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Optimal partition invariancy in multi-parametric linear optimization

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Abstract. In a linear optimization problem, objective function, coefficients matrix, and the right-hand side might be perturbed with distinct parameters independently. For such a problem, we are interested in finding the region that contains the origin, and the optimal partition remains invariant. A computational methodology is presented here for detecting the boundary of this region. The cases where perturbation occurs only in the coefficients matrix and right-hand side vector or the objective function are specified as special cases. The findings are illustrated with some simple examples.

Keywords: Optimal partition, multi-parametric linear optimization, invariancy analysis.
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1 Introduction

Analyzing the effect of data perturbation on the optimal solution in a Linear optimization (LO) produces insightful interpretations, especially when this variation occurs via parameters. Measurement and forecasting errors of data or fluctuating market situations may pose such problems, and having a clearer view would be more valuable in the decision-making process. This problem may be originated from the managerial point of view in a production plan. In an unperturbed model, the objective would be maximizing the profit, while there are some limitations on available resources, capacities, and demand. Here, variables are potential productions levels. These data and variables are incorporated in technological constraints. Therefore, optimal values for the variables refer to the best production levels. Managers want to know the effect of such perturbations on the current optimal production plan.

Parametric LO has been studied from diverse standpoints. Here, we consider the version that parameters are present in the coefficients matrix, right-hand side vector, and the objective function, which can vary independently. Now, consider the primal problem

$$P(\varepsilon,\lambda,\gamma): \quad \min\{(c+\gamma\triangle c)^T x: \ s.t.\ (A+\varepsilon\triangle A)x = b+\lambda\triangle b,\ x\geq 0\},$$

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and its dual

$$D(\varepsilon,\lambda,\gamma): \quad \max\{(b+\lambda\triangle b)^T y \ : \ s.t.\ (A+\varepsilon\triangle A)^T y+s=c+\gamma\triangle c,\ s\geq 0\},$$

where $\varepsilon,\lambda,\gamma\in\mathbb{R}$ are parameters, $\triangle c\in\mathbb{R}^n$, $\triangle b\in\mathbb{R}^m$ and $\triangle A\in\mathbb{R}^{m\times n}$ are any corresponding perturbations. The vectors $x,s\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$ are unknowns of the problems which will be determined for any given values of the parameters. To indicate that $\triangle A,\triangle b$ or $\triangle c$ is zero in a problem, we set the corresponding coefficient to zero.

It is apparent that γ in the constraints of $P(\varepsilon,\lambda,\gamma)$ and λ in the constraints of $D(\varepsilon,\lambda,\gamma)$ are absent. Thus, for simplicity, we drop the corresponding parameters in the notation likewise. Further, when one of the perturbing directions $\triangle A$, $\triangle b$, or $\triangle c$ is zero, or the parameter value is ineffective on the problem, we delete the corresponding parameter from the notations.

A point $x\geq 0$ is called a feasible solution of $P(\varepsilon,\lambda,\gamma)$ if $(A+\varepsilon\triangle A)x=b+\lambda\triangle b$, and a vector $(y,s),s\geq 0$ is called a feasible solution of $D(\varepsilon,\lambda,\gamma)$ if $(A+\varepsilon\triangle A)^T y+s=c+\gamma\triangle c$. They are optimal correspondingly if and only if $x^T s=0$ by the complementary slackness theorem. The feasible solution sets of these problems are denoted by $\mathscr{P}(\varepsilon,\lambda,\gamma)$ and $\mathscr{D}(\varepsilon,\lambda,\gamma)$, respectively. Any feasible solution with superscript with asterisk indicates its optimality. The primal-dual optimal solution (x^*,y^*,s^*) is strictly complementary if $x_j^*+s_j^*>0, j=1,\ldots,n$, and it exists by the Goldman-Tuckers theorem [7] if the feasible solution sets of both problems are non-empty. A strictly complementary optimal solution can be obtained by an interior point method [15], and it will partition the index set $\{1,\ldots,n\}$ uniquely as $\pi_{\varepsilon\lambda\gamma}=(B_{\varepsilon\lambda\gamma},N_{\varepsilon\lambda\gamma})$, where

$$B_{\varepsilon\lambda\gamma}:=\{j|x_j^*>0,\ \text{for some } x^*\in\mathscr{P}^*(\varepsilon,\lambda,\gamma)\},$$

$$N_{\varepsilon\lambda\gamma}:=\{j|s_j^*>0,\ \text{ for some } (y^*,s^*)\in\mathscr{D}^*(\varepsilon,\lambda,\gamma)\}.$$

This partition is known as an optimal partition, which is unique agsinst the basic partition that would not be unique in general.

