Journal of Mathematical Modeling
Vol. 10, No. 3, 2022, pp. 499-514. Research Article

JMM

A numerical method for solving stochastic linear quadratic problem with a finance application

Mohammad Hossein Fotoohi Bafghi[†], **Sohrab Effati**[‡*], **Omid Solaymani Fard**[§]

[†] *Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran*
[‡] *Center of Excellence on Soft Computing and Intelligent Information Processing*
[§] *Ferdowsi University of Mashhad, Mashhad, Iran*
Email(s): mohammadhossein.fotoohibafghi@mail.um.ac.ir, s-effati@um.ac.ir, omidsfard@gmail.com

Abstract. This paper is concerned with the stochastic linear quadratic regulator (LQR) optimal control problem in which dynamical systems have control-dependent diffusion coefficients. In fact, providing the solution to this problem leads to solving a matrix Riccati differential equation as well as a vector differential equation with boundary conditions. The present work mainly proposes not only a novel method but also an efficient fixed-point scheme based on the spline interpolation for the numerical solution to the stochastic LQR problem. Via implementing the proposed method to the corresponding differential equation of the stochastic LQR optimal control problem, not only is the numerical solution gained, but also a suboptimal control law is obtained. Furthermore, the method application is illustrated by means of an optimal control example with the financial market problems, including two investment options.

Keywords: stochastic, quadratic, optimal, control, Riccati equation, approximation, financial market.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

The stochastic linear quadratic regulator (LQR) problem is an optimal control of a linear stochastic differential equation with a quadratic cost criterion. Generally, stochastic LQR control is one of the most important classes of optimal control problems, which can be widely applied in various fields (e.g., modern engineering, economic problems, and mathematical finance), especially in the control of financial markets. For instance, Merton initiated the study of financial markets using continuous-time stochastic models [22]. Regarding the classical Merton portfolio optimization problem, there are two investment options: a risk-less asset with a constant interest rate and a risky asset whose price regularly fluctuates.

*Corresponding author.
Received: 22 October 2021 / Revised: 10 March 2022/ Accepted: 11 March 2022
DOI: 10.22124/JMM.2022.20887.1826

The LQR problem has been proposed by Wonham [24] and elaborated by many scholars, among whom we can mention [4, 12, 16].

The optimal feedback control of the stochastic LQR problem will be obtained if the problem can be reduced to solving a Riccati differential equation and a backward differential equation. Thus, regarding stochastic LQR problems, it is evident to study the associated differential equations [1]. As a result, the numerical solution to the stochastic LQR relies on solving these equations efficiently. The linear version of backward differential equation was first introduced by Bismut [3], and the nonlinear version was independently introduced by Pardoux and Peng [17] as well as Duffie and Epstein [6]. These equations present a proper structure and wide applicability in a number of different areas, specifically in mathematical finance. Hence, they have received considerable research attention in recent years (see, e.g., [5, 7, 13, 15, 23, 27]). For instance, the Black-Scholes formula for options pricing can be recovered via a system of forward-backward stochastic differential equations.

In the scientific sources on the stochastic LQR problem, control weight is usually assumed as a positive definite matrix in the cost functional. Indeed, this assumption is essential for the problem to be well posed, since positive definiteness is required to prepare it as a convex optimization problem. However, in the stochastic LQR problem, when the diffusion term depends on the control, it has recently been proposed that the control weight may be negatively definite, yet the problem remains well-posed.

The stochastic maximum principle for the LQR problem has been investigated since the 1960s [2, 8, 9, 14, 26]. As all the results demonstrate, the diffusion term does not depend on the control variable. This concept was introduced by Peng [18] and Zhou [27] with respect to the systems with control-dependent diffusion coefficients. Furthermore, regarding the usual first-order adjoint equation, the second-order adjoint equation needs to be introduced in order to represent the risk factor due to implying uncertainty. The maximum principle applied to the problem is indeed an extension of the Hamiltonian principle for a status in which a quadratic term exists in the diffusion coefficient. However, Zhou [28] revealed that Peng's maximum principle is sufficient under certain convex conditions leading to an optimal solution for some stochastic LQR problems, though the control weight is a negative definite matrix.

The present study is mainly intended to propose a new numerical approach concerning a class of stochastic LQR optimal control problems with systems that have control-dependent diffusion coefficients, specifically via using the technique for the numerical solution of nonlinear stochastic Ito-Volterra integral equations [10, 11]. As a matter of fact, the method is based on a combination of the fixed-point method and linear spline interpolation [20]. It is worth noting that, since we could not detect any solved numerical example in the literature review of this kind of problem, we cannot compare this method with other methods. Thus, in this study, relative error is used in order to investigate the convergence of the method.

The study outline is as follows: In Section 2, the stochastic LQR optimal control problem with indefinite control weight costs is reviewed. A successive approximation method (SAM) is introduced in Section 3. In Section 4, we practically utilize our findings to obtain a suboptimal control law and an efficient algorithm with low computational complexity. In Section 5, we solve a financial example. Ultimately, this paper is ended with conclusions in Section 6.

2 Solution to LQR Problem

Let $\left(\Omega, F, \{F_t\}_{t_0 \le t \le t_f}, \mathbb{P}\right)$ be given and fixed complete filtered probability space [8] and let $\mathscr{C}\left([t_0,t_f];S\right)$ be the Banach space of S-valued continuous functions on $[t_0,t_f]$ endowed with the maximum norm $\|\cdot\|$ for a given Hilbert space S. Consider the following linear stochastic differential equation:

$$dx(t) = \Big[A(t)x(t) + B(t)u(t) + f(t)\Big]dt + \big[C(t)u(t)\big]dW(t), \qquad t_0 \le t \le t_f, \tag{1}$$

with initial condition $x(t_0) = x_0$. Here $W(\cdot)$ is an n-dimensional standard Brownian motion on the interval $[t_0,t_f]$ and $x(\cdot) \in \mathbb{R}^n$ (n-dimensional euclidean space). Furthermore, the entries of $n \times n$ matrix $A(t)$ and $n \times m$ matrices $C(t)$ and $B(t)$ belong to $\mathscr{C}\left([t_0,t_f];\mathbb{R}\right)$.

