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On the inverse eigenvalue problem for a specific symmetric matrix

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Abstract. The aim of the current paper is to study a partially described inverse eigenvalue problem of a specific symmetric matrix, and prove some properties of such matrix. The problem includes the construction of the matrix by the minimal eigenvalue of all leading principal submatrices and eigenpair $(\lambda_2^{(n)}, x)$ such that $\lambda_2^{(n)}$ is the maximal eigenvalue of the required matrix. We investigate conditions for the solvability of the problem, and finally an algorithm and its numerical results are presented.

Keywords: Eigenvalue, eigenpair, leading principal submatrices, inverse eigenvalue problem.
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1 Introduction

Constructing a matrix with a particular structure from total or a partial eigendata is regarded as inverse eigenvalue problem arising in some applications. In [1] inverse eigenvalue problems are described with details. Special types of inverse eigenvalue problems have been studied in [2–4]. The problem in this paper involves the construction of a specific symmetric matrix. This is carried out through the minimal eigenvalue of each of its leading principal submatrices and an eigenpair of the matrix. The symmetric matrix will be of the following form

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$$A_n = \begin{pmatrix} a_1 & b_1 & \dots & b_{p+1} & & & & & & & & \\ b_1 & a_2 & & & & & & & & & & \\ \vdots & & \ddots & & & & & & & & & \\ b_{p+1} & & & a_{p+1} & b_{p+2} & & & & & & & \\ & & & b_{p+2} & a_{p+2} & b_{p+3} & & & & & & \\ & & & & b_{p+3} & \ddots & \ddots & & & & & \\ & & & & & \ddots & \ddots & b_{p+m} & & & & \\ & & & & & & b_{p+m} & a_{p+m} & b_{p+m+1} & \dots & b_{n-1} \\ & & & & & & & b_{p+m+1} & a_{p+m+1} & & \\ & & & & & & & \vdots & & \ddots & \\ & & & & & & & b_{n-1} & & & a_n \end{pmatrix} \tag{1}$$

where $b_j > 0$ for $j = 1,\dots,n-1$ and $n = p+m+q$.

In the current paper, we attempt to solve the following IEP, denoted by IEP1, consisting of constructing a matrix of form (1) with partial eigendata.

IEP1: The list of real numbers $\lambda_2^{(n)}$, $\lambda_1^{(j)}$, $j = 1,\dots,n$, and real vector $x = (x_1,x_2,\dots,x_n)$ are given. Find a matrix A_n of the form (1) such that $\lambda_2^{(n)}$ is the maximal eigenvalue of A_n, $\lambda_1^{(j)}$ is the minimal eigenvalue of the leading principal submatrix A_j of A_n and $(\lambda_2^{(n)},x)$ is an eigenpair of A_n.

The paper is organized as follows. Section 2 will provide a brief outline of some lemmas used all over this paper. In Section 3, IEP1 is discussed and an algorithm is presented. In Section 4, we investigate conditions for the existence of the nonnegative matrix A_n. In Section 5, we report a numerical example to illustrate the solutions of IEP1. Finally, Section 6 concludes the paper.

2 Preliminaries

What follows is necessary lemmas that are used in the paper.

Lemma 1. *[5] Let $\varphi(\lambda)$ be a monic polynomial of degree n with all real zeroes. If $\lambda_1^{(n)}$ and $\lambda_n^{(n)}$ are, respectively, the minimal and the maximal zero of $\varphi(\lambda)$, then*

(i) if $x < \lambda_1^{(n)}$, we have $(-1)^n\varphi(x) > 0$;

(ii) if $x > \lambda_n^{(n)}$, we have $\varphi(x) > 0$.

Lemma 2. *[6] (**Cauchy's interlacing theorem**) Let $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \dots \le \lambda_n^{(n)}$ be the eigenvalues of a real symmetric matrix $A_{n\times n}$ and $\mu_1^{(n-1)} \le \mu_2^{(n-1)} \le \dots \le \mu_{n-1}^{(n-1)}$ be the eigenvalues of an $(n-1)\times(n-1)$ principal submatrix of $A_{n\times n}$. Then, $\lambda_1^{(n)} \le \mu_1^{(n-1)} \le \dots \le \mu_{n-1}^{(n-1)} \le \lambda_n^{(n)}$.*

Lemma 3. *[4] Let A_n be an $n \times n$ matrix of the form (1) and $\lambda_1^{(j)}$, $\lambda_2^{(j)}$ be the minimal and maximal eigenvalue of the leading principal submatrix A_j of A_n, $j = 1, 2, \ldots, n$. Then*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_2^{(n)}, \tag{2}$$

and

$$\lambda_1^{(j)} < a_k < \lambda_2^{(j)}, \quad k = 1, 2, \ldots, j, \quad j = 2, \ldots, n. \tag{3}$$

3 The existence of the solution of the problem IEP1

Let $\{\varphi_j(\lambda) = \det(\lambda I_j - A_j)\}_{j=1}^n$ be the sequence of characteristic polynomials of A_n. We obtain the recurrence relation $\varphi_j(\lambda)$ of the matrix A_n in Lemma 4.

