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Parameter-uniform fitted operator method for singularly perturbed Burgers-Huxley equation

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Abstract. We develop a robust uniformly convergent numerical scheme for singularly perturbed time dependent Burgers-Huxley partial differential equation. We first discretize the time derivative of the equation using the Crank-Nicolson finite difference method. Then, the resulting semi-discretized nonlinear ordinary differential equations are linearized using the quasilinearization technique, and finally, design a fitted operator upwind finite difference method to resolve the layer behavior of the solution in the spatial direction. Our analysis has shown that the presented method is second order parameter uniform convergent in time and first order in space. Numerical experiments are conducted to validate the theoretical results.

Keywords: Singularly perturbed problem, Burgers-Huxley equation, Crank-Nicolson finite difference scheme, fitted operator method, parameter uniform convergence.
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1 Introduction

In this article, we examine the numerical solution of the following singularly perturbed time dependent Burgers-Huxley equation on the domain $D \equiv \Omega \times (0,T] \equiv (0,1) \times (0,T]$

$$\begin{cases} \frac{\partial u}{\partial t} + \mathscr{L}_\varepsilon u(x,t) = 0, & (x,t) \in D, \\ u(x,0) = f(x), & x \in \Omega, \\ u(0,t) = h_0(t), \ \ u(1,t) = h_1(t), & t \in [0,T], \end{cases} \tag{1}$$

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where

$$\mathscr{L}_\varepsilon u(x,t) \equiv -\varepsilon u_{xx} + \alpha u u_x - \beta(1-u)(u-\gamma)u,$$

$0 < \varepsilon \ll 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \in (0,1)$ are real parameters, and the prescribed functions $h_0(t)$, $h_1(t)$ and $f(x)$ are sufficiently smooth and satisfy the compatibility conditions $h_0(0) = f(0)$ and $h_1(0) = f(1)$. This equation was investigated by Satsuma [28] in 1986. It describes the interaction between reaction mechanism, convection effect and diffusion transport. It is the combination of Burgers [8] and Hodgkin-Huxley [14] equations. They have various applications in nonlinear physics.

For $\varepsilon = 1$, [1,2,32] presented a number of numerical techniques that can be used to solve the Burgers-Huxley equation in the framework of nonstandard and exponential finite difference methods. The authors analyzed the performance of the different techniques relative to various combinations of parameters involved. However, these methods cannot be said to be parameter-uniform as they are only designed for the case $\varepsilon = 1$.

When the perturbation parameter $\varepsilon \to 0$ and for suitable values of α and β, the problem (1) has a unique solution $u(x,t)$ which exhibits one or two boundary layer(s) [17].

The singularly perturbed Burgers-Huxley problem (1) models various situations that occur in applied mathematics and engineering such as in population genetics [3], fluid mechanics [4], turbulence flow [6], nerve propagation and wall in liquid crystals [29], and shock waves [31]. In order to solve the problem (1), asymptotic expansion methods were proposed by several scholars, see for instance [28–30]. But, there are difficulties in finding the appropriate asymptotic expansions in the inner and the outer regions. It is, therefore, necessary to develop numerical methods that generate better approximations of the solution.

In the last few decades several standard/classical numerical methods have been developed to solve the Burgers-Huxley equation. Some of these methods are the Adomian decomposition methods [13, 15], finite difference methods [25, 27], finite element methods [5, 18, 23, 24], and spectral methods [10, 16]. However, these standard numerical methods lead to severe restrictions on the mesh size due to the presence of boundary layer(s) to attain stability for small values of the perturbation parameter ε. To avoid these restrictions, parameter uniform convergent numerical schemes are constructed. These methods are known as fitted numerical methods [22] and are categorized into fitted operator methods (FOMs) and fitted mesh methods (FMMs).

In recent studies various parameter uniformly convergent methods have been developed to solve (1) using FMMs. To cite a few, Kaushik *et al.* [17] developed a uniform convergent scheme for the problem (1). They used the backward-Euler scheme to discretize the temporal direction on a uniform mesh and an upwind finite difference scheme for the spatial discretization on the Shishkin mesh. They obtained first order in the time and almost first order in the space except for a logarithmic factor. Kadalbajoo *et al.* [12] modified this scheme using a hybrid finite difference operator to discretize the space derivatives on a Shishkin mesh. They obtained first order parameter uniform in time direction. For the space direction, they have shown first order convergence on the outside and almost second order uniform convergence inside the boundary layer regions.

In [19], Liu *et al.* used classical backward-Euler finite difference for the time derivative on a uniform mesh and an upwind finite difference scheme for the space derivative on an equidistributed grid to develop an ε-uniform convergent scheme for (1). Their analysis has shown a first order rate of convergence both in time and space.

Up to the best of our knowledge, the finite difference schemes available in the literature for the singularly perturbed Burgers-Huxley equation (1) are based on fitted mesh methods and are first order uniformly convergent both in time and space up to a logarithmic factor. It is known that the logarithm factor alters the convergence rate and the accuracy of such methods (especially those built on Shishkin type meshes) [22]. The FOMs, on the other hand, enjoy a high nodal accuracy.

In this work, we present an ε-uniform convergent numerical method for solving the problem (1). The proposed method utilizes the Crank-Nicolson scheme for the time discretization and a FOM for the space discretization. This guarantees that the time discretization is second order accurate while the space discretization is (exactly) first order accurate. The nonlinearity of the problem will be dealt with via the Newton-Kantorovich quasilinearization technique [7].

Remark 1. *Throughout the paper C denotes a generic positive constant which is independent of the parameter ε and the mesh sizes. $C^{2,1}(\bar{D})$ denotes a collection of functions $\Phi(x,t)$ on $\bar{D}$ which are once differentiable in time and twice differentiable in space. Also, $\|\cdot\|$ is used as the maximum norm.*

2 A priori bounds

Here, we present a priori bounds for the solution of the continuous problem (1) and its time derivatives which are important for the convergence analysis of the time discrete scheme. First, let us present the following maximum principle for the operator $\frac{\partial}{\partial t} + \mathscr{L}_\varepsilon$ defined in (1).

Lemma 1. *Let $\Phi(x,t) \in C^{2,1}(\bar{D})$. If $\Phi(x,t) \geq 0$, $\forall (x,t) \in \partial D$ (boundary of D) and $\left(\frac{\partial}{\partial t} + \mathscr{L}_\varepsilon\right)\Phi(x,t) \geq 0$, $\forall (x,t) \in D$, then $\Phi(x,t) \geq 0$, $\forall (x,t) \in \bar{D}$.*

Proof. See the proof in [12, Lemma 1]. □

This lemma leads to the following uniform stability result [12].

Lemma 2. *Let $u(x,t)$ be the solution of (1), then $\|u\|_{\bar{D}} \leq \|u\|_{\partial D}$.*

Lemma 3. *The bounds for the solution $u(x,t)$ of (1) and its time derivatives are given by*

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \leq C, \qquad \forall (x,t) \in D, \text{ for } i = 0,1,2.$$

Proof. Refer to [19, Lemma 2.3]. □

3 Time semi-discretization

The time domain $[0,T]$ is discretized into N number of uniform meshes, each of length $\Delta t = T/N$. Let $0 = t_0,\ t_N = T$, and $t_n = n\Delta t,\ n = 0,1,2,\dots,N$, be the mesh points.