The cost functional associated with system (1) is

$$J(t_0,x_0,u(\cdot)) = \mathbb{E}\left\{\frac{1}{2}\int_{t_0}^{t_f}\Big[x^T(t)M(t)x(t) + u^T(t)N(t)u(t)\Big]dt + \frac{1}{2}x^T(t_f)Dx(t_f)\right\}, \tag{2}$$

where $\mathbb{E}$ indicates the mathematical expectation and the symbol T denotes the transpose operation. Let the $n \times n$ matrices $M(t)$ and D be symmetric positive semidefinite, the $m \times m$ matrix $N(t)$ be symmetric, and t_f be an exit time or a terminal time. In addition, the matrices $M(t)$ and $N(t)$ have continuous entries. Note that here we do not assume that $N(t)$ is positive definite.

A feedback control law is a piecewise continuous function $\mathbb{U}(\cdot,\cdot)$ from $[t_0,t_f] \times \mathbb{R}^n$ into U, where U is a closed subset of $\mathbb{R}^m$. Here, the class of admissible controls is denoted by $\mathscr{U}$. The control applied at time t by using the feedback control $\mathbb{U}$ is $u(t) = \mathbb{U}(t,x(t))$. The minimization of stochastic LQR optimal control problem is regarded as the task of finding an optimal control $u^*(\cdot) \in \mathscr{U}$ on the interval $[t_0,t_f]$ by minimizing J and the associated optimum performance index $J^*(t_0,x_0)$ is the value of J resulted by using the optimal control [8].

Consider the matrix Riccati differential equation

$$\begin{aligned}\dot{K}(t) &= -K^T(t)A(t) - A^T(t)K(t) - M(t)\\ &\quad + K^T(t)B(t)\Big(N(t) + C^T(t)K(t)C(t)\Big)^{-1}B^T(t)K(t), \qquad t_0 \le t \le t_f,\end{aligned} \tag{3}$$

with boundary condition $K(t_f) = D$. Also, consider the vector differential equation

$$\dot{g}(t) = -A^T(t)g(t) - K(t)f(t) + K^T(t)B(t)\Big(N(t) + C^T(t)K(t)C(t)\Big)^{-1}B^T(t)g(t), \qquad t_0 \le t \le t_f, \tag{4}$$

with boundary condition $g(t_f) = 0$ [29]. Here $N(t) + C^T(t)K(t)C(t)$ is a positive definite matrix.

Assume that Eqs. (3) and (4) have unique solutions $K(t)$ and $g(t)$, respectively, such that the entries of $n \times n$ matrix $K(t)$ belong to $\mathscr{C}\left([t_0,t_f];\mathbb{R}\right)$ and $g(t) \in \mathscr{C}\left([t_0,t_f];\mathbb{R}^n\right)$. Then the stochastic LQR problem can be reduced to solving these differential boundary problems.

Theorem 1. *Assume that Eqs.* (3) *and* (4) *have unique solutions. Then an optimal control for problem* (1)–(2) *is*

$$u^*(t) = -P(t)^{-1}B^T(t)\big(K(t)x(t) + g(t)\big), \qquad t_0 \le t \le t_f, \tag{5}$$

where

$$P(t)=N(t)+C^T(t)K(t)C(t), \qquad t_0 \le t \le t_f. \tag{6}$$

Furthermore, the optimal cost value is

$$J^*(t_0,x_0) = \frac{1}{2}\int_{t_0}^{t_f}\Big[2f^T(t)g(t)+g^T(t)B(t)P^{-1}(t)B^T(t)g(t)dt\Big]dt+\frac{1}{2}x_0^T K(t_0)x_0+x_0 g(t_0). \tag{7}$$

Proof. Consider the stochastic linear differential Eq. (1). Using Ito's formula, we have

$$\begin{aligned} d(x^T(t)K(t)x(t)) &= x^T(t)\dot{K}(t)x(t)dt+2x^T(t)K(t)dx(t)+\big(u^T(t)C^T(t)K(t)C^T(t)u(t)\big)\\ &= \Big\{x^T(t)\big(-M(t)+K^T(t)B(t)P^{-1}(t)B^T(t)K(t)\big)x(t)+2u^T(t)B^T(t)K(t)x(t)\\ &\quad +2x^T(t)K(t)f(t)+\big(u^T(t)C^T(t)K(t)C^T(t)u(t)\big)\Big\}dt\\ &\quad +\{2x^T(t)K(t)C(t)u(t)\}dW(t) \end{aligned} \tag{8}$$

and

$$\begin{aligned} d(x^T(t)g(t)) = {} & \Big\{x^T(t)K^T(t)B(t)P^{-1}(t)B^T(t)g(t)-x^T(t)K(t)f(t)+u^T(t)B^T(t)g(t)+f^T(t)g(t)\Big\}dt\\ & +\{u^T(t)C^T(t)g(t)\}dW(t). \end{aligned} \tag{9}$$

The boundary conditions of (3) and (4) imply

$$x^T(t_f)Dx(t_f)=\int_{t_0}^{t_f} d\big(x^T(t)K(t)x(t)\big)+x_0^T K(t_0)x_0,$$

and

$$\int_{t_0}^{t_f} d\big(x^T(t)g(t)\big)+x_0 g(t_0)=0.$$

Thus by integrating both (8) and (9) from t_0 to t_f, taking expectations, and adding them together, the cost functional (2) gets

$$\begin{aligned} J(t_0,x_0,u(\cdot)) &= \frac{1}{2}\mathbb{E}\Bigg\{\int_{t_0}^{t_f}\Big[x^T(t)M(t)x(t)+u^T(t)\big(P(t)-C^T(t)K(t)C(t)\big)u(t)\Big]dt\\ &\quad +\int_{t_0}^{t_f} d\big(x^T(t)K(t)x(t)\big)+d\big(x^T(t)g(t)\big)\Bigg\}+x_0^T K(t_0)x_0+x_0 g(t_0)\\ &= \frac{1}{2}\mathbb{E}\Bigg\{\int_{t_0}^{t_f}\Big[u^T(t)P(t)u(t)+2u^T(t)B^T(t)\big(K(t)x(t)+g(t)\big)\\ &\quad +x^T(t)K(t)B(t)P^{-1}(t)B^T(t)K(t)x(t)+2f^T(t)g(t)\\ &\quad +2x^T(t)K(t)B(t)P^{-1}(t)B^T(t)g(t)\Big]dt\Bigg\}+\frac{1}{2}x_0^T K(t_0)x_0+x_0 g(t_0) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\mathbb{E}\Bigg\{\int_{t_0}^{t_f}\Big[\Big(u(t)+P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)\Big)^T P(t)\\
&\Big(u(t)+P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)\Big)\\
&+2f^T(t)g(t)-g(t)B(t)P^{-1}(t)B^T(t)g(t)\Big]dt\Bigg\}+\frac{1}{2}x_0^T K(t_0)x_0+x_0 g(t_0). \qquad (10)
\end{aligned}
$$