Lemma 4. *Let A_n be an $n \times n$ matrix of the form (1). Then the sequence of $\{\varphi_j(\lambda)\}_{j=1}^n$ satisfies the following recurrence relations*

(i) $\varphi_1(\lambda) = (\lambda - a_1)$;

(ii) $\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \prod_{k=2}^{j-1}(\lambda - a_k), \quad j = 2, \ldots, p+2;$

(iii) $\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \varphi_{j-2}(\lambda), \quad j = p+3, \ldots, p+m+1;$

(iv) $\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \varphi_{p+m-1}(\lambda) \prod_{k=p+m+1}^{j-1}(\lambda - a_k), \quad j = p+m+2, \ldots, n,$

where $\varphi_0(\lambda) = 1$.

Proof. The relations can be verified by expanding the determinant. □

In the following lemma, we show how to gain the component x_j of x from elements A_{j-1} and x_1.

Lemma 5. *If $x = (x_1, x_2, \ldots, x_n)^T$ is eigenvector of A_n corresponding to maximal eigenvalue $\lambda_2^{(n)}$, then $x_1 \neq 0$ and components of this eigenvector are given by:*

$$x_j = \frac{-b_{j-1}x_1}{(a_j - \lambda_2^{(n)})}, \qquad j = 2, \ldots, p+1, \tag{4}$$

$$x_j = \frac{(-1)^j \varphi_{j-1}(\lambda_2^{(n)})x_1}{\prod_{i=p+1}^{j-1} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}, \qquad j = p+2,, \ldots, p+m, \tag{5}$$

$$x_j = \frac{-b_{j-1}\varphi_{p+m-1}(\lambda_2^{(n)})x_1}{(a_j - \lambda_2^{(n)})\prod_{i=p+1}^{p+m-1} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}, \qquad j = p+m+1 \ldots, n. \tag{6}$$

Proof. Since $(\lambda_2^{(n)}, x)$ is an eigenpair of A_n, so $A_n x = \lambda_2^{(n)} x$, which can be transformed into the form

$$(a_1 - \lambda_2^{(n)})x_1 + \sum_{i=2}^{p+2} b_{i-1}x_i = 0, \tag{7}$$

$$b_{j-1}x_1 + (a_j - \lambda_2^{(n)})x_j = 0, \quad j = 2, \ldots, p+1, \tag{8}$$

$$b_{p+1}x_1 + (a_{p+2} - \lambda_2^{(n)})x_{p+2} + b_{p+2}x_{p+3} = 0, \tag{9}$$

$$b_{j-1}x_{j-1} + (a_j - \lambda_2^{(n)})x_j + b_j x_{j+1} = 0, \quad j = p+3, \ldots, p+m-1, \tag{10}$$

$$b_{p+m-1}x_{p+m-1} + (a_{p+m} - \lambda_2^{(n)})x_{p+m-1} + \sum_{i=p+m+1}^{n} b_{i-1}x_i = 0, \tag{11}$$

$$b_{j-1}x_{p+m} + (a_j - \lambda_2^{(n)})x_j = 0, \quad j = p+m+1, \ldots, n. \tag{12}$$

By Lemma 3 for $j = 2, \ldots, p+1$ we have $(a_j - \lambda_2^{(n)}) \neq 0$. Therefore, according to (8), x_j can be written as follows

$$x_j = \frac{-b_{j-1}x_1}{(a_j - \lambda_2^{(n)})}, \quad j = 2, \ldots, p+1.$$

As a result, (4) holds.