To approximate the time derivative term of the Burgers-Huxley equation (1), we use the Crank-Nicolson finite difference, which gives a system of two point nonlinear boundary value problems

$$\begin{cases} u^0 = u(x,0) = f(x), & x \in \Omega, \\ (I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)u^{n+1} = (I - \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)u^n, & \\ u^{n+1}(0) = h_0(t_{n+1}),\ \ u^{n+1}(1) = h_1(t_{n+1}), & n = 0,1,\dots,N-1, \end{cases} \tag{2}$$

where

$$\bar{\mathscr{L}}_\varepsilon u^{n+1}(x) \equiv -\varepsilon u_{xx}^{n+1} + \alpha u^{n+1} u_x^{n+1} - \beta(1-u^{n+1})(u^{n+1}-\gamma)u^{n+1},$$

and $u^n(x)$ is the semi-discrete approximation to the exact solution $u(x,t_n)$ of (1) at time level $t_n = n\Delta t$, $0 \leq n \leq N-1$.

3.1 Convergence analysis for the time semi-discretization

In order to analyze the uniform convergence of the solution $u^n(x)$ of (2) to the exact solution $u(x,t_n)$, we will do the stability analysis and the consistency result of the scheme (2). First, let us observe the following semi-discrete maximum principle for the operator $I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon$ defined in (2).

Lemma 4. *If $\Phi^{n+1}(0) \geq 0$, $\Phi^{n+1}(1) \geq 0$ and $(I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)\Phi^{n+1}(x) \geq 0$ for all $x \in \Omega$, then $\Phi^{n+1}(x) \geq 0$ for all $x \in \bar{\Omega}$.*

Proof. Suppose there is $x^* \in \bar{\Omega}$ such that $\Phi^{n+1}(x^*) = \min_{x\in\bar{\Omega}} \Phi^{n+1}(x) < 0$. From the given hypothesis and second derivative test, we have $x^* \neq 0$, $x^* \neq 1$, $\Phi_x^{n+1}(x^*) = 0$ and $\Phi_{xx}^{n+1}(x^*) > 0$. Then, from (2) we have

$$\begin{aligned}
(I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)\Phi^{n+1}(x^*) = & \ \Phi^{n+1}(x^*) + \frac{\Delta t}{2}\left(-\varepsilon\Phi_{xx}^{n+1} + \alpha\Phi^{n+1}\Phi_x^{n+1} - \beta(1-\Phi^{n+1})(\Phi^{n+1}-\gamma)\Phi^{n+1}\right)(x^*) \\
= & \ \Phi^{n+1}(x^*) + \frac{\Delta t}{2}\left(-\varepsilon\Phi_{xx}^{n+1}(x^*) - \beta(1-\Phi^{n+1}(x^*))(\Phi^{n+1}(x^*)-\gamma)\Phi^{n+1}(x^*)\right) \\
< & \ 0,
\end{aligned}$$

which contradicts the given assumption and thus $\Phi^{n+1}(x) \geq 0$ for all $x \in \bar{\Omega}$. □

This maximum principle leads to the following stability result

$$\|(I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)^{-1}\| \leq C. \tag{3}$$

The local truncation error e_{n+1} associated to the semi-discrete scheme (2) is given by $e_{n+1} = u(x,t_{n+1}) - \hat{u}^{n+1}(x)$, where $\hat{u}^{n+1}(x)$ is the solution of the following boundary value problem

$$\begin{cases} (I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)\hat{u}^{n+1} = (I - \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)u(x,t_n), & x \in \Omega, \\ \hat{u}^{n+1}(0) = h_0(t_{n+1}), \ \ \hat{u}^{n+1}(1) = h_1(t_{n+1}), & n = 0,1,\ldots,N-1. \end{cases} \tag{4}$$

Lemma 5 (Local error estimate). *Estimate of the local error e_{n+1} is given by*

$$\|e_{n+1}\| \leq C\Delta t^3. \tag{5}$$

Proof. Since the solution $u(x,t)$ of (1) is sufficiently smooth, by using Taylor series expansion, we have

$$u(x,t_n) = u(x,t_{n+1}) - \Delta t u_t(x,t_{n+\frac{1}{2}}) - \frac{\Delta t^3}{24}u_{ttt}(x,\eta), \tag{6}$$

for some $\eta \in [t_n, t_{n+1}]$, and $u(x,t_n)$ is the exact solution of (1) at nth time level.

Using (2), Eq. (6) leads to

$$u(x,t_n) = u(x,t_{n+1}) + \frac{\Delta t}{2}\left(\bar{\mathscr{L}}_\varepsilon u(x,t_{n+1}) + \bar{\mathscr{L}}_\varepsilon u(x,t_n)\right) - \frac{\Delta t^3}{24} u_{ttt}(x,\eta). \tag{7}$$

This implies that

$$(I - \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)u(x,t_n) = (I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon)u(x,t_{n+1}) - \frac{\Delta t^3}{24} u_{ttt}(x,\eta). \tag{8}$$

Subtracting (4) from (8) suggests that e_{n+1} is the solution of the boundary value problem

$$\left(I + \frac{\Delta t}{2}\bar{\mathscr{L}}_\varepsilon\right) e_{n+1} = O(\Delta t^3), \tag{9}$$
$$e_{n+1}(0) = 0,\ e_{n+1}(1) = 0.$$

Using the stability result (3), we obtain $\|e_{n+1}\| \le C\Delta t^3$. □

The global error of the time semi-discretization is given by

$$E_{n+1} = u(x,t_{n+1}) - u^{n+1}(x) = \sum_{i=1}^{n+1} e_i.$$

Using the local error estimate result (5) and triangular inequality the following consistency result follows.

Theorem 1 (Global error estimate). *Estimate of the global error E_{n+1} is given by*

$$\|E_{n+1}\| \le C\Delta t^2, \quad n\Delta t \le T. \tag{10}$$

Therefore, the semi-discrete scheme (2) is a second order uniformly convergent in time.

3.2 Quasilinearization

The nonlinear boundary value problem (2) can be linearized using Newton-Kantorovich quasilinearization approach [7] and followed by simplification yields the following sequence of linear boundary value problems

$$\begin{cases} u^{0,(j+1)} = f(x), \qquad x \in \bar{\Omega} \\ -\varepsilon u_{xx}^{n+1,(j+1)} + \alpha u^{n+1,(j)} u_x^{n+1,(j+1)} + \left(\frac{2}{\Delta t} + \alpha u_x^{n+1,(j)}(x)\right. \\ +\beta\left(3[u^{n+1,(j)}]^2 - 2(1+\gamma)u^{n+1,(j)} + \gamma\right)\Big) u^{n+1,(j+1)} \\ \quad = \varepsilon u_{xx}^{n,(j+1)} + \left(\frac{2}{\Delta t} - \alpha u_x^{n,(j+1)}\right) u^{n,(j+1)} + \beta\left(1 - u^{n,(j+1)}\right)(u^{n,(j+1)} - \gamma)\, u^{n,(j+1)} \\ \qquad + \left(\alpha {u_x}^{n+1,(j)} + \beta\left(2[u^{n+1,(j)}]^2 - (1+\gamma)u^{n+1,(j)}\right)\right) u^{n+1,(j)} \qquad x \in \Omega, \\ u^{n+1,(j+1)}(0) = h_0(t_{n+1}),\ \ u^{n+1,(j+1)}(1) = h_1(t_{n+1}), \qquad n = 0,1,\ldots,N-1, \end{cases} \tag{11}$$

where $u^{n+1,(j)}$ is the jth nominal solution of the semi-discrete scheme (2) with initial guess $u^{n+1,(0)}$ and $j = 0,1,2,\ldots,$ is the iteration index. It is shown in [17] that the convergence of the Newton-Kantorovich quasilinearization (11) is quadratic.