In this paper we assume that the problem

$$\min\{c^T x:\ Ax=b,\ x\geq 0\},$$

and its dual

$$\max\{b^T y:\ A^T y+s=c,\ s\geq 0\},$$

are given with the optimal partition $\pi=(B,N)$. We want to find the region in which for the parameters' values in it we have $\pi_{\varepsilon\lambda\gamma}=\pi$. This region is referred to as the invariancy region and denoted by $\Lambda_{\varepsilon\lambda\gamma}$. A computational methodology is presented to determine this region, along with some numerical examples for illustration.

Let us explain the importance of the study and its results in a prototype example. Let the problem $P(\varepsilon,\lambda,\gamma)$ be modeled for a production problem. The objective function defines the negation of the profit, and constraints characterize some technological and financial limitations. In this way, the right-hand side denotes available resources. Variation in the left-hand side of constraints (parameterized by ε along with a direction $\triangle A$) can be viewed as technological change in the production plan. On the other hand, variability in the available source would be depicted by variation along with a direction $\triangle b$ (parameterized by λ). Moreover, profit fluctuation in the ever-developing market could be considered as a perturbation along with a direction $\triangle c$ (parameterized by γ). Observe that positive variables in an optimal solution refer to active productions lines, and their values reflect the production levels.

The first realization is that these parameters play their roles independently in practice. Knowing the region for these parameters where the optimal partition is invariant, delivers some managerial insights. First, non of the productions with indices in N are produced in any optimal production plans for parameters values in the recognized region. Moreover, if B in the optimal partition is associated with a basic optimal solution, the basis is invariant in this region. Additionally, if this solution is nondegenerate, the active production lines are active but with other production levels. This interpretation is also valid when B corresponds with an underdetermined matrix.

The rest of paper is organized as follows. Some related results that appeared previously are reviewed in Section 2. The necessary assumption on the perturbing direction $\triangle A$, and some preliminary concepts are stated in Section 3. Some fundamental properties of the presented method are described in Section 4. The method of detecting the boundaries of the desired invariancy region are expressed in Section 5. Some examples are provided in Section 6 to illustrate the procedure. Some concluding remarks are stated in Section 7.

2 Literature review

Some works corresponding to optimal partition invariancy in an LO problem are reviewed here. When either the objective function or the coefficients matrix, A had been perturbed and then analyzed thoroughly in [15]. It has been proved that optimal partition invariancy intervals are open, and the representation of the optimal value function is different at two sides of the joining endpoint. The authors referred to these points as breakpoints; we may refer to them as pass points and the two intervals with a common point as adjacent intervals. The optimal value function is a continuous piecewise linear function over these intervals and is not differentiable at the pass point. Since the slope of the objective function is constant on each invariancy interval, these intervals as also referred to as linearity intervals.

Optimal partition invariancy analysis of simultaneous perturbation with identical parameters in the objective function and the right-hand side vectors has been studied in [6]. In this case, each invariancy region is also an open interval or a singleton. The optimal value function is a continuous piecewise quadratic function with different representations on two adjacent intervals and thus is not differentiable at the pass point. Moreover, optimal partitions are different on these intervals and their endpoints, too. These points are then referred to as transition points instead of breakpoints. This case has also been studied when the problem is in canonical form [9]. Though the concept of the invariancy region in the canonical form is somehow dissimilar to its definition in the standard form with a different interpretation, the obtained results are almost the same.

The case when these two parameters vary independently has been investigated in [5]. The authors proved that invariancy regions are rectangles (in special cases, lines and points), making a mesh-like convex area that might be unbounded. Moreover, the optimal value function is a bivariate piecewise linear over this area, and optimal partitions are different over two adjacent regions and the pass lines. Particularly, the pass lines are exactly determined in polynomial time. The case when the objective function and the right-hand side of constraints are stated with different parameters that vary independently has been studied in [10]. A special case has been investigated in [11] that there are two parameters in the objective function and two in the right-hand side vector of the constrains, each of which varies independently. The authors presented an algorithm just for identifying the invariancy region involving the origin when the right-hand side is perturbed. They also proved that this region is a polyhedron and presented a closed form of the optimal value function on this region.

