the given game G.

Theorem 1 ([28])**.** *Every two person zero-sum matrix game* $G = (S^m, S^n, A)$ *has a solution.*

Theorem 2 ([28])**.** *The triplet* $(x^*, y^*, v^*) \in S_m \times S_n \times \mathbb{R}$ *is a solution of the game G if and only if* x^* *is optimal for (LP),* y^* *is optimal for (LD) and* v^* *is the common value of (LP) and its dual (LD):*

$$(LP)\begin{cases}\max\ v\\ \text{subject to}\\ \sum_{i=1}^{m} a_{ij}x_i \geq v,\\ x \in S^m,\end{cases} \qquad (LD)\begin{cases}\min\ w\\ \text{subject to}\\ \sum_{j=1}^{m} a_{ij}y_j \leq w,\\ y \in S^n.\end{cases}$$

In some real-world matrix game problems, player strategies are subject to meet linear inequalities rather than being in S^m or S^n. These situations lead to constrained matrix games initially studied by Charnes [9].

Mathematically, a constrained matrix game with crisp payoffs is obtained by replacing the constraint set of strategies $S^m = \{x \in \mathbb{R}^m_+,\ e^T x = 1\}$ of player I with $S_1 = \{x \in \mathbb{R}^m_+,\ e^T x = 1,\ \ Bx \leq b\}$ and $S^n = \{y \in \mathbb{R}^n_+,\ e^T y = 1\}$ of player II with $S_2 = \{y \in \mathbb{R}^n_+,\ e^T y = 1,\ \ Dy \geq d\}$, where $B \in \mathbb{R}^{p\times m}$, $b \in \mathbb{R}^p$, $d \in \mathbb{R}^q$, $D \in \mathbb{R}^{q\times n}$, and p and q are a positive integers. The constrained matrix game CG is denoted $CG = (S_1,\ S_2,\ A)$.

Definition 2. *[29] Assume that there exist mixed strategies* $x^* \in S_1$ *and* $y^* \in S_2$ *so that*

$${x^*}^T A y^* = \max_{x\in S_1}\min_{y\in S_2}(x^T Ay) = \min_{y\in S_2}\max_{x\in S_1}(x^T Ay).$$

Then, (x^*, y^*) *and* ${x^*}^T A y^*$ *are called a saddle point and a value of the constrained matrix game CG, respectively.*

The constraint $\sum_{i\in I} x_i = 1$ is equivalent to both $\sum_{i\in I} x_i \leq 1$ and $-\sum_{i\in I} x_i \leq -1$.
Similarly, the constraint $\sum_{j\in J} y_i = 1$ is equivalent to both $\sum_{j\in J} y_i \leq 1$ and $-\sum_{j\in J} y_j \leq -1$.
Let us define the following notations as in [29]:

- $e_1^T = (1,\ldots,1)$ and $e_2^T = (-1,\ldots,-1)$ as m vectors; $c^T = (b^T, 1, -1) \in \mathbb{R}^{1\times(p+2)}$ and $H^T = (B^T, e_1, e_2) \in \mathbb{R}^{m\times(p+2)}$.
- $e_3^T = (1,\ldots,1)$ and $e_4^T = (-1,\ldots,-1)$ as n vectors; $u^T = (d^T, 1, -1) \in \mathbb{R}^{1\times(q+2)}$ and $E^T = (D^T, e_3, e_4) \in \mathbb{R}^{n\times(q+2)}$.

Using these notations, the strategic sets of players I and II, are given, by $S_1 = \{x \in \mathbb{R}^m,\ Hx \leq c,\ x \geq 0\}$ and $S_2 = \{y \in \mathbb{R}^n,\ Ey \geq u,\ y \geq 0\}$, respectively.

The main theorem of the constrained matrix game theory asserts that every constrained matrix game CG is equivalent to two linear programming problems $(C-LP)$ and $(C-LD)$ which are dual to each other [9, 29].

Theorem 3. *An element $(x^*,\ y^*) \in S_1 \times S_2$ is a solution of the constrained game $CG = (S_1,\ S_2,\ A)$ if and only if there exist $z^* \in \mathbb{R}^{q+2}$, $s^* \in \mathbb{R}^{p+2}$ such that $(x^*,\ z^*)$ and $(s^*,\ y^*)$ are optimal for the mutually dual pair of linear programming problems $(C-LP)$ and $(C-LD)$:*

$$(C-LP)\begin{cases}\max\ u^T z\\ \text{subject to}\\ E^T z - A^T x \le 0,\\ Hx \le c,\\ z \ge 0,\\ x \ge 0,\end{cases} \qquad (C-LD)\begin{cases}\min c^T s\\ \text{subject to}\\ H^T s - Ay \ge 0,\\ Ey \ge u,\\ s \ge 0,\\ y \ge 0.\end{cases} \tag{1}$$

2.2 *LR*-fuzzy numbers and ranking criteria

Here, we include some properties and concepts of fuzzy set theory, which are applied in the following sections.

Definition 3. *([47]) Let U be a universe set. A fuzzy set $\widetilde{A}$ in U is a set of ordered pairs, $\widetilde{A} = \{(x,\ \mu_{\widetilde{A}}(x)) \mid x \in U\}$, where the function $\mu_{\widetilde{A}}\ :\ U \mapsto [0,\ 1]$ is called membership function, which assigns to each elements $x \in U$ a real number $\mu_{\widetilde{A}}(x)$ in the interval $[0,\ 1]$. The value $\mu_{\widetilde{A}}(x)$ represents the degree of membership of x in $\widetilde{A}$. Thus, the closer the value $\mu_{\widetilde{A}}(x)$ to unity, the higher the degree of membership of x in $\widetilde{A}$.*

Definition 4. *([14]). A fuzzy number $\widetilde{a}$ is said to be a LR-fuzzy number if*

$$\mu_{\widetilde{a}}(t) = \begin{cases} L(\frac{a-t}{\alpha}), & \text{if } t \le a,\\ R(\frac{t-a}{\beta}), & \text{if } t \ge a,\end{cases}$$

where a is the mean value of $\widetilde{a}$, α and β (non negative) are its left and right spreads, respectively, L and R are the left and right shape functions, respectively, defined as follows $L,\ R\ :\ [0,\ 1] \longrightarrow [0,\ 1]$ with $L(1) = R(1) = 0$ and $L(0) = R(0) = 1$ are decreasing and continuous. Using its mean value, left and right spreads, and shape functions, a LR-fuzzy number $\widetilde{a}$ is symbolically written as $\widetilde{a} = (a,\ \alpha,\ \beta)_{LR}$. In addition, we assume that a LR-fuzzy number degenerates to a real number as $\alpha = \beta = 0$, i.e., $(a,\ 0,\ 0)_{LR} = a$.