For simplicity, we use the following notations

$$
\begin{aligned}
U^{n+1}(x) &= u^{n+1,(j+1)}(x),\\
a^{(j)}(x) &= \alpha u^{n+1,(j)}(x),\\
b^{(j)}(x) &= \frac{2}{\Delta t} + \alpha u_x^{n+1,(j)}(x) + \beta\left(3[u^{n+1,(j)}]^2 - 2(1+\gamma)u^{n+1,(j)} + \gamma\right),\\
F^{(j)}(x) &= \varepsilon u_{xx}^{n,(j+1)} + \left(\frac{2}{\Delta t} - \alpha u_x^{n,(j+1)}\right)u^{n,(j+1)} + \beta\left(1-u^{n,(j+1)})(u^{n,(j+1)}-\gamma\right)u^{n,(j+1)}\\
&\quad + \left(\alpha {u_x}^{n+1,(j)} + \beta\left(2[u^{n+1,(j)}]^2 - (1+\gamma)u^{n+1,(j)}\right)\right)u^{n+1,(j)}.
\end{aligned}
$$

Using the above notations, (11) can be written as

$$
\begin{cases}
U^0(x) = f(x),\\
\mathscr{L}_\varepsilon^N U^{n+1}(x) = F^{(j)}(x), \qquad x \in \Omega,\\
U^{n+1}(0) = h_0(t_{n+1}),\ \ U^{n+1}(1) = h_1(t_{n+1}),
\end{cases} \tag{12}
$$

where

$$
\mathscr{L}_\varepsilon^N U^{n+1}(x) = -\varepsilon U_{xx}^{n+1} + a^{(j)}(x)U_x^{n+1} + b^{(j)}(x)U^{n+1}.
$$

Let p and q be positive numbers such that $a^{(j)}(x) \geq p > 0,\ b^{(j)}(x) \geq q > 0$.

Assuming that $a^{(j)}(x)$, $b^{(j)}(x)$ and $F^{(j)}(x)$ are sufficiently smooth functions, then the linear singularly perturbed boundary value problem (12) has a unique solution that exhibits a boundary layer at $x = 1$ when $\varepsilon \to 0$ (See in [22]).

In the uniform convergence analysis of the fully-discrete problem, we require the following asymptotic behavior of the solution $U^{n+1}(x)$ of (12) and its derivatives in space.

Lemma 6. *Let $U^{n+1}(x)$ be the solution of the linearized semi-discrete scheme (12), the bounds of $U^{n+1}(x)$ and its derivatives are*

$$
\left|\frac{\partial^i U^{n+1}}{\partial x^i}\right| \leq C\left(1 + \varepsilon^{-i}\exp(-p(1-x)/\varepsilon)\right),\ x \in \bar{\Omega},\ i = 0,1,2,3. \tag{13}
$$

Proof. The proof can be found in [9]. □

4 The spatial discretization

The spatial domain $\Omega = [0,1]$ is discretized into M equal number of mesh elements each of length $h = 1/M$. This gives the spatial mesh

$$
\bar{\Omega}^M = \{x_m = mh,\ m = 0,1,2,\ldots,M,\ x_0 = 0,\ x_M = 1,\ h = 1/M\},
$$

where the x_m's are mesh points. The semi-discrete problem (12) can be fully-discretized using the non-standard finite difference methodology of Mickens [20] as follows

$$
\begin{cases}
U_m^0 = f(x_m), \qquad 1 \leq m \leq M-1,\\
\mathscr{L}_\varepsilon^{M,N} U_m^{n+1} \equiv -\varepsilon\left[\frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{\psi_m^2(h,\varepsilon)}\right] + a_m^{(j)}\left[\frac{U_m^{n+1} - U_{m-1}^{n+1}}{h}\right] + b_m^{(j)} U_m^{n+1} = F_m^{(j)}, \qquad 1 \leq m \leq M-1,\\
U_0^{n+1} = h_0(t_{n+1}),\ \ U_M^{n+1} = h_1(t_{n+1}),
\end{cases} \tag{14}
$$

where the denominator function ψ_m is

$$\psi_m^2(h,\varepsilon) = \frac{h\varepsilon}{a_m^{(j)}}\left(\exp(\frac{a_m^{(j)}h}{\varepsilon}) - 1\right), \quad m = 1,2,\ldots,M-1.$$

For details about how this denominator function is derived, interested readers may refer to [21]. Here, U_m^{n+1} is the numerical approximation of the exact solution $u(x_m,t_{n+1})$ at $j+1$ iteration and

$$a_m^{(j)} = a^{(j)}(x_m),\ b_m^{(j)} = b^{(j)}(x_m),\ F_m^{(j)} = F^{(j)}(x_m).$$

The fully-discretized equation (14) can be written as:

$$\begin{cases} U_m^0 = f(x_m), \qquad 1 \le m \le M-1, \\ E_m^{n+1}U_{m-1}^{n+1} + F_m^{n+1}U_m^{n+1} + G_m^{n+1}U_{m+1}^{n+1} = H_m^{n+1}, \qquad 1 \le m \le M-1, \\ U_0^{n+1} = h_0(t_{n+1}),\ \ U_M^{n+1} = h_1(t_{n+1}), \end{cases} \tag{15}$$

where

$$\begin{aligned} E_m^{n+1} &= -\frac{\varepsilon}{\psi_m^2} - \frac{a_m^{(j)}}{h}, \\ F_m^{n+1} &= \frac{2\varepsilon}{\psi_m^2} + \frac{a_m^{(j)}}{h} + b_m^{(j)}, \\ G_m^{n+1} &= -\frac{\varepsilon}{\psi_m^2}, \\ H_m^{n+1} &= \frac{\varepsilon}{\psi_m^2}(U_{m-1}^n - 2U_m^n + U_{m+1}^n) + \left(\frac{2}{\Delta t} - \alpha\frac{(U_m^n - U_{m-1}^n)}{h}\right)U_m^n + \beta(1-U_m^n)(U_m^n-\gamma)U_m^n \\ &\quad + \left(\alpha\frac{(u_m^{n+1,(j)} - u_{m-1}^{n+1,(j)})}{h} + \beta\left(2[u_m^{n+1,(j)}]^2 - (1+\gamma)u_m^{n+1,(j)}\right)\right)u_m^{n+1,(j)}. \end{aligned}$$

Eq. (15) is a tridiagonal system which can be solved using Thomas Algorithm.

5 Convergence analysis of the fully-discrete scheme

The fully-discrete finite difference operator $\mathscr{L}_\varepsilon^{M,N}$ of (14) satisfies the following maximum principle.