Here $P(t)$ is a positive definite matrix, and so

$$
\Big(u(t)+P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)\Big)^T P(t)\Big(u(t)+P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)\Big)\geq 0.
$$

Thus, the minimum value of the cost functional occurs when

$$
u(t)+P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)=0.
$$

It follows immediately that the optimal feedback control is resulted using (5) and the optimal cost value is resulted using (7). The optimal control is given by (2) provided that the corresponding Eq. (1) under (5) has a solution. However, under (5), Eq. (1) is reduced to

$$
\begin{cases}
dx(t)=\Big[A(t)x(t)-B(t)P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)\Big]dt\\
\qquad\qquad -C(t)P^{-1}(t)B^T(t)\big(K(t)x(t)+g(t)\big)dW(t), \qquad t_0\leq t\leq t_f,\\
x(t_0)=x_0.
\end{cases} \qquad (11)
$$

Note that $K\in\mathscr{C}\big([t_0,t_f];\mathbb{R}\big)$, $g\in\mathscr{C}\big([t_0,t_f];\mathbb{R}\big), P\in\mathscr{C}\big([t_0,t_f];\mathbb{R}\big)$. Eq. (11) is a nonhomogeneous linear stochastic differential equation, and consequently, it has one and only one solution. Thus, the proof is completed (we refer the reader to [29, Theorem 4.1]). □

3 Numerical solution

In the previous section, it was demonstrated that solving stochastic LQR optimal control problem (1)–(2) is equivalent to solving two differential equations of (3) and (4). Furthermore, in order to provide approximate solutions to (1), we need to solve these equations. We can write (1) with initial condition $x(t_0)=x_0$ as a stochastic integral equation

$$
x(t)=x_0+\int_{t_0}^{t}[A(s)x(s)+B(s)u(s)+f(s)]\,ds+\int_{t_0}^{t}C(s)u(s)dW(s), \qquad t_0\leq t\leq t_f, \qquad (12)
$$

where $u(s)=-\big(N(s)+C^T(s)K(s)C(s)\big)^{-1}B^T(s)\big(K(s)x(s)+g(s)\big)$. Here, two functions $K(s)$ and $g(s)$ are the solutions to (3) and (4). The second part of (12) is a stochastic integral. For any $t\geq 0$, the Brownian motion is almost definitely continuous but not differentiable at t. Thus, in this section, a new approximations method is presented, which is based on the linear spline interpolation.

The SAM [25] is regarded as one of the well-known and applicable classical methods to solve the initial value problems and integral equations. This method is often used to prove the existence and uniqueness of the solutions to integral equations.

The following recurrence relation is introduced by the SAM:

$$x_n(t) = x_0 + \int_{t_0}^{t} [A(s)x_{n-1}(s) + B(s)u_{n-1}(s) + f(s)]\, ds + \int_{t_0}^{t} C(s)u_{n-1}(s)dW(s), \quad t_0 \le t \le t_f, \tag{13}$$

where $u_{n-1}(s) = -\big(N(s) + C^T(s)K(s)C(s)\big)^{-1}B^T(s)\big(K(s)x_{n-1}(s) + g(s)\big)$.

Theorem 2. *Let x_n be the solutions sequence produced by the successive approximation (13). Then x_n converges to x as $n \to \infty$, and x is the unique solution to (12).*

Proof. The entries of matrices $A(t)$, $B(t)$, and $C(t)$ involve continuous functions, and thus the terms in integrals satisfy Lipschitz and linear growth conditions:

$$\begin{aligned} \Big\| \Big(A(s)x(s) + B(s)u\big(s,x(s)\big)\Big) - \Big(A(s)y(s)B(s)u\big(s,y(s)\big)\Big)\Big\|^2 &\le L\|x(s)-y(s)\|^2, \\ \Big\| C(s)u\big(s,x(s)\big) - C(s)u\big(s,y(s)\big)\Big\|^2 &\le L\|x(s)-y(s)\|^2, \end{aligned}$$

and

$$\begin{aligned} \Big\| A(s)x(s) + B(s)u\big(s,x(s)\big)\Big\|^2 &\le L_1\big(1 + \|x(s)\|^2\big), \\ \Big\| C(s)u\big(s,x(s)\big)\Big\|^2 &\le L_1\big(1 + \|x(s)\|^2\big), \end{aligned}$$

for all $t_0 \le s \le t_f$. Here $u(s,x(s)) = P^{-1}(s)B^T(s)\big(K(s)x(s)(s) + g(s)\big)$ and L and L_1 are constants. □

Regarding linear spline interpolation, a new modification of the SAM is presented. First, we take a partition Δ with nodal points on $[t_0,t_f]$ as $\Delta : t_0 < t_1 < \cdots < t_{m-1} < t_m = t_f$, where $h_i = t_i - t_{i-1}, i = 1,2,\ldots,m$. Then the zeroth approximation $x_0(t) = x_0$ is considered. Using the recurrence relation (13), we have

$$x_1(t) = x_0 + \int_{t_0}^{t} [A(s)x_0 + B(s)u_0(s) + f(s)]\, ds + \int_{t_0}^{t} C(s)u_0(s)dW(s), \quad t_0 \le t \le t_f, \tag{14}$$

where $u_0(s) = -\big(N(s) + C^T(s)K(s)C(s)\big)^{-1}B^T(s)\big(K(s)x_0 + g(s)\big)$. The function $x_1(t)$ on $[t_0,t_f]$ can be approximated by a linear spline interpolation as

$$x_1(t) \approx S^1_\Delta(t) = \sum_{i=1}^{m-1} \psi^1_i(t)\chi_{[t_i,t_{i+1}]}(t), \tag{15}$$

where

$$\psi^1_i(t) = \frac{1}{h_{i+1}}\big[x_1(t_i)(t_{i+1}-t) + x_1(t_{i+1})(t-t_i)\big], \qquad i = 0,1,2,\ldots,m-1. \tag{16}$$