Definition 5. *([13]). The α-cut, $\alpha \in [0,\ 1]$, of a LR-fuzzy number $\widetilde{a} = (a,\ \alpha_1,\ \beta_1)$ is a closed interval defined by $a_\alpha = \{x|\mu_A(x) \ge \alpha\} = [a^L_\alpha, a^R_\alpha]$, where a^L_α and a^R_α are the left and right endpoints of a_α, respectively. For an LR-fuzzy number with invertible and non-increasing functions L and R, the α-cut is characterized by $[a^L_\alpha, a^R_\alpha] = [a - \alpha_1 L^{-1}(\alpha),\ a + \beta_1 R^{-1}(\alpha)]$.*

Definition 6. *([13]). (LR-Fuzzy numbers arithmetics) Let $\widetilde{a} = (a_1,\ \alpha_1,\ \beta_1)$ and $\widetilde{b} = (a_2, \alpha_2, \beta_2)$ be two fuzzy numbers of LR type. Then, the addition, the subtraction and the scalar multiplication, are respectively defined as $\widetilde{a} + \widetilde{b} = (a_1 + a_2,\ \alpha_1 + \alpha_2,\ \beta_1 + \beta_2)$, $\widetilde{a} - \widetilde{b} = (a_1 - a_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2)$, and*

$$\begin{cases} k.\widetilde{a} = (ka_1,\ k\alpha_1,\ k\beta_1), & \text{if } k \ge 0,\ k \in \mathbb{R},\\ k.\widetilde{a} = (ka_1, -k\beta_1, -k\alpha_1), & \text{if } k < 0,\ k \in \mathbb{R}.\end{cases}$$

Definition 7. *([48]). Given a universe set U, let $\mathscr{P}(U)$ be the family of crisp subsets of U and the set function $Pos\ :\ \mathscr{P}(U) \longrightarrow [0,\ 1]$. The triplet $(U,\ \mathscr{P}(U),\ Pos)$ is called a possibility space, where Pos satisfies $Pos(U) = 1$, $Pos(\emptyset) = 0$ and $Pos(\cup_i E_i) = \sup_i Pos(E_i)$ for any collection (E_i) in $\mathscr{P}(U)$.*

Remark 1.

1. *For each $E \in \mathscr{P}(U)$, $Pos(E)$ is the measure of the best case of occurrence of the event E, i.e it is the maximal chance that the event E occurs. It expresses the decision maker's optimism about E.*
2. *$Nec(E) = 1 - Pos(E)$ gives the measure of the worst case of occurrence of the event E, i.e., it is the minimal chance that E occurs. It expresses the decision maker's pessimism about E.*

We recall the following lemma for ordering fuzzy numbers due to Sakawa [32].

Lemma 1. *([32]). Let $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ be two fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$:*

$$Pos\{\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2\} \geq \alpha \ \text{ if and only if } \ \lambda^R_{1,\alpha} \geq \lambda^L_{2,\alpha},$$

$$Nec\{\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2\} \geq \alpha \ \text{ if and only if } \ \lambda^L_{1,1-\alpha} \geq \lambda^R_{2,\alpha},$$

where $\lambda^R_{1,\alpha}, \lambda^L_{1,\alpha}$ and $\lambda^R_{2,\alpha}, \lambda^L_{2,\alpha}$ are the left and the right side endpoints of the α-level sets (α-cuts) $[\lambda^R_{1,\alpha}, \lambda^L_{1,\alpha}]$ and $[\lambda^R_{2,\alpha}, \lambda^L_{2,\alpha}]$, of $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$, respectively.

An efficient approach for comparing fuzzy numbers is by use of ranking functions. We consider a ranking function $\mathscr{R} : F(\mathbb{R}) \mapsto \mathbb{R}$, which maps each fuzzy number into real line, where a natural order exists [6, 16, 25, 46, 49].

Definition 8. *Let $\widetilde{a}$ and $\widetilde{b}$ be two fuzzy numbers, we define orders on $F(\mathbb{R})$ by*

$$\widetilde{a} \underset{\mathscr{R}}{\geqslant} \widetilde{b} \ \text{ if and only if } \ \mathscr{R}(\widetilde{a}) \geqslant \mathscr{R}(\widetilde{b}),$$

$$\widetilde{a} \underset{\mathscr{R}}{>} \widetilde{b} \ \text{ if and only if } \ \mathscr{R}(\widetilde{a}) > \mathscr{R}(\widetilde{b}),$$

$$\widetilde{a} \underset{\mathscr{R}}{=} \widetilde{b} \ \text{ if and only if } \ \mathscr{R}(\widetilde{a}) = \mathscr{R}(\widetilde{b}),$$

$$\widetilde{a} \underset{\mathscr{R}}{\leqslant} \widetilde{b} \quad \text{ if and only if } \quad \widetilde{b} \underset{\mathscr{R}}{\geqslant} \widetilde{a}.$$

There are several ranking functions in the literature [6, 16, 25, 42, 46]. For example, we recall the following ranking functions

1. Malekis ranking function given in [25]:

$$\mathscr{R}(\tilde{a}) = \int_0^1 (\inf_{x \in \tilde{a}_r} + \sup_{x \in \tilde{a}_r}) dr,$$

where $\inf_{x\in\tilde{a}}$ and $\sup_{x\in\tilde{a}}$ are the lower and upper bounds of any r-level set $\tilde{a}$ of $\tilde{a}$ and both are represented by finite numbers. According to the Definition 4 and Definition 5, an easy computation gives the expression of $\mathscr{R}(\tilde{a}) = 2a + \frac{1}{2}(\beta - \alpha)$ for $\tilde{a} = (a, \alpha, \beta)_{LR}$ and $L(t) = R(t) = \max(0, 1-t)$.