Lemma 7. *Let Φ_m^{n+1}, $m = 0,1,\ldots,M$ be discrete mesh functions. If $\Phi_0^{n+1} \ge 0$, $\Phi_M^{n+1} \ge 0$ and $\mathscr{L}_\varepsilon^{M,N}\Phi_m^{n+1} \ge 0$, for $1 \le m \le M-1$, then $\Phi_m^{n+1} \ge 0$, for $0 \le m \le M$.*

Proof. Assume there is m^* such that $\Phi_{m^*}^{n+1} = \min_{0\le m\le M}\Phi_m^{n+1} < 0$. From the given hypothesis, we have $m^* \ne 0$, $m^* \ne M$, $\Phi_{m^*+1}^{n+1} - \Phi_{m^*}^{n+1} \ge 0$, and $\Phi_{m^*-1}^{n+1} - \Phi_{m^*}^{n+1} \ge 0$. Then, from (14), we have

$$\begin{aligned} \mathscr{L}_\varepsilon^{M,N}\Phi_{m^*}^{n+1} &= -\varepsilon\left[\frac{\Phi_{m^*+1}^{n+1} - 2\Phi_{m^*}^{n+1} + \Phi_{m^*-1}^{n+1}}{\psi_{m^*}^2}\right] + a_{m^*}^{(j)}\left[\frac{\Phi_{m^*}^{n+1} - \Phi_{m^*-1}^{n+1}}{h}\right] + b_{m^*}^{(j)}\Phi_{m^*}^{n+1} \\ &= -\varepsilon\left[\frac{\Phi_{m^*+1}^{n+1} - \Phi_{m^*}^{n+1} + \Phi_{m^*-1}^{n-1} - \Phi_{m^*}^{n+1}}{\psi_{m^*}^2}\right] + a_{m^*}^{(j)}\left[\frac{\Phi_{m^*}^{n+1} - \Phi_{m^*-1}^{n+1}}{h}\right] + b_{m^*}^{(j)}\Phi_{m^*}^{n+1} \\ &< 0, \end{aligned}$$

which contradicts the given assumption $\mathscr{L}_\varepsilon^{M,N}\Phi_m^{n+1} \geq 0$ for $1 \leq m \leq M-1$ and thus $\Phi_m^{n+1} \geq 0$, for $0 \leq m \leq M$. □

The following fully-discrete stability result is important to prove the ε-uniform convergence of the proposed method.

Lemma 8. *Let Φ_m^{n+1}, $m = 0,1,\dots,M$ be any mesh functions such that $\Phi_0^{n+1} = \Phi_M^{n+1} = 0$, then*

$$|\Phi_m^{n+1}| \leq \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}|, \ \ 0 \leq m \leq M,$$

where $b^{(j)}(x_m) \geq q > 0$.

Proof. Define

$$[\Psi^{\pm}]_m^{n+1} = \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}| \pm \Phi_m^{n+1}.$$

Then, we have

$$\begin{aligned}
[\Psi^{\pm}]_0^{n+1} &= \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}| \pm \Phi_0^{n+1} = \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}| \geq 0,\\
[\Psi^{\pm}]_M^{n+1} &= \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}| \pm \Phi_M^{n+1} = \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}| \geq 0,
\end{aligned}$$

and, for $m = 1,2,\dots,M-1$,

$$\begin{aligned}
\mathscr{L}_\varepsilon^{M,N}[\Psi^{\pm}]_m^{n+1} &= \mathscr{L}_\varepsilon^{M,N}\big(\frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}| \pm \Phi_m^{n+1}\big)\\
&= \frac{b^{(j)}(x_m)}{q}\max_{0\leq m\leq M}|\mathscr{L}_\varepsilon^{M,N}\Phi_m^{n+1}| \pm \mathscr{L}_\varepsilon^{M,N}\Phi_m^{n+1}\\
&\geq \max_{0\leq m\leq M}|\mathscr{L}_\varepsilon^{M,N}\Phi_m^{n+1}| \pm \mathscr{L}_\varepsilon^{M,N}\Phi_m^{n+1}, \qquad \text{since } b^{(j)}(x_m) \geq q\\
&\geq 0.
\end{aligned}$$

Here, $[\Psi^{\pm}]_m^{n+1}$ satisfies the hypothesis of the fully-discrete maximum principle given in Lemma 7, hence $[\Psi^{\pm}]_m^{n+1} \geq 0, \ 0 \leq m \leq M$. This gives the required result

$$|\Phi_m^{n+1}| \leq \frac{1}{q}\max_{1\leq i\leq M-1}|\mathscr{L}_\varepsilon^{M,N}\Phi_i^{n+1}|, \ \ 0 \leq m \leq M.$$

Hence, the proof is complete. □

Theorem 2 (Error estimate in the spatial direction). *Let $U^{n+1}(x)$ be the solution of the semi-discrete problem (12) and U_m^{n+1} be the solution of the fully-discrete problem (14). Then, the error estimate is given by*

$$\left|U^{n+1}(x_m) - U_m^{n+1}\right| \leq Ch, \ 0 \leq m \leq M.$$

Proof. The truncation error of the fully-discrete scheme (14) is given by

$$\begin{aligned}\mathscr{L}_\varepsilon^{M,N}\left(U^{n+1}(x_m)-U_m^{n+1}\right) = &\left[-\varepsilon(U^{n+1})''(x_m)+a^{(j)}(x_m)(U^{n+1})'(x_m)+b^{(j)}(x_m)U^{n+1}(x_m)\right]\\ &-\left[-\varepsilon\frac{h^2}{\psi_m^2}D^+D^-U^{n+1}(x_m)+a^{(j)}(x_m)D^-U^{n+1}(x_m)+b^{(j)}(x_m)U^{n+1}(x_m)\right]\\ =\,&\varepsilon(\frac{h^2}{\psi_m^2}D^+D^- - \frac{d^2}{dx^2})U^{n+1}(x_m)+a^{(j)}(x_m)(\frac{d}{dx}-D^-)U^{n+1}(x_m), \end{aligned} \tag{16}$$

where

$$D^-U^{n+1}(x_m)=\frac{U^{n+1}(x_m)-U^{n+1}(x_{m-1})}{h},\quad D^+U^{n+1}(x_m)=\frac{U^{n+1}(x_{m+1})-U^{n+1}(x_m)}{h},$$
$$D^+D^-U^{n+1}(x_m)=\frac{U^{n+1}(x_{m+1})-2U^{n+1}(x_m)+U^{n+1}(x_{m-1})}{h^2}.$$

Using triangular inequality, equation (16) leads to

$$\begin{aligned}\left|\mathscr{L}_\varepsilon^{M,N}(U^{n+1}(x_m)-U_m^{n+1})\right| \le\ &\varepsilon\left|(\frac{h^2}{\psi_m^2}-1)D^+D^-U^{n+1}(x_m)\right|+\varepsilon\left|(D^+D^- - \frac{d^2}{dx^2})U^{n+1}(x_m)\right|\\ &+\left|a^{(j)}(x_m)(\frac{d}{dx}-D^-)U^{n+1}(x_m)\right|. \end{aligned}\tag{17}$$

We use the Taylor series expansion for some terms of the above equation to simplify as,

$$\frac{h^2}{\psi_m^2}-1=-\frac{a^{(j)}(\eta_1)h}{2\varepsilon+ha^j(\eta_1)}, \tag{18}$$
$$D^+D^-U^{n+1}(x_m)=\frac{d^2U^{n+1}}{dx^2}(\eta_2), \tag{19}$$
$$(D^+D^- - \frac{d^2}{dx^2})U^{n+1}(x_m)=\frac{h^2}{16}\frac{d^4U^{n+1}}{dx^4}(\eta_3), \tag{20}$$
$$(\frac{d}{dx}-D^-)U^{n+1}(x_m)=\frac{h}{2}\frac{d^2U^{n+1}}{dx^2}(\eta_4), \tag{21}$$

for some η_i, $x_{m-1}\le\eta_i\le x_{m+1}$, $i=1,2,3,4$.