The case when only coefficient matrix is perturbed with a single parameter has been investigated in [4]. The authors applied the concept of admissible direction [8] and proved that with this property of the perturbing direction guarantees the convexity of invariancy regions that are open intervals. Presented tools could identify these regions exactly. Further, the optimal value function is identified as a fractional function over each interval. It is necessary to mention that the domain of the optimal value function might be open in this case, while it is closed when the right-hand side or the objective function data is perturbed. Optimal partition invariancy analysis with two independently perturbing parameters on the constraints matrix A has been studied [14]. The authors presented an algorithm that can determine the invariancy region containing the origin. Then, this algorithm has been used to present a solution approach for the non-linear non-convex pooling problem. It is worth noting that the obtained solutions enjoy insightful interpretations for this problem.

As far as we noticed, the effect of the perturbation on the constraints matrix A along with the right-hand side vector b and cost coefficients c has not been considered simultaneously. In [12], the authors investigated a family of the uni-parametric LO problem when the right-hand side vector b and the coefficients matrix A are linearly perturbed with the same parameter. They introduced the concepts of induced optimal partition and change point. These points can be also refereed to as pass point equivalently. Existence of such points in the definition of induced optimal partition distinguishes it from the well-known optimal partition. Moreover, they presented a computational algorithm to determine some intervals where induced optimal partitions are invariant. The algorithm was implemented on some large-scale test problems to illustrate its behavior. A closed form of the optimal value function is also presented on each invariancy region.

The effect on the known optimal partition and the behavior of the optimal value function has been investigated when the objective function and the coefficients matrix are perturbed with the same parameter [13]. A prototype example of dynamical systems has been considered to show the usefulness of the obtained results in practice. As before, the optimal induced partition invariancy intervals are identified for each considered parameter value. It was shown that the optimal value function is continuous piecewise fractional in the interior of its domain, while it may not be necessarily continuous at the endpoints.

3 Preliminaries

Let $(\varepsilon,\lambda,\gamma)$ run throughout a nonempty set $\Lambda_{\varepsilon\lambda\gamma}$. Since optimal solutions set $P(\varepsilon,\lambda,\gamma)$ is a subset of the corresponding feasible solution set, then connectivity of $\Lambda_{\varepsilon\lambda\gamma}$ refers back to the continuity of the solution sets of $(A+\varepsilon\triangle A)x = b+\lambda\triangle b$ with respect to ε and λ. If $A+\varepsilon\triangle A$ is invertible over a dense open subset $U\subseteq\mathbb{R}$ or rank$(A+\varepsilon\triangle A)$ is constant for all $\varepsilon\in\mathbb{R}$, the continuity of the solution set with respect to ε and λ is a straightforward result in standard linear algebra. When $(A+\varepsilon\triangle A)x = b+\lambda\triangle b$ is underdetermined and rank$(A+\varepsilon\triangle A)$ differs for distinct values of ε, to guarantee this continuity, one must impose some regularity conditions on the problem. As a result, the invariancy region is not necessarily connected in general. Here, we are interested only in finding the largest connected set that includes the origin $(\varepsilon,\lambda,\gamma) = (0,0,0)$.

Let us elaborate the concept of continuity of the feasible solution set $\mathscr{P}(\varepsilon,\lambda,\gamma)$. Analogous explanation can be provided for $\mathscr{D}(\varepsilon,\lambda,\gamma)$. Consider the multifunction Φ defined on $\mathbb{R}^3$ that for a given parameter $(\varepsilon,\lambda,\gamma)$ produces the set $\mathscr{P}(\varepsilon,\lambda,\gamma)$ a subset of $\mathbb{R}^n$. The domain of Φ is the set

$$Dom(\Phi)=\{(\varepsilon,\lambda,\gamma)|\Phi(\varepsilon,\lambda,\gamma)\neq\varnothing\},$$

and shows the region for the values of parameters where $P(\varepsilon,\lambda,\gamma)$ is feasible. For any subset C of $\mathbb{R}^3$, we write $\Phi(C)$ for the image $\bigcup_{(\varepsilon,\lambda,\gamma)\in C}\Phi(\varepsilon,\lambda,\gamma)$, and the range of Φ is the set $R(\Phi)=\Phi(\mathbb{R}^3)$. The inverse multifunction $\Phi^{-1}:P(\mathbb{R}^n)\rightarrow\mathbb{R}^3$ is defined by

$$(\varepsilon,\lambda,\gamma)\in\Phi^{-1}(x)\Leftrightarrow x\in\Phi(\varepsilon,\lambda,\gamma)\text{ for }(\varepsilon,\lambda,\gamma)\ \in\mathbb{R}^3.$$