2. Hosseinzadeh's ranking function given in [16] :

$$\mathscr{R}(\tilde{a}) = a + \frac{1}{4}(\beta - \alpha), \text{ for } \tilde{a} = (a, \alpha, \beta)_{LR}.$$

Let us consider the linearity property restricted to positive scalars which is essential to establish our main results:

(P_1) $\mathscr{R}(\gamma\tilde{a} + \tilde{b}) = \gamma\mathscr{R}(\widetilde{a}) + \mathscr{R}(\widetilde{b})$ for any LR-Fuzzy numbers $\widetilde{a}, \widetilde{b} \in F(\mathbb{R})$ and any $\gamma \in [0, +\infty[$.

Remark 2. *It is easy to remark that:*

- *Maleki's ranking function* $\mathscr{R}(\tilde{a}) = 2a + \frac{1}{2}(\beta - \alpha)$ *for* $\tilde{a} = (a, \alpha, \beta)_{LR}$ *and* $L(t) = R(t) = \max(0, 1-t)$ *satisfies property* (P_1).
- *Hosseinzadeh's ranking function* $\mathscr{R}(\tilde{a}) = a + \frac{1}{4}(\beta - \alpha)$ *for* $\tilde{a} = (a, \alpha, \beta)_{LR}$ *satisfies property* (P_1).

3 Constrained matrix games with fuzzy payoffs and fuzzy linear constraints

In this section, we formally introduce the constrained matrix game with payoffs and linear constraints expressed with LR-fuzzy numbers. To difuzzify this game, we proceed as follows. Using the possibility measure, we formulate the fuzzy constraints of each player as chance constraints. In our methodology for solving this game, we propose the $\mathscr{R}$-saddle point equilibrium solution which is based on a ranking function. Then, we establish sufficient existence conditions of the solution. To this end, we use Lemma 1 for ordering fuzzy numbers due to Sakawa [32] and the ranking function $\mathscr{R}$ (Definition 8).

3.1 Problem Description and its Solution

A constrained matrix game in which the payoffs and the constraints matrices are LR-fuzzy numbers is given as follows. Let $\tilde{A} = [\tilde{a}_{ij}]$ denote the fuzzy matrix payoff of player I (the payoff matrix of player II is $-\tilde{A}$), and $\tilde{B} = (\tilde{b}_{ki})_{p\times m}$ and $\tilde{D} = (\tilde{d}_{lj})_{q\times n}$ denote the fuzzy matrices which define the constraints of player I and player II, respectively.

Thus, $\tilde{S}_1 = \{x \in X,\ \tilde{B} \preceq b\}$ and $\tilde{S}_2 = \{y \in Y,\ \tilde{D}y \succeq d\}$ represent the fuzzy constraint sets of strategies for players I and II, respectively, where $d \in \mathbb{R}^q$, $b \in \mathbb{R}^p$, p and q are a positive integers. Here $\preceq$ and $\succeq$ are fuzzy versions of symbols $\geq$ and $\leq$ respectively (see Zimmermann [51]). We denote this game by $G(\tilde{A}) = (\tilde{S}_1,\ \tilde{S}_2)$.

Each player is interested in maximizing her/his payoff. Since the fuzzy constraints $\tilde{B}x \preceq b$ and $\tilde{D}y \succeq d$ do not define a crisp feasible set, we assume that the constraints will hold with a possibility levels. Let $\delta^1 = (\delta^1_k)^p_{k=1} \in [0,\ 1]^p$ and $\delta^2 = (\delta^2_k)^q_{l=1} \in [0,\ 1]^q$, where $\delta^1_k \in [0,\ 1]$ is the possibility level for k^{th} constraint of player I, and $\delta^2_l \in [0, 1]$ is the possibility level for l^{th} constraint of player II. Thus, the fuzzy constraints of each player are replaced with the individual chance constraints. Hence, we obtain the player's strategies sets by:

$$S_1(\delta^1) = \{x \in \mathbb{R}^m \mid x \in X,\ Pos\{\tilde{B}_k x \preceq b_k\} \geq \delta_k^1,\ \forall k \in \mathscr{J}_1\},$$

$$S_2(\delta^2) = \{y \in \mathbb{R}^n \mid y \in Y,\ Pos\{\tilde{D}_l y \geq d_l\} \succeq \delta_l^2,\ \forall l \in \mathscr{J}_2\},$$

with $\mathscr{J}_1 = \{1,2,\ldots,p\}$ and $\mathscr{J}_2 = \{1,2,\ldots,q\}$; $\tilde{B}_k = (\tilde{B}_{k1}, \tilde{B}_{k2}, \ldots, \tilde{B}_{km})$, $k \in \mathscr{J}_1$ is a k^{th} row of the fuzzy matrix $\tilde{B}$; $\tilde{D}_l = (\tilde{D}_{l1}, \tilde{D}_{l2}, \ldots, \tilde{D}_{ln})$, $l \in \mathscr{J}_2$ is a l^{th} row of the fuzzy matrix $\tilde{D}$.

We denote the above zero-sum game with fuzzy payoffs and individual chance constraints by $G(\delta, \tilde{A})$.

The game $G(\delta, \tilde{A})$ is a constrained matrix game with fuzzy payoffs. When each of the players chooses a strategy, a gain for each one of them is represented as a fuzzy number. The outcome of the game has a zero-sum structure. So, one player's loss in a transaction is equivalent to another player's gain (see Sakawa [33]).