Substituting Eqs. (18)-(21) in (17), it leads to

$$\begin{aligned}\left|\mathscr{L}_\varepsilon^{M,N}(U^{n+1}(x_m)-U_m^{n+1})\right| \le\ &\varepsilon\frac{a^{(j)}(\eta_1)h}{2\varepsilon+ha^j(\eta_1)}\Big|\frac{d^2U^{n+1}}{dx^2}(\eta_2)\Big|+\varepsilon\frac{h^2}{16}\Big|\frac{d^4U^{n+1}}{dx^4}(\eta_3)\Big|\\ &+a^{(j)}(x_m)\frac{h}{2}\Big|\frac{d^2U^{n+1}}{dx^2}(\eta_4)\Big|. \end{aligned}\tag{22}$$

Let $\eta=\max\{\eta_1,\eta_2,\eta_3\}$. Using the bounds of derivatives in (13) and the boundedness of $a^{(j)}(x)$, for

$m = 0, 1, \ldots, M$, we obtain

$$
\begin{aligned}
\left|\mathscr{L}^{M,N}(U^{n+1}(x_m) - U_m^{n+1})\right| &\leq Ch\left(1+\varepsilon^{-2}\exp\left(\frac{-p(1-\eta_2)}{\varepsilon}\right)\right) + Ch^2\varepsilon\left(1+\varepsilon^{-4}\exp\left(\frac{-p(1-\eta_3)}{\varepsilon}\right)\right) \\
&\quad + Ch\left(1+\varepsilon^{-2}\exp\left(\frac{-p(1-\eta_4)}{\varepsilon}\right)\right) \\
&\leq Ch\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-\eta)}{\varepsilon}\right)\right) + Ch^2\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-\eta)}{\varepsilon}\right)\right) \\
&\quad + Ch\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-\eta)}{\varepsilon}\right)\right) \\
&\leq Ch\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-\eta)}{\varepsilon}\right)\right).
\end{aligned}
$$

Apply limit as ε approaches to zero and using [26, Lemma 7], we have

$$\lim_{\varepsilon\to 0}|\mathscr{L}_\varepsilon^{M,N}(U^{n+1}(x_m) - U_m^{n+1})| \leq Ch, \quad m = 0, 1, \ldots, M.$$

Lemma 8 leads to $\lim\limits_{\varepsilon\to 0}|U^{n+1}(x_m) - U_m^{n+1}| \leq Ch$. □

The temporal and spatial error estimates in Theorems 1 and 2, respectively, give the the following main result of this paper.

Theorem 3. *Let $u(x,t)$ be the exact solution of (1) and U_m^n be the solution of the numerical scheme (14). The error estimate for the totally discrete scheme is $|u(x_m,t_n) - U_m^n| \leq C(\Delta t^2 + h)$, $0 \leq m \leq M$.*

Therefore, the proposed method is ε-uniform convergent of second order in time and first order in space.

6 Numerical results and discussion

In this section, we present numerical results using the proposed FOM in (15) and compare them with the numerical results of FMMs given in [17, 19]. Three test examples are considered to validate the theoretical results.

Since the exact solution of (1) is not known, we compute the maximum absolute error $E_\varepsilon^{M,N}$ by using the double mesh principle [11] given by

$$E_\varepsilon^{M,N} = \max_{(x_m,t_n)\in D_M^N} |u_{m;M}^{n;N} - u_{m;2M}^{n;2N}|,$$

where $u_{m;M}^{n;N}$ and $u_{m;2M}^{n;2N}$ are approximate solutions to problem (1) on D_M^N and D_{2M}^{2N}, respectively. The ε-uniform maximum absolute error is given by $E^{M,N} = \max\limits_{\varepsilon} E_\varepsilon^{M,N}$.

The corresponding rate of convergence $R_\varepsilon^{M,N}$ is defined by

$$R_\varepsilon^{M,N} = \log_2\left(\frac{E_\varepsilon^{M,N}}{E_\varepsilon^{2M,2N}}\right),$$

and the ε-uniform rate of convergence is $R^{M,N} = \max_{\varepsilon} R_{\varepsilon}^{M,N}$. In addition, for the linearization, we used $u_m^{n+1,(0)} = f(x_m)$ as the initial guess, and we iterated up to 10^{-6} tolerance for stopping criteria. We consider the following three examples.

Example 1. Consider the following singularly perturbed Burgers-Huxley problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} - (1-u)(u-0.5)u = 0, \quad (x,t) \in (0,1) \times (0,1], \\ \quad u(x,0) = x(1-x^2), \quad 0 < x < 1, \\ \quad u(0,t) = 0 = u(1,t), \quad 0 \le t \le 1. \end{cases}$$

Example 2. Consider the following singularly perturbed Burgers problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad (x,t) \in (0,1) \times (0,1], \\ \quad u(x,0) = x(1-x^2), \quad 0 < x < 1, \\ \quad u(0,t) = 0 = u(1,t), \quad 0 \le t \le 1. \end{cases}$$

Example 3. Consider the following Burgers-Huxley problem:

$$\begin{cases} u_t - \varepsilon u_{xx} + \alpha u u_x - \beta(1-u)(u-\gamma)u = 0, \quad (x,t) \in (0,1) \times (0,1], \\ \quad u(x,0) = \frac{\gamma}{2} + \frac{\gamma}{2}\tanh(A_1 x), \quad 0 < x < 1, \\ \quad u(0,t) = \frac{\gamma}{2} + \frac{\gamma}{2}\tanh(-A_1 A_2 t), \;\; u(1,t) = \frac{\gamma}{2} + \frac{\gamma}{2}\tanh(A_1(1-A_2 t)), \quad 0 \le t \le 1. \end{cases}$$

The numerical solution of Example 1 is plotted in Figure 1 for $T = 1$, $M = 64$, $N = 40$, and $\varepsilon = 10^{-4}$. Also, Figures 2 and 3 are the numerical solutions for different values of T and ε, respectively. From these Figures 1-3, we observe that the numerical solution of Example 1 for small values of ε has boundary layer on the right side of the spatial domain with the time $t \to 1$.

In Tables 1,5, and 9, we tabulate the maximum absolute errors and numerical rate of convergences for different values of ε and the temporal mesh size N with fixed spatial mesh size $M = 64$. As we observe from the results, the newly proposed method converges uniformly in time with respect to ε and the rate of convergence shows second order in time. In Tables 2, 6, and 10, we also tabulate the maximum absolute errors and numerical rate of convergences for different values of spatial mesh size M with fixed temporal mesh size N. The first order ε-uniform convergence in space is observed.