The graph of this multifunction is defined as

$$G(\Phi)=\{(\varepsilon,\lambda,\gamma,x)\in\mathbb{R}^3\times\mathbb{R}^n,x\in\Phi(\varepsilon,\lambda,\gamma)\}.$$

We say the multifunction Φ is Lower Semi-Continuous (LSC) at a point $(\varepsilon_0,\lambda_0,\gamma_0,x)$ in its graph if, for all neighbourhoods V of x, the image $\Phi(\varepsilon_0,\lambda_0,\gamma_0)$ intersects V for all points $(\varepsilon,\lambda,\gamma)$ close to $(\varepsilon_0,\lambda_0,\gamma_0)$. This means that $(\varepsilon_0,\lambda_0,\gamma_0)$ must lie in $int(Dom(\Phi))$. Equivalently, for any sequence of points $(\varepsilon_k,\lambda_k,\gamma_k)$ approaching $(\varepsilon_0,\lambda_0,\gamma_0)$, there is a sequence of points $x^k\in\Phi(\varepsilon_k,\lambda_k,\gamma_k)$ approaching x. If, for $(\varepsilon_0,\lambda_0,\gamma_0)$ in the domain, this property holds for all points x in $\Phi(\varepsilon_0,\lambda_0,\gamma_0)$, we say Φ is LSC at $(\varepsilon_0,\lambda_0,\gamma_0)$.

We say Φ is open at a point $(\varepsilon,\lambda,\gamma,x^0)$ in its graph if, for all neighbourhoods U of $(\varepsilon,\lambda,\gamma)$, the point x^0 lies in $int(\Phi(U))$. In particular, x^0 must lie in $int(R(\Phi))$. Equivalently, for any sequence of points (x^k) approaching x^0 there is a sequence of points $(\varepsilon_k,\lambda_k,\gamma_k)$ approaching $(\varepsilon,\lambda,\gamma)$ such that $x^k\in\Phi(\varepsilon_k,\lambda_k,\gamma_k)$ for all k. If, for x^0 in the range, this property holds for all points $(\varepsilon,\lambda,\gamma)$ in $\Phi^{-1}(x^0)$, we say Φ is open at x^0. In some documents, openess is referred to as Upper Semi-Continuity (USC) [1]. A multifunction is continuous when it is LSC and USC. We refer to [3] for more detail.

For fixed values $\hat{\lambda}$ and $\hat{\gamma}$, a given real ε, and perturbing direction $\triangle A$, $\varepsilon\triangle A$ is said an admissible change if the problem $P(\varepsilon,\hat{\lambda},\hat{\gamma})$ has optimal solution. It can be shown that $\varepsilon\triangle A$ is not generally an admissible change for all $\varepsilon\in(0,\varepsilon^*)$ just because $\varepsilon^*\triangle A$ is an admissible change. A perturbing direction $\triangle A$ is an admissible direction if there exists $\varepsilon^*>0$, such that $\varepsilon\triangle A$ is an admissible change for all $\varepsilon\in[0,\varepsilon^*)$ [8]. Let $\mathscr{A}$ be the set of admissible changes of problem $P(\varepsilon,\hat{\lambda},\hat{\gamma})$ and be nonempty. This set is not convex in general [8]. For an admissible direction $\triangle A$, let $\hat{\varepsilon}:=\sup\{\varepsilon^*:\ \varepsilon\triangle A\in\mathscr{A},\varepsilon\in[0,\varepsilon^*)\}$, and $\hat{\Lambda}_{\hat{\lambda}\hat{\gamma}}:=\{\varepsilon:\ \varepsilon\triangle A\in\mathscr{A}\}$. It is proved [8] that if $\mathscr{A}=\bigcup_{k=1}^{K}\mathscr{P}_k$, and each $\mathscr{P}_k$ is a polyhedron containing the origin, then $\hat{\Lambda}_{\hat{\lambda}\hat{\gamma}}$ is simply an interval. Thus, we impose the following property on $\triangle A$ for it.