Definition 9. *(Fuzzy Expected Payoff) For any pair of the mixed strategies* $(x,\ y) \in S_1(\delta^1) \times S_2(\delta^2)$, *fuzzy expected payoff of player I is defined as the fuzzy number* $\tilde{E}(x,\ y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j \tilde{a}_{ij}$.

Definition 10. *($\mathscr{R}$-saddle point equilibrium) A strategy profile $(x^*,\ y^*)$ is a $\mathscr{R}$-saddle point equilibrium at $(\delta^1,\ \delta^2)$ possibility levels in the game $G(\delta, \tilde{A})$ if and only if*

$$\tilde{E}(x^*,\ y) \underset{\mathscr{R}}{\geqslant} \tilde{E}(x^*,\ y^*) \underset{\mathscr{R}}{\geqslant} \tilde{E}(x,\ y^*),\ \ \forall (x,\ y) \in S_1(\delta^1) \times S_2(\delta^2),$$

where $\mathscr{R}$ is a given ranking function (Definition 8).

3.2 Theoretical results

In Lemma 2, the strategy sets of the players under some conditions on the constraint matrices in $G(\delta, \tilde{A})$ are given.

Lemma 2. *Assume that* $\widetilde{B}_{ki} = (B_{ki},\ \alpha_{ki}^{(1)},\ \beta_{ki}^{(1)})$ *and* $\widetilde{D}_{lj} = (D_{lj},\ \alpha_{lj}^{(2)},\ \beta_{lj}^{(2)})$, $i \in \mathbb{I}$, $j \in \mathbb{J}$, $k \in \mathscr{J}_1$ *and* $l \in \mathscr{J}_2$, *are LR-fuzzy numbers. Then, for all* $\delta = (\delta^1, \delta^2) \in [0,1]^p \times [0,\ 1]^q$, *we have*

$$S_1(\delta^1) = \{x \in \mathbb{R}^m | x \in X,\ [B_k - L^{-1}(\delta_k^1)\alpha_k^{(1)}]x \leq b_k,\ \forall k \in \mathscr{J}_1\},$$

$$S_2(\delta^2) = \{y \in \mathbb{R}^n | y \in Y,\ -[D_l + R^{-1}(\delta_l^2)\beta_l^{(2)}]y \leq -d_l, \forall l \in \mathscr{J}_2\},$$

where $\beta_k^{(1)} = (\beta_{k1}^{(1)}, \beta_{k2}^{(1)}, \ldots, \beta_{km}^{(1)}), k \in \mathscr{J}_1$, $\beta_l^{(2)} = (\beta_{l1}^{(2)}, \beta_{l2}^{(2)}, \ldots, \beta_{ln}^{(2)}), l \in \mathscr{J}_2$, $\alpha_k^{(1)} = (\alpha_{k1}^{(1)}, \alpha_{k2}^{(1)}, \ldots, \alpha_{km}^{(1)}), k \in \mathscr{J}_1$ *and* $\alpha_l^{(2)} = (\alpha_{l1}^{(2)}, \alpha_{l2}^{(2)}, \ldots, \alpha_{ln}^{(2)}), l \in \mathscr{J}_2$, *in which* $B_k = (B_{k1}, B_{k2}, \ldots, B_{km})$, $k \in \mathscr{J}_1$ *and* $D_l = (D_{l1}, D_{l2}, \ldots, D_{ln})$, $l \in \mathscr{J}_2$.

Proof. For all $k \in \mathscr{J}_1$ and $l \in \mathscr{J}_2$, we write

1) $Pos\{\tilde{B}_k x \leq b_k\} \geq \delta_k^1 \Longleftrightarrow Pos\{\sum_{i=1}^{m} \tilde{B}_{ki} x_i \leq b_k\} \geq \delta_k^1$. Using Lemma 1, we obtain

$$Pos\{\sum_{i=1}^{m} \tilde{B}_{ki} x_i \leq b_k\} \geq \delta_k^1 \Longleftrightarrow \sum_{i=1}^{m} x_i B_{ki} - L^{-1}(\delta_k^1) \sum_{i=1}^{n} x_i \alpha_{ki}^{(1)} \leq b_k.$$

Then, $S_1(\delta^1) = \{x \in \mathbb{R}^m \mid x \in X,\ [B_k - L^{-1}(\delta_k^1)\alpha_k^{(1)}]x \leq b_k, \forall k \in \mathscr{J}_1\}$, where $B_k = (B_{k1},\ B_{k2}, \ldots,\ B_{km})$, $k \in \mathscr{J}_1$.

2) For the second player, similarly for all $l \in \mathscr{J}_2$, we have

$$Pos\{\tilde{D}_l^{\omega} y \geq d_l\} \geq \delta_l^2 \Longleftrightarrow \sum_{j=1}^{n} y_j D_{lj}(\omega) + R^{-1}(\delta_l^2) \sum_{j=1}^{m} y_j \beta_{lj}^2 \geq d_l,\ \forall l \in \mathscr{J}_2.$$

So, $S_2(\delta^2) = \{y \in \mathbb{R}^n \mid y \in Y, -[D_l + R^{-1}(\delta_l^2)\beta_l^{(2)}]y \leq -d_l,\ \forall l \in \mathscr{J}_2\}$, where $D_l = (D_{l1}, D_{l2}, \ldots, D_{ln})$, $l \in \mathscr{J}_2$. □

Associated with the fuzzy game $G(\delta,\ \tilde{A})$, consider the crisp two zero-sum matrix game $G(\delta,\ A) = (S_1(\delta^1),\ S_2(\delta^2),\ A)$, where $A = [a_{ij}],\ a_{ij} = \mathscr{R}(\tilde{a}_{ij}),\ i \in \mathbb{I},\ j \in \mathbb{J}$ are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij},\ i \in \mathbb{I},\ j \in \mathbb{J}$ with respect to a given ranking function $\mathscr{R}$ (Definition 8).