For comparison purposes, the maximum absolute errors and CPU time using the new proposed method and the fitted mesh methods in [17, 19], are given in Tables 3, 4, 7 and 8 for Examples 1 and 2. As we observe from these tables our method gives more accurate results than the schemes in [17, 19] and the CPU time of our method is slightly smaller than the method in [17]. The proposed method can be applied to other singularly perturbed quasilinear problems of type (1).

7 Concluding remarks

In this paper, we have constructed a parameter-uniform convergent numerical method for solving the singularly perturbed Burgers-Huxley problem that has the right boundary layer in the spatial direction. We have applied the Crank-Nicolson method to discretize the time derivative and a fitted operator upwind

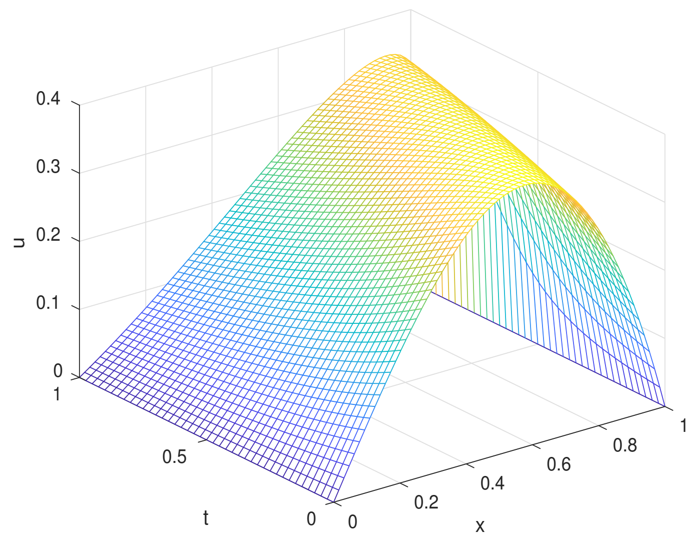

Figure 1: Behavior of the solution for Example 1 at $M = 64,\ N = 40$ and $\varepsilon = 10^{-4}$.

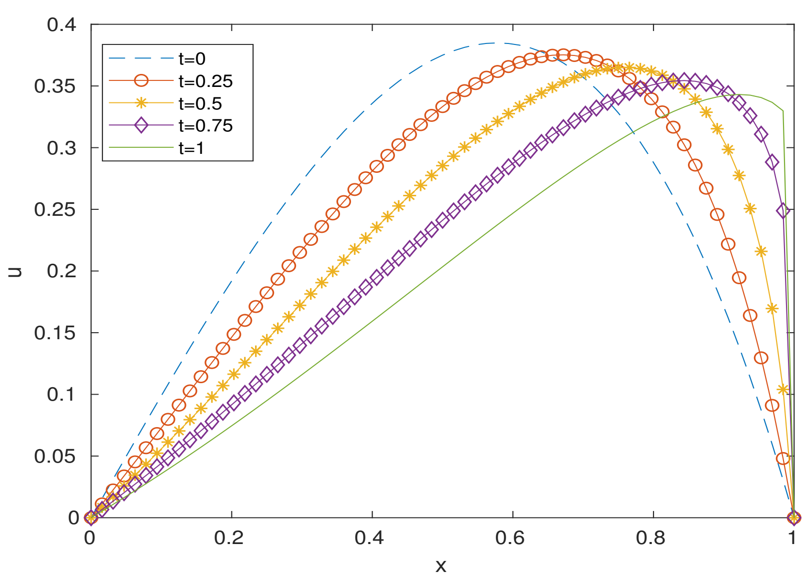

Figure 2: Numerical solutions of Example 1 for different values of t with $M = 64,\ N = 40$ and $\varepsilon = 10^{-4}$.

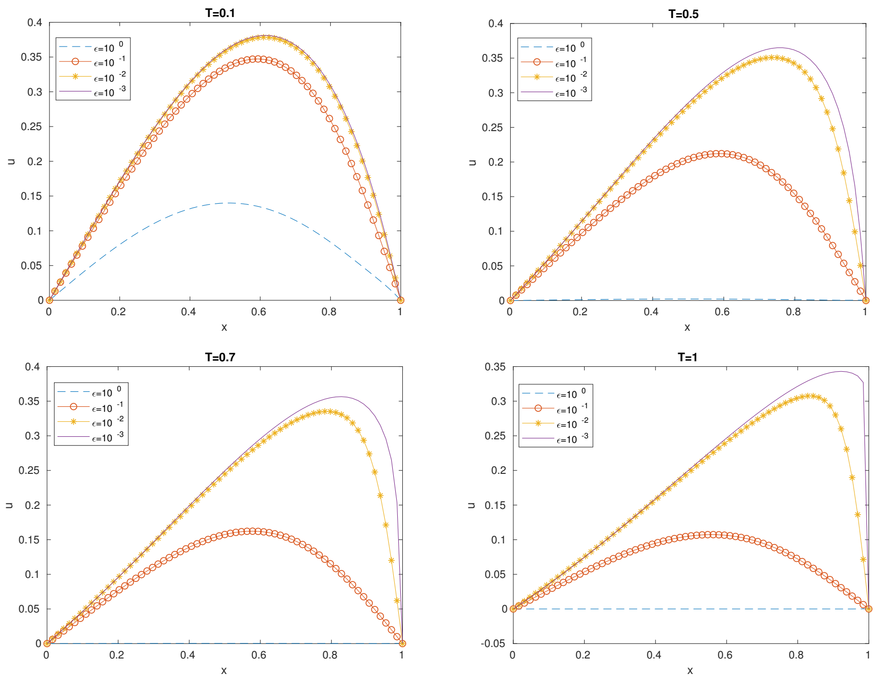

Figure 3: Numerical solutions of Example 1 for different values of ε with $M = 64,\ N = 40$.

Table 1: Maximum absolute errors and rate of convergence for Example 1 at the number of intervals N with fixed spatial mesh size $M = 64$.

$\varepsilon \downarrow$	$N = 20$	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$
10^{0}	1.73e-02	8.83e-03	4.44e-03	2.23e-03	1.04e-03	5.29e-04
	0.97	0.99	1.00	1.10	0.97	
10^{-2}	1.10e-04	3.44e-05	8.34e-06	1.78e-06	4.33e-07	1.08e-07
	1.68	2.05	2.22	2.04	2.01	
10^{-4}	7.82e-04	1.96e-04	4.91e-05	1.23e-05	3.07e-06	7.68e-07
	1.99	2.00	2.00	2.00	2.00	
10^{-6}	7.82e-04	1.96e-04	4.91e-05	1.23e-05	3.07e-06	7.68e-07
	1.99	2.00	2.00	2.00	2.00	
10^{-8}	7.82e-04	1.96e-04	4.91e-05	1.23e-05	3.07e-06	7.68e-07
	1.99	2.00	2.00	2.00	2.00	
10^{-10}	7.82e-04	1.96e-04	4.91e-05	1.23e-05	3.07e-06	7.68e-07
	1.99	2.00	2.00	2.00	2.00	
10^{-12}	7.82e-04	1.96e-04	4.91e-05	1.23e-05	3.07e-06	7.68e-07
	1.99	2.00	2.00	2.00	2.00	
$E^{N,M}$	7.82e-04	1.96e-04	4.91e-05	1.23e-05	3.07e-06	7.68e-07
$R^{M,N}$	1.99	2.00	2.00	2.00	2.00	

Table 2: Maximum absolute errors and rate of convergence for Example 1 at the number of intervals M with fixed temporal mesh size $N = 10$.