For $j = p+2, \ldots, p+m$, it can be shown by induction on j that (5) holds. For the base case by (7) we have

$$x_{p+2} = \frac{(\lambda_2^{(n)} - a_1)x_1 - \sum_{i=2}^{p+1} b_{i-1}x_i}{b_{p+1}},$$

by replacing $x_j, j = 2, \ldots, p+1$ we get

$$x_{p+2} = \frac{\prod_{j=1}^{p+1}(a_j - \lambda_2^{(n)})x_1 - \sum_{i=2}^{p+1}(b_{j-1}^2 \prod_{k=2,k\neq j}^{p+1}(a_k - \lambda_2^{(n)})x_1)}{b_{p+1}\prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})} = \frac{(-1)^{p+2}\varphi_{p+1}(\lambda_2^{(n)})x_1}{b_{p+1}\prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}.$$

Now suppose the lemma holds for $j = p+3, \ldots, p+m-1$, we prove it for $j = p+m$. From (10) we obtain

$$x_{p+m} = \frac{-b_{p+m-2}x_{p+m-2} + (\lambda_2^{(n)} - a_{p+m-1})x_{p+m-1}}{b_{p+m-1}}. \tag{13}$$

By induction we have

$$x_{p+m-2} = \frac{(-1)^{p+m-2}\varphi_{p+m-3}(\lambda_2^{(n)})x_1}{\prod_{i=p+1}^{p+m-3} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}, \tag{14}$$

and

$$x_{p+m-1} = \frac{(-1)^{p+m-1}\varphi_{p+m-2}(\lambda_2^{(n)})x_1}{\prod_{i=p+1}^{p+m-2} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}. \tag{15}$$

Therefore

$$x_{p+m} = \frac{-b_{p+m-2}}{b_{p+m-1}} \times \left(\frac{(-1)^{p+m-2}\varphi_{p+m-3}(\lambda_2^{(n)})}{\prod_{i=p+1}^{p+m-3} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})} \right) x_1$$

$$+ \frac{(\lambda_2^{(n)} - a_{p+m-1})}{b_{p+m-1}} \times \left(\frac{(-1)^{p+m-1}\varphi_{p+m-2}(\lambda_2^{(n)})}{\prod_{i=p+1}^{p+m-2} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})} \right) x_1$$

$$= \frac{(-1)^{p+m}\varphi_{p+m-1}(\lambda_2^{(n)})x_1}{\prod_{i=p+1}^{p+m-1} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}.$$

For $j = p+m+1,\ldots,n$ by Lemma 3 we have $(a_j - \lambda_2^{(n)}) \neq 0$. Therefore, according to (12), x_j can be written as follows:

$$x_j = \frac{-b_{j-1}x_{p+m}}{(a_j - \lambda_2^{(n)})},$$

by replacing x_{p+m} in above equation we get

$$x_j = \frac{-b_{j-1}\varphi_{p+m-1}(\lambda_2^{(n)})x_1}{(a_j - \lambda_2^{(n)}) \prod_{i=p+1}^{p+m-1} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}.$$

Since x is an eigenvector, we have $x \neq 0$. If $x_1 = 0$ then from (4), (5) and (6) all other entries of x become zero, hence $x_1 \neq 0$. □

Theorem 1 presents the solution to the IEP1 and the conditions under which the problem is solvable.

Theorem 1. *There are solutions to IPE1 if the following conditions are satisfied*

(a) There is a solution $\alpha > 0$ of the equation

$$\alpha^2 \prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i) - \frac{\alpha x_1 \varphi_{j-1}(\lambda_1^{(j)})}{x_j} + (\lambda_2^{(n)} - \lambda_1^{(j)})\varphi_{j-1}(\lambda_1^{(j)}) = 0,$$

for $j = 2,\ldots,p+1$.

(b) There is a solution $\beta > 0$ of the equation

$$\beta^2 \varphi_{p+m-1}(\lambda_1^{(j)}) \prod_{i=p+m+1}^{j-1}(\lambda_1^{(j)} - a_i) - \frac{\beta x_1 \varphi_{j-1}(\lambda_1^{(j)})}{x_j} + (\lambda_2^{(n)} - \lambda_1^{(j)})\varphi_{j-1}(\lambda_1^{(j)}) = 0,$$

for $j = p+m+1,\ldots,n$.