The relationship between the $\mathscr{R}$-saddle point equilibrium of the game $G(\delta,\tilde{A})$ and the saddle point equilibrium of the game $G(\delta,\ A)$ is characterised by the following theorem.

Theorem 4. *Assume that $\widetilde{B}_{ki} = (B_{ki},\ \alpha_{ki}^{(1)},\ \beta_{ki}^{(1)})$ and $\widetilde{D}_{lj} = (D_{lj},\ \alpha_{lj}^{(2)},\ \beta_{lj}^{(2)})$, $i \in \mathbb{I}$, $j \in \mathbb{J}$, $k \in \mathscr{J}_1$ and $l \in \mathscr{J}_2$, are LR-fuzzy numbers. And let $\mathscr{R}$ be any ranking function (Definition 8) satisfying property (P_1). Then $(x^*,\ y^*) \in S_1(\delta^1) \times S_2(\delta^2)$ is a $\mathscr{R}$-saddle point equilibrium at $(\delta_1,\ \delta_2)$ levels, of game $G(\delta,\ \tilde{A})$ if and only if $(x^*,\ y^*)$ is a saddle point equilibrium of the crisp constrained matrix game $G(\delta,\ A)$.*

Proof. For all $(x,\ y) \in S_1(\delta^1) \times S_2(\delta^2)$, $(x^*,\ y^*)$ is an $\mathscr{R}$-addle point for game $G(\delta,\tilde{A})$ if and only if

$$\begin{aligned}\tilde{E}(x^*,\ y) \underset{\mathscr{R}}{\geqslant} \tilde{E}(x^*,\ y^*) \underset{\mathscr{R}}{\geqslant} \tilde{E}(x,\ y^*) \ &\Leftrightarrow\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j \tilde{a}_{ij} \underset{\mathscr{R}}{\geqslant} \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j^* \tilde{a}_{ij} \underset{\mathscr{R}}{\geqslant} \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j^* \tilde{a}_{ij}\\ &\Leftrightarrow\ \mathscr{R}(\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j \tilde{a}_{ij}) \geq \mathscr{R}(\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j^* \tilde{a}_{ij}) \geq \mathscr{R}(\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j^* \tilde{a}_{ij}).\end{aligned}$$

Making use of the property (P_1), we get the equivalence

$$\begin{aligned}\tilde{E}(x^*,y) \underset{\mathscr{R}}{\geqslant} \tilde{E}(x^*,y^*) \underset{\mathscr{R}}{\geqslant} \tilde{E}(x,y^*) \ &\Leftrightarrow\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j \mathscr{R}(\tilde{a}_{ij}) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j^* \mathscr{R}(\tilde{a}_{ij}) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j^* \mathscr{R}(\tilde{a}_{ij})\\ &\Leftrightarrow\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j a_{ij} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* y_j^* a_{ij} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j^* a_{ij}\\ &\Leftrightarrow\ x^{*T} Ay \geq x^{*T} Ay^* \geq xTAy^*. \end{aligned} \tag{2}$$

Eq. (2) means that (x^*,y^*) is a saddle point equilibrium for the crisp constrained matrix game $G(\delta,\ A)$. □

For the rest of this paper, we assume that the ranking function $\mathscr{R}$ (Definition 8) satisfies property (P_1).

Definition 11. *($\mathscr{R}$-value) If the strategy profile (x^*,y^*) is a $\mathscr{R}$-saddle point equilibrium at (δ^1,δ^2) possibility levels in the game $G(\delta,\tilde{A})$, then $v^*(\delta) = x^{*T}Ay^*$ is called the $\mathscr{R}$-value of the constrained matrix game $G(\delta,\tilde{A})$.*

Theorem 5. *Assume that $\widetilde{B}_{ki} = (B_{ki},\ \alpha_{ki}^{(1)},\ \beta_{ki}^{(1)})$ and $\widetilde{D}_{lj} = (D_{lj},\ \alpha_{lj}^{(2)},\ \beta_{lj}^{(2)})$, $i \in \mathfrak{I}$, $j \in \mathfrak{J}$, $k \in \mathscr{J}_1$ and $l \in \mathscr{J}_2$, are LR-fuzzy numbers. Then, for all $\delta \in [0,\ 1]^p \times [0,\ 1]^q$, there exists a $\mathscr{R}$-saddle point equilibrium $(x^*,\ y^*)$ for the game $G(\delta,\ \tilde{A})$ which solves the linear programming problems.*

$$\begin{cases} \max\ x^T Ay \\ \text{subject to} \\ (B_k - L^{-1}(\delta_k^1)\alpha_k^{(1)})x \le b_k, k \in \mathscr{J}_1, \\ x \in X, \end{cases} \qquad \begin{cases} \min\ x^T Ay \\ \text{subject to} \\ -(D_l + R^{-1}(\delta_l^2)\beta_l^{(2)})y \le -d_l, l \in \mathscr{J}_2, \\ y \in Y. \end{cases} \tag{3}$$

Proof. Let $\delta \in [0,\ 1]^p \times [0,\ 1]^q$. From Lemma 2,

$$S_1(\delta^1) = \{x \in \mathbb{R}^m \mid x \in X,\ [B_k - L^{-1}(\delta_k^1)\alpha_k^{(1)}]x \le b_k,\ \forall k \in \mathscr{J}_1\},$$

$$S_2(\delta^2) = \{y \in \mathbb{R}^n | y \in Y,\ -[D_l + R^{-1}(\delta_l^2)\beta_l^{(2)}]y \le -d_l,\ \forall l \in \mathscr{J}_2\},$$

are compact sets. For any $\delta \in [0,1]^p \times [0,\ 1]^q$, the inequality constraints

$$(B_k - L^{-1}(\delta_k^1)\alpha_k^{(1)})x \le b_k,\ \ k \in \mathscr{J}_1,$$

$$-(D_l + R^{-1}(\delta_l^2)\beta_l^{(2)})y \le -d_l,\ \forall l \in \mathscr{J}_2,$$

are linear. Then, $S_1(\delta^1)$ and $S_2(\delta^2)$ are convex and compact sets. Also, function $x^T Ay$ is a continuous function. Then, using the minimax theorem of Von Neumann [45], we conclude that $G(\delta,\ A)$ has at least one saddle point equilibrium $(x^*,\ y^*)$, which solves the linear programming problems in Eq. (3). By Theorem 4, $(x^*,\ y^*)$ is a $\mathscr{R}$-saddle point of the game $G(\delta,\ \tilde{A})$ and $v(\delta) = x^{*T} Ay^*$. □

4 Methodology for solving the game $G(\tilde{A})$

Based on the results in Section 3, we give an algorithm for the determination of the $\mathscr{R}$-saddle point equilibrium.