By substituting the grid points $t_i, i = 0, 1, \ldots, m$, in (14), we can find the unknown coefficients $x_1(t_i), i = 0, 1, \ldots, m$, as below:

$$x_1(t_i) = x_0 + \int_{t_0}^{t_i} [A(s)x_0 + B(s)u_0(s) + f(s)]\, ds + \int_{t_0}^{t_i} C(s)u_0(s)dW(s), \qquad t_0 \le t \le t_f. \tag{17}$$

By substituting (15) in (13) for $n = 2$, we can obtain

$$\begin{aligned} x_2(t) &= x_0 + \int_{t_0}^{t} [A(s)x_1(s) + B(s)u_1(s) + f(s)]\, ds + \int_{t_0}^{t} C(s)u_1(s)dW(s) \\ &\approx x_0 + \int_{t_0}^{t} \left[A(s)S_\Delta^1(s) + B(s)u_\Delta^1(s) + f(s)\right] ds + \int_{t_0}^{t} C(s)u_\Delta^1(s)dW(s), \qquad t_0 \le t \le t_f, \end{aligned} \tag{18}$$

where $u_\Delta^1(s) = -\left(N(s) + C^T(s)K(s)C(s)\right)^{-1} B^T(s)\left(K(s)S_\Delta^1(s) + g(s)\right)$. The function $x_2(t)$ on $[t_0, t_f]$ can be approximated by a similar way:

$$x_2(t) \approx S_\Delta^2(t) = \sum_{i=1}^{m-1} \psi_i^2(t)\chi_{[t_i,t_{i+1}]}(t), \tag{19}$$

where $\psi_i^2(t) = \left[x_2(t_i)(t_{i+1} - t) + x_2(t_{i+1})(t - t_i)\right]/h_{i+1}$, $i = 0, 1, 2, \ldots, m-1$. Similarly, we can obtain the unknown coefficients $x_2(t_i), k = 0, 1, \ldots, m$, by substituting the grid points $t_i, i = 0, 1, \ldots, m$, in (18), as follows:

$$x_2(t_i) \approx x_0 + \int_{t_0}^{t_i} \left[A(s)S_\Delta^1(s) + B(s)u_\Delta^1(s) + f(s)\right] ds + \int_{t_0}^{t_i} C(s)u_\Delta^1(s)dW(s), \quad t_0 \le t \le t_f. \tag{20}$$

Generally, based on the aforementioned structure, we can approximate the function $x_n(t)$ on $[t_0, t_f]$, for $n \ge 2$, as

$$x_n(t) \approx S_\Delta^n(t) = \sum_{i=1}^{m-1} \psi_i^n(t)\chi_{[t_i,t_{i+1}]}(t), \tag{21}$$

where

$$\psi_i^n(t) = \frac{1}{h_{i+1}}\left[x_n(t_i)(t_{i+1} - t) + x_n(t_{i+1})(t - t_i)\right], \qquad i = 0, 1, 2, \ldots, m-1. \tag{22}$$

In addition,

$$x_n(t_i) \approx x_0 + \int_{t_0}^{t_i} \left[A(s)S_\Delta^{n-1}(s) + B(s)u_\Delta^{n-1}(s) + f(s)\right] ds + \int_{t_0}^{t_i} C(s)u_\Delta^{n-1}(s)dW(s), \quad t_0 \le t \le t_f, \tag{23}$$

where $u_\Delta^{n-1}(s) = \left(N(s) + C^T(s)K(s)C(s)\right)^{-1} B^T(s)\left(K(s)S_\Delta^{n-1}(s) + g(s)\right)$.

Definition 1. *Let f be a function on $[a,b]$. The modulus of continuity of f is defined by*

$$\omega(f, \delta) = \sup_{x,y \in [a,b], |x-y| < \delta} |f(x) - f(y)|.$$

Lemma 1. *([19]) The function $f(t)$ is uniformly continuous on $[a,b]$ if and only if $\lim_{\delta\to 0}\omega(f,\delta)=0$.*

Theorem 3. *([21]) Let $f(t)\in C^1[a,b]$ and $S^n_\Delta(t)$ be approximations of $f(t)$ by a linear spline interpolation. Then, $\|f(t)-S^n_\Delta(t)\|_\infty \le \omega(f,\delta)$.*

Proposition 1. *Let $\{S^n_\Delta(t)\}_{n=1}^\infty$ be the solution sequence produced by the numerical successive approximations (21). Then this sequence converges to the solution $x(t)$ to (12).*

Proof. The entries of matrices $A(t)$, $B(t)$, and $C(t)$ involve continuous functions. Therefore, the terms in integrals satisfy Lipschitz and linear growth conditions. Thus the error function $E^n_\Delta(t)$ and residual function $R^n_\Delta(t)$ are defined as

$$\begin{aligned} E^n_\Delta(t) &= x_n(t)-S^n_\Delta(t), \qquad n\ge 1,\\ R^n_\Delta(t) &= x_n(t)-Z^n_\Delta(t), \qquad n\ge 2, \end{aligned}$$

where

$$Z_n(t)=x_0+\int_{t_0}^t \left[A(s)S^{n-1}_\Delta(s)+B(s)u^{n-1}_\Delta(s)+f(s)\right]ds+\int_{t_0}^t C(s)u^{n-1}_\Delta(s)dW(s), \quad t_0\le t\le t_f.$$

First, we show that

$$\lim_{|\Delta|\to 0, n\to\infty}\|R^n_\Delta(t)\|=0, \qquad n\ge 2.$$

Thus, for $n\ge 2$, by applying the Cauchy-Schwarz inequality and Doob's inequality, we have

$$\begin{aligned} \|R^n_\Delta(t)\|^2 &= \|x_n(t)-Z^n_\Delta(t)\|^2\\ &\le 2\Big\|\int_{t_0}^t \Big[\Big(A(s)x_{n-1}(s)+B(s)u_{n-1}(s)\Big)-\Big(A(s)S^{n-1}_\Delta(s)+B(s)u^{n-1}_\Delta(s)\Big)\Big]ds\Big\|^2\\ &\quad +2\Big\|\int_{t_0}^t C(s)\big(u_{n-1}(s)-u^{n-1}_\Delta(s)\big)dW(s)\Big\|^2\\ &\le 2\int_{t_0}^t \Big\|\Big[\Big(A(s)x_{n-1}(s)+B(s)u_{n-1}(s)\Big)-\Big(A(s)S^{n-1}_\Delta(s)+B(s)u^{n-1}_\Delta(s)\Big)\Big]\Big\|^2 ds\\ &\quad +2\int_{t_0}^t \Big\|C(s)\big(u_{n-1}(s)-u^{n-1}_\Delta(s)\big)\Big\|^2 ds\\ &\le L_3\int_{t_0}^t \Big\|x_{n-1}(s)-S^{n-1}_\Delta\Big\|^2 ds = L_3\int_{t_0}^t \Big\|E^{n-1}_\Delta\Big\|^2 ds \le L_3\omega^2(x_{n-1}(s),|\Delta|). \end{aligned}$$