Proof. Solving the IEP1 is equivalent to solving the equations

$$\varphi_j(\lambda_1^{(j)}) = 0, \tag{16}$$

$$A_n x = \lambda_2^{(n)} x. \tag{17}$$

Moreover, $b_{j-1} > 0$ for all j. From Lemma 4 and (16) we have

$$\varphi(\lambda_1^{(1)}) = 0 \Rightarrow a_1 = \lambda_1^{(1)}.$$

Since $x_j \neq 0$, by (8) we get

$$a_j = \lambda_2^{(n)} - b_{j-1}\frac{x_1}{x_j}, \quad j = 2,\ldots,p+1. \tag{18}$$

Substituting a_j into (16), we have

$$b_{j-1}^2 \prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i) - \frac{b_{j-1}x_1\varphi_{j-1}(\lambda_1^{(j)})}{x_j} + (\lambda_2^{(n)} - \lambda_1^{(j)})\varphi_{j-1}(\lambda_1^{(j)}) = 0, \tag{19}$$

condition (a) holds, which implies $b_{j-1} > 0$ for $j = 2,\ldots,p+1$.

For $j = p+2,\ldots,p+m$, by (16) we have

$$\varphi_j(\lambda_1^{(j)}) = 0 \Rightarrow (\lambda_1^{(j)} - a_j)\varphi_{j-1}(\lambda_1^{(j)}) - b_{j-1}^2\varphi_{j-2}(\lambda_1^{(j)}) = 0$$

$$\Rightarrow a_j = \lambda_1^{(j)} - \frac{b_{j-1}^2\varphi_{j-2}(\lambda_1^{(j)})}{\varphi_{j-1}(\lambda_1^{(j)})}. \tag{20}$$

By (5) we have

$$b_{j-1} = \frac{(-1)^j\varphi_{j-1}(\lambda_2^{(n)})x_1}{x_j\prod_{i=p+1}^{j-2} b_i \prod_{k=2}^{p+1}(a_k - \lambda_2^{(n)})}, \qquad j = p+2,,\ldots,p+m. \tag{21}$$

Since $x_j \neq 0$, then by (21), $b_{j-1} \neq 0$. By successively using (20) and (21) a_j and b_{j-1} are obtained. Finally by (12) we get

$$a_j = \lambda_2^{(n)} - b_{j-1}\frac{x_{(p+m)}}{x_j}, \quad j = p+m+1,\ldots,n. \tag{22}$$

Substituting a_j into (16) we have

$$b_{(j-1)}^2\varphi_{p+m-1}(\lambda_1^{(j)}) \prod_{i=p+m+1}^{j-1}(\lambda_1^{(j)} - a_i) - \frac{b_{(j-1)}x_1\varphi_{j-1}(\lambda_1^{(j)})}{x_j} + (\lambda_2^{(n)} - \lambda^{(j)})\varphi_{j-1}(\lambda_1^{(j)}) = 0, \tag{23}$$

condition (b) holds, which implies $b_{j-1} > 0$ for $j = p+m+1,\ldots,n$. $\square$

The following algorithm solves the IEP1. All recurrent relations involve the square of entries b_{j-1}. In the algorithm we choose the $b_{j-1} > 0$.

Algorithm 1 To solve problem **IEP1**

Input: $p, m, q, \lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_1^{(n)}, \lambda_2^{(n)}$, where $p+m+q=n$.
$a_1 = \lambda_1^{(1)}$.
for $j = 2$ to $p+1$ **do**
    replacing $a_1, a_2, \dots, a_{j-1}$ *and* $b_1, b_2, \dots, b_{j-2}$ *into* (19) *to finding the two solutions* α_1 *and* α_2.
  if $\alpha_1 < 0$ *and* $\alpha_2 < 0$ **then**
    ending the algorithm.
  end if
  if $\alpha_1 > 0$ **then**
    $b_{j-1} = \alpha_1$, *computing* a_j *by* (19).
  end if
  if $\alpha_2 > 0$ **then**
    $b'_{j-1} = \alpha_2$, *computing* a'_j *by* (19).
  end if
end for
for $j = p+2$ to $p+m$ **do**
    $a_j = \lambda_1^{(j)} - \frac{b_{j-1}^2 \varphi_{j-2}(\lambda_1^{(j)})}{\varphi_{j-1}(\lambda_1^{(j)})}. \qquad b_{j-1} = \frac{(-1)^j \varphi_{j-1}(\lambda_2^{(n)}) x_1}{x_j \prod_{i=p+1}^{j-2} b_i \prod_{k=2}^{p+1} (a_k - \lambda_2^{(n)})}$
end for
for $j = p+m+1$ to n **do**
    replacing $a_{p+m}, a_{p+m+1}, \dots, a_{j-1}$ *and* $b_{p+m-1}, b_{p+m}, \dots, b_{j-2}$ *into* (23) *to finding the two solutions* β_1 *and* β_2.
  if $\beta_1 < 0$ *and* $\beta_2 < 0$ **then**
    ending the algorithm.
  end if
  if $\beta_1 > 0$ **then**
    $b_{j-1} = \beta_1$, *computing* a_j *by* (23).
  end if
  if $\beta_2 > 0$ **then**
    $b'_{j-1} = \beta_2$, *computing* a'_j *by* (23).
  end if
end for

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4 The nonnegative case

In this section, we examine the conditions for the existence of nonnegative matrix A_n of the form (1).