Assumption: For the given optimization problem $P(\varepsilon,\lambda,\gamma)$, $\triangle A$ is an admissible direction for all $(\varepsilon,\lambda,\gamma)\in\Lambda_{\varepsilon\lambda\gamma}$. That is for such $(\varepsilon,\lambda,\gamma)$, $\hat{\Lambda}_{\lambda\gamma}$ is an interval.

It is clear that feasible solution sets of the primal and dual linear optimization problems are convex for any given parameter. The problem is the convexity of the region for ε, where the problem has an optimal solution with known optimal partition. Otherwise, the convexity of the region for ε is not guaranteed. Moreover, being full-row rank of A and $A+\varepsilon\triangle A$ is necessary to find optimal partition using an interior point method. Otherwise, we can find the desired matrix by some computations to remove some of the rows. An analogous method can be used if we need a full-column rank. Therefore, we do not mention this condition explicitly. Recall that when either the right-hand or left-hand side is perturbed, with the proposed assumption, the region is an interval. For the case when both are perturbed, the region is neither a rectangle (See Example 1) nor convex (See Example 2).

The Moore-Penrose inverse of a real matrix $X \in \mathbb{R}^{m\times n}$ always exists. It is a unique matrix $X^\dagger \in \mathbb{R}^{n\times m}$, where

$$X^\dagger X X^\dagger = X^\dagger, \quad XX^\dagger X = X, \quad (X^\dagger X)^T = X^\dagger X, \quad (XX^\dagger)^T = XX^\dagger.$$

In general, $XX^\dagger$ is not necessarily an identity matrix, while it maps all column vectors of X to themselves, and $(X^\dagger)^\dagger = X$. In solving $Ax = b$, when A has full-row rank, from minimum norm problem we have $x = A^\dagger b = A^T(AA^T)^{-1}b$. On the other hand, when A is of full-column rank, then $x = A^\dagger b = (A^TA)^{-1}A^Tb$ from least squares problem. We refer to [2] for more detail.

In realization theory for $b, c \in \mathbb{R}^l$ and $C \in \mathbb{R}^{l\times l}$, a rational function $f(\varepsilon) = 1 + \varepsilon c^T(I_l + \lambda C)^{-1}b$ can be described in terms of eigenvalues of two matrices C and $C^\times = C + bc^T$ [16]. It was proved that

$$\begin{aligned} f(\varepsilon) = \det f(\varepsilon) &= \det(1 + \varepsilon c^T(I_l + \varepsilon C)^{-1}b) = \det(I_l + bc^T(I_l + \varepsilon C)^{-1}) \\ &= \frac{\det(I_l + \varepsilon(C + bc^T))}{\det(I_l + \varepsilon C)} = \frac{\det(I_l + \varepsilon C^\times)}{\det(I_l + \varepsilon C)}, \end{aligned}$$

where I_l is an $l \times l$ identity matrix. An also we have,

$$f(\varepsilon) = \prod_{j=1}^{l} \frac{1 + \varepsilon \alpha_j^\times}{1 + \varepsilon \alpha_j}, \tag{1}$$

where, $\alpha_1, \ldots, \alpha_l$ are eigenvalues of C and $\alpha_1^\times, \ldots, \alpha_l^\times$ are eigenvalues of $C^\times$, counted according to their multiplicities. If C and $C^\times$ do not have common eigenvalues, then the factors in numerator and denominator, i.e., the number l on the right-hand-side of (1) is the least.

4 Fundamental properties

In this section, some fundamental properties are proved. By the next theorem, strictly complementary property is redefined as two useful inequalities necessary to some of the following theorems.