Furthermore, $\|E^n_\Delta(t)\|^2=\|x_n(t)-Z^n_\Delta(t)+Z^n_\Delta(t)-S^n_\Delta(t)\|^2$. Hence,

$$\|E^n_\Delta(t)\|^2 \le 2\|Z^n_\Delta(t)-S^n_\Delta(t)\|^2+2\|R^n_\Delta(t)\|^2 \le 2\omega^2(Z^n_\Delta(t),|\Delta|)+2L_3\omega^2(x_{n-1}(s),|\Delta|).$$

Moreover, Z^n_Δ is uniformly continuous on $[t_0,t_f]$, and from Lemma 1, we obtain

$$\lim_{|\Delta|\to 0, n\to\infty}\|R^n_\Delta(t)\|=0, \qquad n\ge 2.$$

Thus, by Theorem 1, we have $\lim_{|\Delta|\to 0, n\to\infty}\|x_n(t)-S^n_\Delta(t)\|=0$, for $n\ge 2$. □

Via applying a numerical integration method (e.g., Legendre Gauss method), we can approximate the first integral part of (23). Regarding the stochastic integral part, we can apply the Ito approximation as follows:

$$\int_{t_0}^{t_i} C(t)u_\Delta^1(t)dW(t) = \sum_{j=0}^{j=i-1} C(t_j)u_\Delta^1(t_j)\big(W(t_{j+1}) - W(t_j)\big).$$

Therefore, we obtain the ith optimal control law as follows:

$$u_i(t) = -\big(N(t) + C^T(t)K(t)C(t)\big)^{-1} B^T(t)\Big(K(t)x_i(t) + g(t)\Big).$$

According to Theorem 1, the solution sequence $x_i(t)$ is almost surely uniform convergence. We define $\hat{x}(t)$ as the limits of sequence $x_i(t)$. The control sequence $u_i(t)$ is only related to $x_i(t)$; so it is also uniformly convergent. Assume $u^*(t)$ as the limit of sequence $u_i(t)$. Summarizing the above, we obtain the following theorem.

Theorem 4. *Consider the problem of minimizing the cost functional* (2) *subject to system* (1)*. Then the optimal control law is obtained as follows:*

$$u^*(t) = -\big(N(t) + C^T(t)K(t)C(t)\big)^{-1} B^T(t)\Big(K(t)\hat{x}(t) + g(t)\Big). \tag{24}$$

4 Suboptimal control design strategy

In this section, we intend to elaborate practically on the results presented in the previous section. Since we cannot calculate the optimal control law in (24), via substituting a finite positive integer l in (24) for $n \to \infty$, we can explore a suboptimal control law with respect to its practical applications. Thus, the lth order suboptimal control law is obtained as follows:

$$u_l(t) = -\big(N(t) + C^T(t)K(t)C(t)\big)^{-1}(t) B^T(t)\Big(K(t)x_l(t) + g(t)\Big). \tag{25}$$

Generally, the lth integer in (25) is specified generally on the basis of a concrete control precision. Let $l = i$, and consider the lth order suboptimal control law from (25). Then, the following optimal value of quadratic performance index can be calculated:

$$J_l \;=\; \mathbb{E}\left\{\frac{1}{2}\int_{t_0}^{t_f}\Big[x_l^T(t)M(t)x_l(t) + u_l^T(t)N(t)u_l(t)\Big]dt + \frac{1}{2}x_l^T(t_f)Dx_l(t_f)\right\}, \tag{26}$$

where $u_l(t)$ is obtained from (25) and $x_l(t)$ is the corresponding state trajectory.

Regarding the accuracy analysis, the following criterion is taken into consideration. The suboptimal control (25) has the desirable accuracy, if, for given positive constants $\varepsilon > 0$, the following condition holds jointly:

$$Er_l = \frac{\|x_l - x_{l-1}\|}{\|x_l\|} < \varepsilon,$$

where $\|\cdot\|$ serves as the Euclidean norm. If the tolerance error bound $\varepsilon > 0$ is appropriately selected on small amounts, the lth order suboptimal control law reaches very close to the optimal control law $u^*(t)$

and the value of quadratic performance index in (26) will be very near to its optimal value J^*. According to Theorems 1 and 4, the boundary state conditions will be tightly satisfied.

Now, in order to confirm the accuracy of solutions, the following algorithm of the proposed method is presented having the low computational complexity:

Algorithm 1:

Input: $t_0, t_f \in \mathbb{R}, m, \mathscr{N} \in \mathbb{N}$.

Step 1: Solve the matrix $K(t)$ and vector $g(t)$ from the differential equations (3) and (4).

Step 2: Put $x_0(t) = x_0$ and $u_0(t) = -\big(N(t)+C^T(t)K(t)C(t)\big)^{-1}B^T(t)\Big(K(t)x_0+g(t)\Big)$.

Step 3: Compute $x_1(t_i), i = 0,1,\ldots,m$ from (17).

Step 4: Define the functions $\psi_i^1(t), i = 0,1,\ldots,m-1$ from (16).

Step 5: Put $x_1(t) \approx S_\Delta^1(t)$ from (15).

Step 6: Put $u_1(t) = u_\Delta^1(t)$.

Step 7: For $n = 2,3,\ldots,\mathscr{N}$
Compute $x_n(t_i), i = 0,1,\ldots,m$ from (23).
Define the functions $\psi_i^n(t), i = 0,1,\ldots,m-1$ from (22).
Compute $x_n(t) \approx S_\Delta^n(t)$ from (21).
Put $u_n(t) = u_\Delta^n(t)$.
If $Er_n = \dfrac{\|x_n - x_{n-1}\|}{\|x_n\|} < \varepsilon$, then stop and go to step 8;

Step 8: Stop the algorithm; set $u_n(t)$ as the desirable close-loop suboptimal control law.