Theorem 2. *Let the list of real numbers $\lambda_2^{(n)}$, $\lambda_1^{(j)}$, $j=1,\dots,n$, and real vector $x=(x_1,x_2,\dots,x_n)$. Then, there exists a nonnegative matrix A_n of the form (1), such that $\lambda_2^{(n)}$ is the maximal eigenvalue of A_n, $\lambda_1^{(j)}$ is the minimal eigenvalue of the leading principal submatrix A_j of A_n and $(\lambda_2^{(n)},x)$ is an eigenpair of A_n if the following conditions are satisfied*

$$x_j > 0, \qquad j=1,\dots,n; \tag{24}$$

$$\lambda_1^{(1)} \geq 0; \tag{25}$$

$$\frac{\lambda_2^{(n)}}{b_{j-1}} \geq \frac{x_1}{x_j}, \qquad j=2,\dots,p+2; \tag{26}$$

$$\frac{\lambda_1^{(j)}}{b_{j-1}^2} \geq \frac{\varphi_{j-2}(\lambda_1^{(j)})}{\varphi_{j-1}(\lambda_1^{(j)})}, \qquad j=p+3,\dots,p+m+1; \tag{27}$$

$$\frac{\lambda_2^{(n)}}{b_{j-1}} \geq \frac{x_{p+m}}{x_j}, \qquad j=p+m+2,\dots,n. \tag{28}$$

Proof. Suppose the conditions (24)-(28) and (2) hold. Theorem 1 confirms the existence of the matrix of the form (1) with positive value b_{j-1}, for $j=2,\dots,n$. We need to show that the diagonal elements a_j are nonnegative.

From (25) we have $\lambda_1^{(1)} \geq 0$, then $a_1 = \lambda_1^{(1)} \geq 0$, and from (2) we obtain $0 \leq \lambda_1^{(1)} < \lambda_1^{(j)} < \lambda_2^{(n)}$, for $j=2,\dots,n$.

We consider the following three cases to discuss the nonnegativity of a_j for $j=2,\dots,n$.

(1) For $j=2,\dots,p+2$, from (26) we have

$$m_j = \lambda_2^{(n)} x_j - b_{j-1} x_1 \geq 0.$$

Hence, from the Theorem 1 and (24), we obtain

$$a_j = \frac{m_j}{x_j} \geq 0.$$

(2) For $j=p+3,\dots,p+m+1$, by multiplying the denominator and numerator of the right-hand fraction in inequality (27) by $(-1)^{j-1}$, we gain

$$\frac{\lambda_1^{(j)}}{b_{j-1}^2} \geq \frac{(-1)^{j-1}\varphi_{j-2}(\lambda_1^{(j)})}{(-1)^{j-1}\varphi_{j-1}(\lambda_1^{(j)})}.$$

By Lemma 1 and inequality (2) we have

$$(-1)^{j-1}\varphi_{j-1}(\lambda_1^{(j)}) > 0.$$

Then

$$(-1)^{j-1}\lambda_1^{(j)}\varphi_{j-1}(\lambda_1^{(j)}) \geq (-1)^{j-1} b_{j-1}^2 \varphi_{j-2}(\lambda_1^{(j)}),$$

or

$$t_j = (-1)^{j-1}\left\{\lambda_1^{(j)}\varphi_{j-1}(\lambda_1^{(j)}) - b_{j-1}^2\varphi_{j-2}(\lambda_1^{(j)})\right\} \geq 0.$$

From the Theorem 1 we obtain

$$a_j = \frac{t_j}{(-1)^{j-1}\varphi_{j-1}(\lambda_1^{(j)})} \geq 0.$$

(3) Finally when $j = p+m+2,\ldots,n$, from (28) we have

$$z_j = \lambda_2^{(n)}x_j - b_{j-1}x_{p+m} \geq 0.$$

Hence, from the proof of Theorem 1 and (24), we obtain

$$a_j = \frac{z_j}{x_j} \geq 0.$$

Therefore, the values of $a_j \geq 0$ for all $j = 1,\ldots,n$, and $b_{j-1} > 0$ for all $j = 2,\ldots,n$. This means that the matrix A_n is nonnegative. Thus the proof is completed. □

5 Numerical example

We test **Algorithm 1** for some examples by the MATLAB software. In this section, we provide one of numerical examples.