Remark 3. *Under the conditions of Theorem 5, making use of the same notations as in the problems in Eq. (1), with $B = (B_{ki} - L^{-1}(\delta_k^1)\alpha_{ki}^{(1)})$ and $D = (-(D_{lj} + R^{-1}(\delta_l^2)\beta_{lj}^{(2)})$, the pair of primal-dual associated to the crisp constrained matrix game $G(\delta,\ A)$, is given as follows:*

$$\begin{cases} \max u^T z \\ subject\quad to \\ E^T z - A^T x \le 0, \\ Hx \le c, \\ z \ge 0, \\ x \ge 0, \end{cases} \qquad \begin{cases} \min c^T s \\ subject\quad to \\ H^T s - Ay \ge 0, \\ Ey \ge u, \\ s \ge 0, \\ y \ge 0. \end{cases} \tag{4}$$

According to the previous sections, the process for solving constrained matrix games with the payoffs and the linear constraints are *LR*-fuzzy numbers is summarized as follows.

Algorithm 1

Step 1. Identify players, denoted by I and II.
Step 2. Identify pure strategies of players I and II. Denote the sets of pure strategies of player I and player II by $\mathbb{I} = \{1, \ldots, m\}$ and $\mathbb{J} = \{1, \ldots, n\}$, respectively.
Step 3. Estimate the constraints matrices expressed with *LR*-fuzzy numbers, denoted $\widetilde{B}$ and $\widetilde{D}$.
Step 4. Estimate the players payoffs expressed with *LR*-fuzzy numbers and construct matrix $\widetilde{A} = [\widetilde{a}_{ij}]_{m\times n}$.
Step 5. Each player chooses a possibility level, δ^1 for player I and δ^2 for player II, such that the constraints will hold with a possibility levels.
Step 6. Identify the ranking function $\mathscr{R}$.
Step 7. Construct the matrix $A = [a_{ij}]_{m\times n}$, $a_{ij} = \mathscr{R}(\widetilde{a}_{ij})$.
Step 8. Construct and solve the linear programming problems according to Eq. (4), and obtain the $\mathscr{R}$-value $v^*(\delta)$ and the optimal strategies $(x^*,\ y^*)$.

Remark 4. *We note that obtaining a saddle point equilibrium in* Step *8, is an easy task due to the properties of the primal dual programs. This interactive procedure depends on the choices of the possibility levels provided by the players in* Step *5.*

Next, we provide an example to illustrate our methodology.

5 An illustrative example

We consider the situation in which two competing business firms plan to release a new product with similar functions. In the target market, the demand amount of this new product is fixed. Consequently, one company's market share increases while another company's market share decreases. By selecting from its available marketing strategies, each firm seeks to draw in as many clients as feasible. Assume that firms have two strategies: advertisements (strategy 1) and special offers (strategy 2). That is, the pure strategy sets of player I (firm f_I) and player II (firm f_{II}) are $\mathbb{I} = \{1,2\}$ and $\mathbb{J} = \{1,2\}$, respectively. A mixed strategy determines how to divide its budget among its various marketing alternatives (strategy 1 and strategy 2). Due to the lack of precision of the available information, the manager of firm f_I (resp. f_{II}) is not able to give precisely the funds that the company needs when it takes pure strategies 1, and 2, respectively. Hence *LR*-fuzzy numbers are suitable to represent the these funds. Let $\tilde{B}_{11} = (62,26,41)_{LR}$ and $\tilde{B}_{12} = (43,28,22)_{LR}$ be the funds which firm f_I needs when it takes strategy 1 and 2, respectively. And the firm f_{II} needs the funds $\tilde{D}_{11} = (28,31,30)_{LR}$ and $\tilde{D}_{12} = (2,15,40)_{LR}$ when it takes strategy 1 and 2, respectively. Moreover, due to the lack of funds, the firm f_1 only provides 40 (thousand dollars). Thus, the mixed strategies of the firm f_I may satisfy the constraint condition $\tilde{B}_{11}x_1 + \tilde{B}_{12}x_2 \preceq 40$.

The firm f_{II} only provides 17 (thousand dollars), the mixed strategies of the firm f_2 may satisfy the constraint conditions $\tilde{D}_{11}y_1 + \tilde{D}_{12}y_2 \preceq 17$ or $-\tilde{D}_{11}y_1 - \tilde{D}_{12}y_2 \succeq -17$.

Due to lack of information and precision about the demands of their new products, the two managers of both firms are not able to exactly forecast the sale amount of both firms' product. Then the demands of the products can be characterized by *LR*-fuzzy numbers. Assume that the payoff matrix is given as follows:

$$\tilde{A} = \begin{pmatrix} (-15,12,31)_{LR} & (13,21,21)_{LR} \\ (21,11,17)_{LR} & (-14,22,18)_{LR} \end{pmatrix},$$

with $L(t) = R(t) = \max(0, 1-t)$ and $R^{-1}(t) = L^{-1}(t) = 1-t$, $t \in [0,1]$.