The following examples illuminate how we can find the optimal solution via applying the proposed method. Symbolic computation software MATLAB was utilized in order to perform the codes.

Example 1. Consider the problem of minimizing cost functional

$$J = \mathbb{E}\left\{\frac{1}{2}\int_0^1 x^2(t)dt + \frac{1}{2}x^2(1)\right\},$$

subject to the system of differential equation

$$dx = \big(x(t)+2u(t)+e^{(t-1)}\big)dt + 2u(t)dW(t), \qquad 0 \le t \le 1,$$

with initial condition $x(0) = 5$. We have $A(t) = M(t) = D = 1$, $B(t) = C(t) = 2$, $N(t) = 0$, and $f(t) = e^{(t-1)}$. Therefore, the Riccati differential equation becomes

$$\dot{K}(t) = -K(t) - 1, \qquad 0 \le t \le 1,$$

with boundary condition $K(1) = 1$, whose unique solution is $K(t) = -1+2e^{1-t}$. Also we have

$$\dot{g}(t) = e^{(t-1)} - 2, \qquad 0 \le t \le 1,$$

with boundary condition $g(1) = 0$, whose unique solution is $g(t) = e^{(t-1)} - 2t + 1$.

First, we take a partition Δ with nodal points on $[t_0, t_f]$ as $\Delta : t_0 < t_1 < \cdots < t_{m-1} < t_m = t_f$, Using relation (17) for $i = 1, 2, \ldots, m$, we have

$$x_1(t_i) = 5 + \int_0^{t_i} [5 + u_0(s) + f(s)]\, ds + \int_{t_0}^{t_i} u_0(s) dW(s), \qquad t_0 \leq t \leq t_f,$$

where $u_0(t) = -(-2+4e^{1-t})^{-1}\big(5(-1+2e^{1-t}) + e^{(t-1)} - 2t + 1\big)$. Then we compute the functions $\psi_i^1(t)$ from (16), and using (15), we obtain

$$x_1(t) \approx S_\Delta^1(t) = \sum_{i=1}^{m-1} \psi_i^1(t) \chi_{[t_i, t_{i+1}]}(t).$$

In the following, via applying (21) and (22), we obtain

$$x_n(t) \approx S_\Delta^n(t) = \sum_{i=1}^{m-1} \psi_i^n(t) \chi_{[t_i, t_{i+1}]}(t),$$

where $\psi_i^n(t) = \big[x_n(t_i)(t_{i+1} - t) + x_n(t_{i+1})(t - t_i)\big]/h_{i+1}$, $i = 0, 1, 2, \ldots, m-1$. In addition,

$$x_n(t_i) \approx 5 + \int_0^{t_i} \left[S_\Delta^{n-1}(s) + u_\Delta^{n-1}(s) + e^{(t-1)}\right] ds + \int_{t_0}^{t_i} u_\Delta^{n-1}(s) dW(s), \qquad 0 \leq t \leq 0.$$

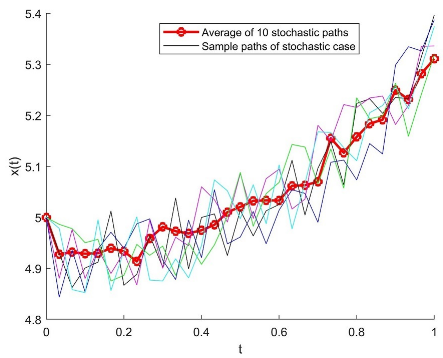
Initial approximation $x_0(t) = x_0$ can be taken into account. Through persistent use of this process, each iteration can result in $x(t)$ and $u(t)$. As a result, a suboptimal cost functional can be obtained. It should be noted that $x(t)$ and $u(t)$ are considered as random variables and that the results are achieved with probability one. Thus, optimal value and suboptimal cost functionals are considered random, as well. Therefore, the simulation 100 times, optimal value, and suboptimal cost functionals are run, and the errors averages are estimated.

As (26) demonstrates, suboptimal cost functionals are acquired. In order to obtain an accurate enough suboptimal control law, the proposed algorithm was applied with the tolerance error bound $\varepsilon = 10^{-4}$. Table 1 illustrates the errors at the different iteration times. As observed in Table 1, after four iterations, the convergence is achieved only; that is, $Er_4 = 7 \times 10^{-6} < 10^{-4}$.

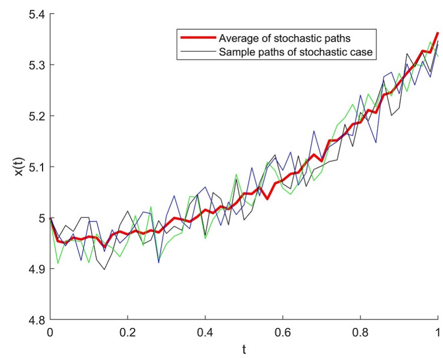
The results demonstrate that in $l = 4$, this method converges to solution $x^*(t) \cong x_4(t)$ in probability. Thus, the optimal control is $u^*(t) \cong u_4(t)$. Moreover, Figures 1 and 2 show plots of $x_4(t)$ and $u_4(t)$.

Table 1: Errors at the different iterations for Example 1.

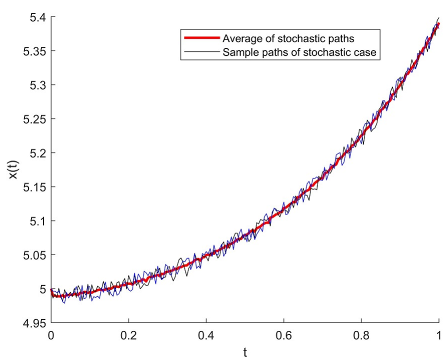
iteration time l	Er_l
1	-
2	0.8707
3	0.4668
4	7×10^{-6}
⋮	⋮



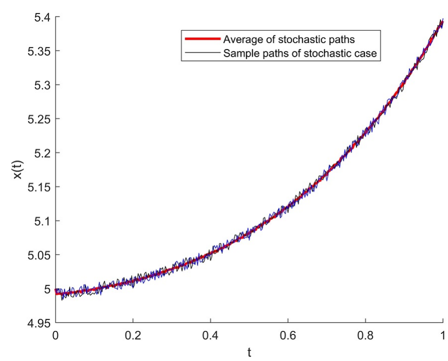
(a) The optimal path for $m = 30$ with 10 runs.



(b) The optimal path for $m = 50$ with 30 runs.