Example 1. For given 10 real numbers

$$\begin{array}{cccccccccc} \lambda_1^{(9)}, & \lambda_1^{(8)}, & \lambda_1^{(7)}, & \lambda_1^{(6)}, & \lambda_1^{(5)}, & \lambda_1^{(4)}, & \lambda_1^{(3)} & \lambda_1^{(2)}, & \lambda_1^{(1)}, & \lambda_2^{(9)}, \\ -9, & -7.3, \quad , & -6, & -3.7, & -3, & -2.5, & -1.1, & -0.5, & 1, & 14.0641, \end{array}$$

and a real vector

$$x = (0.0193,\ 0.0027,\ 0.0031,\ 0.0026,\ 0.0887,\ 0.2396,\ 0.6144,\ 0.3723,\ 0.6467\ )^T.$$

Find a matrix A_9 of the form (1) such that $\lambda_1^{(j)}$ is the minimal of A_j, $j = 1,2,\ldots 9$, moreover $(\lambda_2^{(9)}, x)$ is its eigenpair.

By applying **Algorithm 1**, we gain the following matrix as the solution

$$A_9 = \begin{bmatrix} 1.0000 & 1.7378 & 1.8400 & 1.9619 & 2.6536 & 0 & 0 & 0 & 0 \\ 1.7378 & 1.5134 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.8400 & 0 & 2.4851 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.9619 & 0 & 0 & -0.6390 & 0 & 0 & 0 & 0 & 0 \\ 2.6536 & 0 & 0 & 0 & 3.5002 & 3.6925 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.6925 & -0.4221 & 5.1113 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.1113 & 0.6675 & 6.0407 & 7.3430 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6.0407 & 4.0824 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7.3430 & 0 & 7.0736 \end{bmatrix}.$$

The eigenvalues of all of the leading principal submatrices are

$$
\begin{aligned}
\sigma(A_1) &= \{\underline{1.0000}\}\\
\sigma(A_2) &= \{\underline{-0.5000}, 3.0291\}\\
\sigma(A_3) &= \{\underline{-1.1000}, 1.9581, 4.1838\}\\
\sigma(A_4) &= \{\underline{-2.5000}, 0.4116, 2.0108, 4, 4980\}\\
\sigma(A_5) &= \{\underline{-3.0000}, 0.1641, 1.8716, 2.8758, 6.0323\}\\
\sigma(A_6) &= \{\underline{-3.7000}, -1.9089, 0.3645, 1.9753, 3.7256, 7.0010\}\\
\sigma(A_7) &= \{\underline{-6.0000}, -2.8475, 0.1808, 1.8188, 2.4844, 4.7692, 7.8072\}\\
\sigma(A_8) &= \{\underline{-7.3000}, -3.0532, -0.4441, 0.5591, 1.9924, 3.7467, 6.6812, 10.0506\}\\
\sigma(A_9) &= \{\underline{-9.0000}, -3.2112, -0.9780, 0.4500, 1.9858, 3.7445, 5.1871, 7.0016, \underline{14.0641}\}.
\end{aligned}
$$

To verify $A_9 x_9 = \lambda_2^{(9)} x_9$, we compute both terms

$$A_9 x_9 = (0.2679, 0.0373, 0.0428, 0359, 1.2383, 3.3691, 8.6839, 5.2341, 9.0573)^T$$

and

$$\lambda_2^{(9)} x_9 = (0.2679, 0.0373, 0.0428, 0359, 1.2383, 3.3691, 8.6839, 5.2341, 9.0573)^T.$$

6 Conclusions

In the current paper, a partially described inverse eigenvalue problem was considered for construction of specific symmetric matrix. The problem involves the construction of this matrix by one eigenpair of the required matrix and minimal eigenvalue of all leading principal submatrices. The relation for gaining the element x_j of the given eigenvector x from the elements of leading principal submatrices is important in gaining the solution. The significance of the IEP1 stems in the fact that it partially describes inverse eigenvalue problem while it constructs matrices from partial eigendata. Such partially described problems may be encountered in computations in which obtaining the entire spectrum is difficult.

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