$\varepsilon \downarrow$	$M = 128$	$M = 256$	$M = 512$	$M = 1024$	$M = 2048$	$M = 4096$
10^0	9.098e-06	2.275e-06	5.687e-07	1.422e-07	3.554e-08	8.896e-09
	2.000	2.000	2.000	2.000	1.998	
10^{-2}	1.345e-04	3.368e-05	8.425e-06	2.107e-06	5.266e-07	1.317e-07
	1.997	1.999	2.000	2.000	2.000	
10^{-4}	1.979e-02	1.255e-02	4.861e-03	5.848e-03	1.203e-02	2.865e-03
	0.657	1.369	-0.267	-1.041	2.071	
10^{-6}	1.975e-02	1.285e-02	7.340e-03	4.373e-03	2.469e-03	1.317e-03
	0.620	0.808	0.747	0.825	0.907	
10^{-8}	1.975e-02	1.285e-02	7.340e-03	4.373e-03	2.471e-03	1.322e-03
	0.620	0.808	0.747	0.824	0.902	
10^{-10}	1.975e-02	1.285e-02	7.340e-03	4.373e-03	2.471e-03	1.322e-03
	0.620	0.808	0.747	0.824	0.902	
10^{-12}	1.975e-02	1.285e-02	7.340e-03	4.373e-03	2.471e-03	1.322e-03
	0.620	0.808	0.747	0.824	0.902	
$E^{N,M}$	1.975e-02	1.285e-02	7.340e-03	4.373e-03	2.471e-03	1.322e-03
$R^{M,N}$	0.620	0.808	0.747	0.824	0.902	

Table 3: Maximum absolute errors for Example 1 at the number of intervals M, N and compare with methods in [17, 19].

	$\varepsilon \downarrow$	$M = 32$ $N = 20$	$M = 64$ $N = 40$	$M = 128$ $N = 80$	$M = 256$ $N = 160$	$M = 512$ $N = 320$
	2^0	1.708e-02	8.840e-03	4.482e-03	2.270e-03	1.141e-03
	2^{-4}	8.197e-04	5.174e-04	2.603e-04	1.396e-04	7.045e-05
	2^{-8}	1.042e-02	3.198e-03	8.274e-04	2.087e-04	5.232e-05
	2^{-12}	3.387e-02	2.844e-02	2.130e-02	9.090e-03	8.110e-03
	2^{-16}	3.387e-02	2.848e-02	2.274e-02	1.752e-02	1.316e-02
In [17]	2^0	2.5632e-02	1.6476e-02	9.9288e-03	5.6712e-03	3.1149e-03
	2^{-4}	3.1540e-02	2.0968e-02	1.1706e-02	6.3027e-03	3.3320e-03
	2^{-8}	8.3069e-02	6.3873e-02	3.7579e-02	1.9554e-02	9.7037e-03
	2^{-12}	5.9149e-02	1.2245e-01	1.4419e-01	9.5806e-02	5.5767e-02
	2^{-16}	3.7835e-02	8.8086e-02	1.9473e-01	2.0775e-01	1.3885e-01
In [19]	2^0	2.4400e-02	1.4033e-02	7.5889e-03	3.9547e-03	2.0227e-03
	2^{-4}	2.0658e-02	1.2614e-02	7.0567e-03	3.7952e-03	1.9563e-03
	2^{-8}	3.8607e-02	1.9497e-02	1.1221e-02	6.2852e-03	3.3405e-03
	2^{-12}	1.7017e-01	1.0083e-01	6.2216e-02	3.9526e-02	2.0493e-02
	2^{-16}	2.0450e-01	1.5975e-01	1.2612e-01	6.9772e-02	3.8531e-02

Table 4: Maximum absolute error and CPU time for Example 1 when $\varepsilon = 2^{-8}$.

	Our method		FMM in [17]	
Number of intervals M,N	$M = 32$ $N = 20$	$M = 64$ $N = 40$	$M = 32$ $N = 20$	$M = 64$ $N = 40$
Maximum Absolute Error	1.042e-02	3.198e-03	3.8607e-02	1.9497e-02
CPU Time	0.00776	0.0118	0.00817	0.0148

Table 5: Maximum absolute errors and rate of convergence for Example 2 at the number of intervals N with fixed spatial mesh size $M = 64$.

$\varepsilon \downarrow$	$N = 20$	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$
10^{0}	1.72e-02	8.78e-03	4.42e-03	2.22e-03	1.04e-03	5.29e-04
	0.97	0.99	0.99	1.10	0.97	
10^{-2}	1.14e-04	3.29e-05	8.00e-06	1.80e-06	4.49e-07	1.12e-07
	1.80	2.04	2.15	2.00	2.00	
10^{-4}	7.67e-04	2.00e-04	5.00e-05	1.25e-05	3.13e-06	7.82e-07
	1.94	2.00	2.00	2.00	2.00	
10^{-6}	7.67e-04	2.00e-04	5.00e-05	1.25e-05	3.13e-06	7.82e-07
	1.94	2.00	2.00	2.00	2.00	
10^{-8}	7.67e-04	2.00e-04	5.00e-05	1.25e-05	3.13e-06	7.82e-07
	1.94	2.00	2.00	2.00	2.00	
10^{-10}	7.67e-04	2.00e-04	5.00e-05	1.25e-05	3.13e-06	7.82e-07
	1.94	2.00	2.00	2.00	2.00	
10^{-12}	7.67e-04	2.00e-04	5.00e-05	1.25e-05	3.13e-06	7.82e-07
	1.94	2.00	2.00	2.00	2.00	
$E^{N,M}$	7.67e-04	2.00e-04	5.00e-05	1.25e-05	3.13e-06	7.82e-07
$R^{M,N}$	1.94	2.00	2.00	2.00	2.00	

Table 6: Maximum absolute errors and rate of convergence for Example 2 at the number of intervals M with fixed temporal mesh size $N = 10$.