Since the fuzzy constraints $\tilde{B}_{11}x_1 + \tilde{B}_{12}x_2 \preceq 40$ and $-\tilde{D}_{11}y_1 - \tilde{D}_{12}y_2 \succeq -17$ do not define a crisp feasible set, each player chooses a confidence level δ_1^i, such that the fuzzy constraints hold with possibility level.

Using chance constraints and the ranking function satisfying property (P_1), coefficient matrices and vectors of the primal-dual of Eq. (4) are obtained as follows.

$$A = \begin{pmatrix} -26 & 26 \\ 46 & -30 \end{pmatrix}, H = \begin{pmatrix} 36+26\delta_1^1 & 15+28\delta_1^1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}, E = \begin{pmatrix} -2+30\delta_1^2 & -38+40\delta_1^2 \\ 1 & 1 \\ -1 & -1 \end{pmatrix},$$

and $c^T = (40,\ 1,\ -1)$, $u^T = (-17,\ 1,\ -1)$. Choosing $\delta_1^1 = \delta_1^2 = 0.5$ and solving the problems in Eq. (4), we obtain a $\mathscr{R}$-saddle point equilibrium at $\delta = (\delta_1^1, \delta_1^2)$ possibility levels $(x^*, y^*) = ((0.55,\ 0.45),$ $(0.0323,\ 0.9677))$. Furthermore, we have the value of the game is $v^*(\delta) = 0,9806$.

6 Conclusions

In this paper, we proposed a new approach to deal with constrained matrix game in which the payoffs and the elements of the constraints matrices are fuzzy numbers. The proposed approach combines the chance constraints approach and the concept of comparison of fuzzy numbers. As a solution of the game, we introduced the concept of $\mathscr{R}$-saddle point equilibrium and the $\mathscr{R}$-value of the game. We have established theoretical results to justify the methodology. Indeed, in the case of *LR*-fuzzy numbers we formulated an associated crisp chance constrained matrix game and sufficient existence conditions are obtained. An example of the market competition game which clarifies the theory and the method discussed in our work is given. The model and method obtained along this paper can be similarly obtained with necessity measure.

Although numerous defuzzification methods for fuzzy constrained matrix games have been described in the literature, to the best of our knowledge, till now, our approach is the first in the literature which deals with constrained matrix game with fuzzy payoffs and fuzzy linear constraints using chance constraints. However, the proposed methodology closely depend on the ranking function, for which we loss a lot of fuzzy information. Therefore, introducing a general methodology to solve the considered game with a more general way represents a matter of further investigation and elaboration in a forthcoming research. Besides, our future research directions will concentrate, for example, on the extension of our approach to fuzzy constrained bi-matrix games and multiple criteria fuzzy constrained matrix games. Finally, let us notice that our approach can be applied to real-world decision making problems such as plastic ban problem [37] and natural desaster management [19].

Acknowledgements

We would like to thank the anonymous reviewers and the editor who helped us to substantially improve this article. This work is supported by the General Directorate of Scientific Research and Technological Development (DGRSDT)/MESRS, Algeria.

References

[1] F. Abbasi, T. Allahviranloo, *Estimation of failure using fault tree analysis based on new operations on LR-type flat fuzzy numbers*, New Math. Nat. Comput. **17** (2021) 153-174.

[2] M.D. Banadaki, H. Navid, *A Bernoulli Tau method for numerical solution of feedback Nash differential games with an error estimation*, Comput. Meth. Differ. Equ. **10** (2022) 894–904.

[3] M.D. Banadaki, H. Navidi, *A numerical treatment based on Bernoulli Tau method for computing the open-loop Nash equilibrium in nonlinear differential games*, Iran. J. Num. Anal. Optim. **12** (2022) 467–482.

[4] M.D. Banadaki, H. Navidi, *Numerical Solution of Open-Loop Nash Differential Games Based on the Legendre Tau Method*, Games **11** 3 (2020) 3–28.

[5] C.R. Bector, S. Chandra, *Fuzzy Mathematical Programming and Fuzzy Matrix Games*, Springer, Berlin, 2005.

[6] G. Bortolan, R. Degani, *A review of some methods for ranking fuzzy subsets*, Fuzzy Sets Syst. **15** (1985) 1–19.

[7] S. Chakraborty, A.S. Kumar, *A framework of LR fuzzy AHP and fuzzy WASPAS for health care waste recycling technology*, Appl. Soft Comput. **127** (2022) 109–388.

[8] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets Syst. **32** (1989) 275-289.

[9] A. Charnes, *Constrained games and linear programming*, Proc. Natl. Acad. Sci., **39** (1953) 639–641.

[10] G. Dantzig, *A proof of the equivalence of the programming problem and the game problem*, Act. Anal. Prod. Alloc. **13** (1951) 330–335.

[11] I. Deli, *Matrix games with simplified neutrosophic payoffs*, in Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets. Cham: Springer International Publishing, (2019) 233–246.

[12] M. Dresher, *Games of Strategy Theory and Applications*, Prentice-Hall, 1961.

[13] D. Dubois, H. Prade, *Operations on fuzzy numbers*, Int. J. Syst. Sci. **9** (1978) 613–626.

[14] D. Dubois, H. Prade, *Fuzzy real algebra: some results*, Fuzzy Sets Syst. **2** (1978) 327–348.

[15] M. Ghanbari, R. Nuraei, T. Allahviranloo, W. Pedrycz, *A new effective approximate multiplication operation on LR fuzzy numbers and its application*, Soft. Comput. **26** (2022) 4103–4113.

[16] A. Hosseinzadeh, S.A. Edalatpanah, *A new approach for solving fully fuzzy linear programming by using the lexicography method*, Adv. Fuzzy Syst. **2016** (2016) 1–6.

[17] S. Jafari, H. Navidi, *A game-theoretic approach for modeling competitive diffusion over social networks* , Games **9** (2018) 1–8.

[18] J. Jana, S.K. Roy, *Two-person game with hesitant fuzzy payoff: An application in MADM*, RAIRO Oper. Res. **55** (2021) 3087–3105.