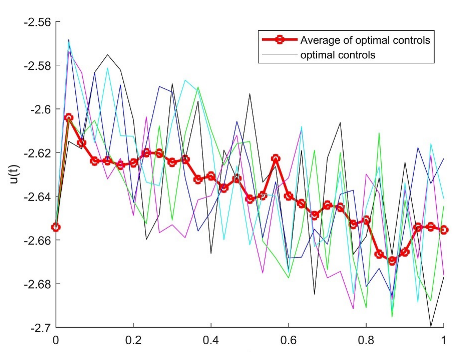


(c) The optimal path for $m = 200$ with 100 runs.

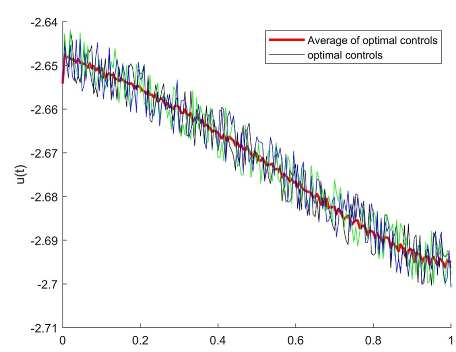


(d) The optimal path for $m = 300$ with 200 runs.

Figure 1: Comparing the optimal path in Example 1.



(a) The optimal control for $m = 30$ with 10 runs.



(b) The optimal control for $m = 200$ with 100 runs.

Figure 2: Comparing the optimal control in Example 1.

5 Application in finance

In this section, we consider a financial market consisting of two investment options. We can either invest money in a bond (a risk-less bank account) with a fixed interest rate r, with prices

$$db(t) = b(t)rdt, \quad b_0 = 1,$$

or we can invest money in a risky asset with an expected rate of return $\mu > r$ and volatility $\sigma(t) > 0$. Let $\pi(t)$ be the proportion of our money invested in the stock at the time t. Assume that $x(t)$ is our money at the time t, which satisfies the following stochastic differential equation:

$$dx(t) = x(t)\Big[r + (\mu - r)\pi(t)\Big]dt + \sigma(t)\pi(t)x(t)dW(t), \qquad t_0 \le t \le t_f, \tag{27}$$

with initial condition $x(t_0) = x_0$, where $x_0 > 0$ is the initial wealth and our control is $\pi(t)$. In this market, the cost function can be the final cost or total cost (holding cost and final cost). The goal is to minimize cost function by changing $\pi(t)$ in during $[t_0, t_f]$.

First, we let final cost for this market be $\frac{1}{2}x^2(t_f)Q$. Thus, the objective function is

$$\min_{\pi(t)} \mathbb{E}\left\{\frac{1}{2}x^2(t_f)Q\right\}. \tag{28}$$

It is evident that the presented model has a continuous optimal control. Here, we explain how the results of previous sections were applied in order to solve this problem using the stochastic LQR problem. Normally, we should turn this problem formula into the stochastic LQR problem (1)–(2). We can rewrite (27) as the following equation

$$dx(t) = \Big[rx(t) + (\mu - r)\pi(t)x(t)\Big]dt + \sigma(t)\pi(t)x(t)dW(t), \qquad t_0 \le t \le t_f.$$

If we get $u(t) = u(t, x(t)) = \pi(t)x(t)$, then this problem is converted to a stochastic LQR problem, and using the obtained results, the problem can be solved. Therefore, (27) is rewritten as

$$dx(t) = \Big[rx(t) + (\mu - r)u(t)\Big]dt + \sigma(t)u(t)dW(t), \qquad t_0 \le t \le t_f, \tag{29}$$

with initial condition $x(t_0) = x_0$. Note that here we have

$$A(t) = r, \quad B(t) = (\mu - r), \quad C(t) = \sigma(t), \quad f(t) = 0, \quad M(t) = N(t) = 0, \quad \text{and} \quad D = Q.$$

As a result, the optimal control problem (1)–(2) is converted to equivalent stochastic LQR problem (28)–(29). Thus two differential equations (2) and (3) are obtained as below:

$$\dot{K}(t) = -2rK(t) + (\mu - r)^2\left(\sigma^2(t)\right)^{-1}K(t), \qquad t_0 \le t \le t_f,$$

with boundary condition $K(t_f) = Q$, and

$$\dot{g}(t) = -rg(t) + (\mu - r)^2\Big(\sigma^2(t)\Big)^{-1}g(t), \qquad t_0 \le t \le t_f,$$

with boundary condition $g(t_f) = 0$. It is evident that $g(t) = 0$ for all $[t_0, t_f]$. Thus, using the LQR problem, we have $u^*(t) = -\big(\sigma^2(t)\big)^{-1}(\mu - r)x(t)$, where $u^*(t) = \pi^*(t)x(t)$. It can be concluded $\pi^*(t) = -(\mu - r)/\sigma^2(t)$, and so the problem is solved.

Now let the cost function be total cost. Consequently, the objective function is as follows:

$$\min_{\pi(t)} \mathbb{E}\left\{\frac{1}{2}\int_{t_0}^{t_f}\Big((\pi(t)x(t))^2 N(t)\Big)dt + \frac{1}{2}x^2(t_f)Q\right\}.$$

As in the previous example, we convert the problem to the stochastic LQR problem (29) with the cost function

$$\min_{u} \mathbb{E}\left\{\frac{1}{2}\int_{t_0}^{t_f}\Big((u(t))^2 N(t)\Big)dt + \frac{1}{2}x^2(t_f)Q\right\},$$

where $u(t) = \pi(t)x(t)$. Hence the differential equation (2) is obtained as below:

$$\dot{K}(t) = -2rK(t) + (\mu - r)^2 K^2(t)\big(N(t) + \sigma^2(t)K(t)\big)^{-1}, \qquad t_0 \le t \le t_f,$$

with boundary condition $K(t_f) = Q$. In this example, $g(t) = 0$ for all $[t_0, t_f]$. Thus, using the LQR problem, we have $u^*(t) = -\big(N(t) + \sigma^2(t)K(t)\big)^{-1}(\mu - r)K(t)x(t)$, where $u^*(t) = \pi^*(t)x(t)$. Consequently $\pi^*(t) = -\big(N(t) + \sigma^2(t)K(t)\big)^{-1}(\mu - r)K(t)$.