$\varepsilon \downarrow$	$M = 128$	$M = 256$	$M = 512$	$M = 1024$	$M = 2048$	$M = 4096$
10^{0}	9.211e-06	2.303e-06	5.758e-07	1.439e-07	3.599e-08	9.003e-09
	2.000	2.000	2.000	2.000	1.999	
10^{-2}	1.218e-04	3.051e-05	7.632e-06	1.908e-06	4.770e-07	1.193e-07
	1.998	1.999	2.000	2.000	2.000	
10^{-4}	1.710e-02	1.102e-02	4.109e-03	5.638e-03	1.456e-02	4.106e-03
	0.635	1.423	-0.457	-1.369	1.826	
10^{-6}	1.712e-02	1.177e-02	7.118e-03	3.908e-03	2.034e-03	1.090e-03
	0.541	0.726	0.865	0.942	0.900	
10^{-8}	1.712e-02	1.177e-02	7.118e-03	3.907e-03	2.030e-03	1.108e-03
	0.541	0.726	0.865	0.945	0.874	
10^{-10}	1.712e-02	1.177e-02	7.118e-03	3.907e-03	2.030e-03	1.108e-03
	0.541	0.726	0.865	0.945	0.874	
10^{-12}	1.712e-02	1.177e-02	7.118e-03	3.907e-03	2.030e-03	1.108e-03
	0.541	0.726	0.865	0.945	0.874	
$E^{N,M}$	1.712e-02	1.177e-02	7.118e-03	3.907e-03	2.030e-03	1.108e-03
$R^{M,N}$	0.541	0.726	0.865	0.945	0.874	

Table 7: Maximum absolute errors for Example 2 at the number of intervals M, N and compare with methods in [17, 19].

	$\varepsilon \downarrow$	$M = 32$ $N = 20$	$M = 64$ $N = 40$	$M = 128$ $N = 80$	$M = 256$ $N = 160$	$M = 512$ $N = 320$
	2^{0}	1.698e-02	8.788e-03	4.464e-03	2.264e-03	1.139e-03
	2^{-4}	7.821e-04	5.092e-04	2.586e-04	1.391e-04	7.030e-05
	2^{-8}	9.214e-03	4.877e-03	1.138e-03	2.969e-04	7.358e-05
	2^{-12}	2.946e-02	2.501e-02	1.884e-02	7.766e-03	7.939e-03
	2^{-16}	2.946e-02	2.505e-02	2.017e-02	1.567e-02	1.188e-02
In [17]	2^{0}	2.4502e-02	1.4057e-02	7.5969e-03	3.9589e-03	2.0249e-03
	2^{-4}	3.7645e-02	2.2072e-02	1.2090e-02	6.4115e-03	3.3584e-03
	2^{-8}	9.9532e-02	5.4894e-02	2.7274e-02	1.2963e-02	6.0337e-03
	2^{-12}	2.2602e-01	2.6665e-01	1.8203e-01	1.0258e-01	5.2694e-02
	2^{-16}	4.4024e-02	3.2475e-01	3.2607e-01	2.5174e-01	1.6644e-01
In [19]	2^{0}	2.4502e-02	1.4057e-02	7.5969e-03	3.9589e-03	2.0249e-03
	2^{-4}	2.2788e-02	1.3365e-02	7.2769e-03	3.8441e-03	1.9728e-03
	2^{-8}	4.4450e-02	2.0109e-02	1.0519e-02	5.9653e-02	3.1896e-03
	2^{-12}	1.8762e-01	8.4106e-02	5.7234e-02	3.8309e-02	1.9041e-02
	2^{-16}	2.1340e-01	1.5016e-01	1.1582e-01	6.1958e-02	3.3441e-02

Table 8: Maximum absolute error and CPU time for Example 2 when $\varepsilon = 2^{-8}$.

	Our method		FMM in [17]	
Number of intervals M,N	$M = 32$ $N = 20$	$M = 64$ $N = 40$	$M = 32$ $N = 20$	$M = 64$ $N = 40$
Maximum Absolute Error	9.214e-02	4.877e-03	4.4450e-02	2.0109e-02
CPU Time	0.00807	0.00814	0.00817	0.0133

Table 9: Maximum absolute errors and rate of convergence for Example 3 at the number of intervals N with fixed spatial mesh size $M = 64$.

$\varepsilon \downarrow$	$N = 20$	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$
10^{0}	9.18e-05	4.61e-05	2.28e-05	1.13e-05	5.22e-06	2.65e-06
	0.99	1.02	1.02	1.11	0.98	
10^{-2}	8.82e-05	2.79e-05	6.82e-06	1.45e-06	3.51e-07	8.76e-08
	1.66	2.03	2.23	2.05	2.00	
10^{-4}	6.80e-05	1.49e-05	3.46e-06	8.67e-07	2.16e-07	5.39e-08
	2.19	2.11	2.00	2.01	2.00	
10^{-6}	6.80e-05	1.49e-05	3.46e-06	8.67e-07	2.16e-07	5.39e-08
	2.19	2.11	2.00	2.01	2.00	
10^{-8}	6.80e-05	1.49e-05	3.46e-06	8.67e-07	2.16e-07	5.39e-08
	2.19	2.11	2.00	2.01	2.00	
10^{-10}	6.80e-05	1.49e-05	3.46e-06	8.67e-07	2.16e-07	5.39e-08
	2.19	2.11	2.00	2.01	2.00	
10^{-12}	6.80e-05	1.49e-05	3.46e-06	8.67e-07	2.16e-07	5.39e-08
	2.19	2.11	2.00	2.01	2.00	
$E^{N,M}$	6.80e-05	1.49e-05	3.46e-06	8.67e-07	2.16e-07	5.39e-08
$R^{M,N}$	2.19	2.11	2.00	2.01	2.00	

Table 10: Maximum absolute errors and rate of convergence for Example 3 at the number of intervals M with fixed temporal mesh size $N = 10$.

$\varepsilon \downarrow$	$M = 128$	$M = 256$	$M = 512$	$M = 1024$	$M = 2048$	$M = 4096$
10^0	5.753e-08	1.438e-08	3.596e-09	8.992e-10	2.230e-10	5.670e-11
	2.000	2.000	2.000	2.011	1.976	
10^{-2}	1.637e-05	4.114e-06	1.030e-06	2.576e-07	6.440e-08	1.610e-08
	1.992	1.998	1.999	2.000	2.000	
10^{-4}	3.630e-04	2.062e-04	1.087e-04	4.728e-05	5.536e-05	5.420e-05
	0.816	0.923	1.201	-0.228	0.030	
10^{-6}	3.630e-04	2.062e-04	1.108e-04	5.759e-05	2.937e-05	1.483e-05
	0.816	0.895	0.945	0.971	0.985	
10^{-8}	3.630e-04	2.062e-04	1.108e-04	5.759e-05	2.937e-05	1.483e-05
	0.816	0.895	0.945	0.971	0.985	
10^{-10}	3.630e-04	2.062e-04	1.108e-04	5.759e-05	2.937e-05	1.483e-05
	0.816	0.895	0.945	0.971	0.985	
10^{-12}	3.630e-04	2.062e-04	1.108e-04	5.759e-05	2.937e-05	1.483e-05
	0.816	0.895	0.945	0.971	0.985	
$E^{N,M}$	3.630e-04	2.062e-04	1.108e-04	5.759e-05	2.937e-05	1.483e-05
$R^{M,N}$	0.816	0.895	0.945	0.971	0.985	

finite difference method to discretize the spatial derivatives. Newton-Kantorovich quasilinearization technique was used for the linearization. Theoretically, we have shown that the proposed method is ε-uniform convergent of order two in time and one in space.

Numerical results are performed in Figures 1-3 and in Tables 1-10 for some test examples to confirm the theoretical results. We have observed from these figures and tables that the numerical results are in agreement with the theoretical findings. We have also observed that the proposed method gives a better numerical accuracy compared to the fitted mesh methods in [17, 19].

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