[19] S. Karmakar, M.R. Seikh, *Bimatrix games under dense fuzzy environment and its application to natural disaster management*, Artif. Intell. Rev. **56** (2022) 2241–2278.

[20] T. Kawaguchi, Y. Maruyama *A note on minimax (maximin) programming*, Manag. Sci. (1976) 670–676.

[21] D.F. Li, *An effective methodology for solving matrix games with fuzzy payoffs*, IEEE Trans. Cyber. **43** (2013) 610-621.

[22] D. Li, C. Cheng, *Fuzzy multiobjective programming methods for fuzzy constrained matrix games with fuzzy payoffs*, Int. J. Uncert. Fuzzy Bas. Syst. **10** (2002) 385–400.

[23] D. Li, F. Hong, *Alfa-cut based linear programming methodology for constrained matrix games with payoffs of trapezoidal fuzzy numbers*, Fuzzy Optim. Deci. Mak. **12** (2013) 191–213.

[24] D. Li, F. Hong, *Solving constrained matrix games with payoffs of triangular fuzzy numbers*, Comput. Math. Appl. **64** (2012) 432–446.

[25] H.R. Maleki, M. Tata, M. Mashinchi, *Linear programming with fuzzy variables*, Fuzzy Sets Syst., **109** (2000) 21–33.

[26] A. Mansoori, M. Eshaghnezhad, S. Effati, *Recurrent neural network model: A new strategy to solve fuzzy matrix games*, IEEE Trans. Neur. Netw. Learn. Syst. **30** (2019) 2538–2547.

[27] J. X. Nan, D. Li, *Linear programming technique for solving interval-valued constraint matrix games*, J. Ind. Manag. Optim. **10** (2014) 1059–1070.

[28] G. Owen, *Game Theory*, Academic Press, San Diego, 1995.

[29] G. Owen, *Game theory*, Second Edition, Academic Press, New York, 1982.

[30] B. Pérez-Cañedo, A. Rosete, J.L. Verdegay, E.R.Concepción-Morales, *A fuzzy goal programming approach to fully fuzzy linear regression*, Inf. Proc. Manag. Uncer. Know. Bas. Syst. **18** (2020) 677–688.

[31] S. K. Roy, A. Bhaumik, *Intelligent water management: a triangular type-2 intuitionistic fuzzy matrix games approach*, Wat. Resour. Manag. **32** (2018) 949-968.

[32] M. Sakawa, *Fuzzy Sets and Interactive Multiobjective Optimization*, Plenum Press, New York, 1993.

[33] M. Sakawa, I. Nishizaki, *Max-min solutions for fuzzy multiobjective matrix games*, Fuzzy Sets Syst. **67** (1994) 53–69.

[34] M.R. Seikh, S. Dutta, D.F. Li, *Solution of matrix games with rough interval pay-offs and its application in the telecom market share problem*, Int. J. Intell. Syst. **36** (2021) 6066–6100.

[35] M. R. Seikh, S. Karmakar, *Credibility equilibrium strategy for matrix games with payoffs of triangular dense fuzzy lock sets*, Sadhana **46** (2021) 158.

[36] M. R. Seikh, S. Karmakar, O. Castillo, *A novel defuzzification approach of type-2 fuzzy variable to solving matrix games: An application to plastic ban problem*, Iran. J. Fuzzy Syst **18** (2021) 155–172.

[37] M.R. Seikh, S. Karmakar, P.K. Nayak, *Matrix games with dense fuzzy payoffs*, Int. J. Intell. Syst. **36** (2021) 1770–1799.

[38] M. R. Seikh, S. Karmakar, M. Xia, *Solving matrix games with hesitant fuzzy pay-offs*, Iran. J. Fuzzy Syst. **17** (2020) 25–40.

[39] M. R. Seikh, P.K. Nayak, M. Pal, *Application of intuitionistic fuzzy mathematical programming with exponential membership and quadratic non-membership functions in matrix games*, Annal. Fuzzy Math. Info. **9** (2015) 183–195.

[40] M.R. Seikh, P.K. Nayak, M. Pal, *Aspiration level approach to solve matrix games with I-fuzzy goals and I-fuzzy pay-offs*, Pac. Sci. Rev. Nat. Sci. Eng. **18** (2016) 5–16.

[41] M.R. Seikh, P.K. Nayak, M. Pal, *Matrix games in intuitionistic fuzzy environment*, Inter. J. Math. Oper. Res. **5** (2013) 693–708.

[42] S. Suneela, S. Chakraverty, *New ranking function for fuzzy linear programming problem and system of linear equations*, J. Inf. Opt. Sci. **40** (2019) 141-156.

[43] T. Verma, *A novel method for solving constrained matrix games with fuzzy payoffs*, J. Intell. Fuzzy Syst. **40** (2021) 191–204.

[44] T. Verma, A. Kumar, *Fuzzy Solution Concepts for Non-cooperative Games: Interval, Fuzzy and Intuitionistic Fuzzy Payoffs*, Springer, New York, 2020.

[45] J. Von Neumann, *On the theory of games*, Math. Ann. **100** (1928) 295-320.

[46] R.R. Yager, *A procedure for ordering fuzzy subsets of the unit interval*, Inf. Sci. **24** (1981) 143–161.

[47] L.A. Zadeh, *Fuzzy sets*, Inf. Control **8** (1965) 338–353.

[48] L.A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets Syst. **1** (1978) 3–28.

[49] O. Zerdani, F. Achemine, *On optimisation over the integer efficient set in fuzzy linear multicriteria programming*, Int. J. Math. Oper. Res. **13** (2018) 281–302.

[50] J. Zhou, F. Yang, K. Wang, *Fuzzy arithmetic on LR fuzzy numbers with applications to fuzzy programming*, J. Intell. Fuzzy Syst. **30** (2015) 71–87.

[51] H.J. Zimmermann, *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets Syst. **1** (1978) 45–55.