Example 2. Consider a financial market with the following stochastic differential equation:

$$dx = x(t)\Big[2 + 3\pi(t)\Big]dt + 2e^t\pi(t)x(t)dW(t), \qquad 0 \le t \le 1,$$

with initial condition $x(0) = 10$. The objective function is considered as

$$\min_{\pi(t)} \mathbb{E}\left\{\frac{1}{2}\int_{0}^{1}\Big((\pi(t)x(t))^2 e^{2t}\Big)dt + \frac{1}{4}x^2(1)\right\}.$$

Here, we have $r = 2$, $\mu = 5$, $\sigma(t) = 2e^t$, $N(t) = e^{2t}$, and $Q = \dfrac{1}{2}$. Therefore, the Riccati differential equation is $\dot{K}(t) = -4K(t) + 9K^2(t)\big(e^{2t} + 4e^{2t}\big)^{-1}$, $0 \le t \le 1$, with boundary condition $K(1) = \dfrac{1}{2}$, whose unique solution is

$$K(t) = \frac{10e^4}{3e^{(4-2t)} - 3e^{(4t-2)} + 20e^{4t}},$$

As a result,

$$\pi^*(t) = -3\Big(\frac{10e^{(2t+4)}}{3e^{(4-2t)} - 3e^{(4t-2)} + 20e^{4t}} + 4e^{-2t}\Big).$$

6 Conclusion

In this paper, we have studied a general class of stochastic LQR, in which the state equation has been formulated in terms of control-dependent diffusion coefficients. It was observed that the optimal control problem might be well-posed even when the control weight costs are indefinite and that the diffusion term in the state equation depends on the control term. The stochastic LQR problem will be solvable if there are solutions to two differential equations. Indeed, using these two differential equations, a new numerical approach has been presented to solve this problem on the basis of linear spline interpolation, which eliminates the calculation (computational) complexities. Ultimately, we have demonstrated the performance of this algorithm in a financial market in order to minimize the final cost. It is worth noting that the underlying purpose of this study is to embed the original problems into the stochastic LQR optimal control problem.

References

[1] P. Benner, J.R. Li, T. Penzl, *Numerical solution of largescale Lyapunov equations, Riccati equations, and linearquadratic optimal control problems*, Numer Linear Algebra Appl. **15** (2008) 755–777.

[2] A. Bensoussan, *Lecture on Stochastic Control, Part I, in Nonlinear Filtering and Stochastic Control*, Lecture Notes in Math, 972, Springer-Verlag, Berlin, (1983) 1–39.

[3] J.M. Bismut, *An introductory approach to duality in optimal stochastic control*, SIAM review **20** (1978) 62–78.

[4] S. Chen, X.J. Li, M.H.A. Davis, *Linear Estimation and Stochastic Control*, Chapman and Hall, London, 1977.

[5] S.P. Chen, X.J. Li, X.Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim. **36** (1998) 1685–1702.

[6] D. Duffie, L. Epstein, *Stochastic differential utility, econometrica*, J. Econom. **60** (1992) 353–394.

[7] D. Duffie, J. Ma, J. Yong, *Black's consol rate conjecture*, Ann. Appl. Probab. **5** (1995) 356–382.

[8] W.H. Fleming, *Optimal control of partially observable diffusions*, SIAM J. Control **6** (1968) 194–214.

[9] U.G. Haussmann, *A Stochastic Maximum Principle for Optimal Control of Diffusions*, Longman, Scientific and Technical, 1986.

[10] M.H. Heydari, M.R. Hooshmandasl, C. Cattani, F.M. Maalek Ghaini, *An efficient computational method for solving nonlinear stochastic It integral equations*, Application for stochastic problems in physics, J. Comput. Phys. **283** (2015) 148–168.

[11] M.H. Heydari, M.R. Hooshmandasl, A. Shakiba, C. Cattani, *Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations*, Nonlinear Dyn. **85** (2016) 1185–1202.

[12] A. Ichikawa, *Dynamic programming approach to stochastic evolution equations*, SIAM J. Control Optim. **17** (1979) 152–174.

[13] N. El Karoui, S. Peng, M.C. Quenez, *Backward stochastic differential equations in finance*, Math. Financ. **7** (1997) 1–71.

[14] H.J. Kushner, *Necessary conditions for continuous parameter stochastic optimization problems*, SIAM J. Control **10** (1972) 550–565.

[15] J. Ma, J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications*, Springer Science and Business Media, 1999.

[16] P. McLane, *Optimal stochastic control of linear systems with state-and control-dependent disturbances*, IEEE Trans. Automat. Contr. **16** (1971) 793–798.

[17] E. Pardoux, S. Peng, *Adapted solution of a backward stochastic differential equation*, Syst. Control. Lett. **14** (1990) 55–61.

[18] S. Peng, *A general stochastic maximum principle for optimal control problems*, SIAM J. Control Optim. **28** (1990) 966–979.

[19] T.J. Rivlin, *An Introduction To the Approximation of Functions*, Dover Publications, 1981.

[20] M. Saffarzadeh, G.B. Loghmani, M. Heydari, *An iterative technique for the numerical solution of nonlinear stochastic Ito-Volterra integral equations*, J. Comput. Appl. Math. **333** (2018) 74–86.

[21] L.L. Schumaker, *Spline Functions: Computational Methods*, SIAM, 2015.

[22] S.M. Sundaresan, *Continuoustime methods in finance: A review and an assessment*, J. Financ. **55** (2000) 1569–1622.

[23] J.C Vitanic, J. Ma, *Hedging options for a large investor and forward-backward SDE's*, Ann. Appl. Probab. **6** (1996) 370–398.

[24] W.M. Wonham, *On a matrix Riccati equation of stochastic control*, SIAM J. Control **6** (1968) 681–697.

[25] A.M. Wazwaz, *Linear and Nonlinear Integral Equations, Methods and Applications*, First Edition, Springer, 2011.

[26] J. Yong, X.Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer Science and Business Media, 1999.

[27] X.Y. Zhou, *A unified treatment of maximum principle and dynamic programming in stochastic controls*, Int. J. Prob. Stoch. Process. **36** (1991) 137–161.

[28] X.Y. Zhou, *Sufficient conditions of optimality for stochastic systems with controllable diffusions*, IEEE Trans. Automat. Contr. **41** (1996) 1176–1179.

[29] X.Y. Zhou and D. Li, *Continuous-time mean-variance portfolio selection: A stochastic LQ framework*, Appl. Math. Optim. **42** (2000) 19–33.