Theorem 1. *Let $(\varepsilon, \lambda, \gamma) \in \Lambda_{\varepsilon\lambda\gamma}$ and the optimal partition $\pi_{\varepsilon\lambda\gamma} = \pi = (B, N)$ be known. Then*

$$(A_B + \varepsilon \triangle A_B)^\dagger (b + \lambda \triangle b) > 0,$$

and

$$(c_N + \gamma \triangle c_N)^T - (c_B + \gamma \triangle c_B)^T (A_B + \varepsilon \triangle A_B)^\dagger (A_N + \varepsilon \triangle A_N) > 0.$$

Proof. It is known that any strictly complementary optimal solution (x^*, y^*, s^*) leads to the optimal partition π [15]. On the other hand, for the given index set B, we have $x_B^* = A_B^\dagger b$ and for the index set N, we have $s_N^* = c_N^T - c_B^T A_B^\dagger A_N$. Moreover, $B \cap N = \varnothing$, $B \cup N = \{1, \ldots, n\}$, and $x_B^* + s_N^* > 0$. The proof will be completed by adapting these inequalities for the perturbed problem $P(\varepsilon, \lambda, \gamma)$. □

When just the left-hand side matrix A is perturbed, has been investigated in [4]. It was proved that the invariancy region is an open interval provided that the necessary assumption for the convexity of this region holds true (See the Assumption in Section 3). Here, we adapt their methodology for our problem when λ and γ are fixed parameters. In the following two theorems, ε_0 is arbitrary in the invariancy region. Especially, it can be set as $\varepsilon_0 = 0$ to facilitate the computation.

Theorem 2. *Let $\hat{\lambda}$ and $\hat{\gamma}$ be fixed and $\pi = (B,N)$ be known for ε_0. Further, let $\beta_{q,1},\dots,\beta_{q,l}$ be eigenvalues of $A_B^{\dagger}(\varepsilon_0)(\triangle A_B + (b+\hat{\lambda}\triangle b)e_q^T)$, and $\alpha_1,\dots,\alpha_l$ be eigenvalues of $A_B^{\dagger}(\varepsilon_0)\triangle A_B$ for $1 \le q \le l = |B|$. Here, $A_B(\varepsilon) = A_B + \varepsilon\triangle A_B$ and $e_q \in \mathbb{R}^n$ is a vector with all zero elements but its q-th element is one. Then, the primal strictly feasibility condition $x_B(\varepsilon,\hat{\lambda}) > 0$ for $P(\varepsilon,\hat{\lambda},\hat{\gamma})$ is transformed to*

$$\prod_{j=1}^{l} \frac{1+(\varepsilon-\varepsilon_0)\beta_{q,j}}{1+(\varepsilon-\varepsilon_0)\alpha_j} \begin{cases} \ge 1, & \varepsilon-\varepsilon_0 \ge 0, \\ \le 1, & \varepsilon-\varepsilon_0 \le 0. \end{cases}$$

Proof. By Theorem 1, we have $x_B(\varepsilon,\hat{\lambda}) = (A_B + \varepsilon\triangle A_B)^{\dagger}(b+\lambda\triangle b) > 0$ for fixed $\hat{\lambda}$ and $\pi = (B,N)$. If $1 \le q \le l$, then the inequality $e_q^T A_B^{\dagger}(\varepsilon)(b+\hat{\lambda}\triangle b) > 0$ leads to

$$1+(\varepsilon-\varepsilon_0)e_q^T A_B^{\dagger}(\varepsilon)(b+\hat{\lambda}\triangle b) \begin{cases} \ge 1, & \varepsilon-\varepsilon_0 \ge 0, \\ \le 1, & \varepsilon-\varepsilon_0 \le 0. \end{cases} \tag{2}$$

To determine the matrix $A_B^{\dagger}(\varepsilon)$, the following possibilities are considered. For $m < l$, $A_B(\varepsilon_0)A_B^{\dagger}(\varepsilon_0) = I_m$, and then

$$A_B^{\dagger}(\varepsilon) = (A_B(\varepsilon_0) + (\varepsilon-\varepsilon_0)\triangle A_B)^{\dagger} = (I_l + (\varepsilon-\varepsilon_0)A_B^{\dagger}(\varepsilon_0)\triangle A_B)^{-1}A_B^{\dagger}(\varepsilon_0). \tag{3}$$

For $m > l$, we have $A_B^{\dagger}(\varepsilon_0)A_B(\varepsilon_0) = I_l$, and therefore

$$A_B^{\dagger}(\varepsilon) = A_B^{\dagger}(\varepsilon_0)(I_m + (\varepsilon-\varepsilon_0)\triangle A_B A_B^{\dagger}(\varepsilon_0))^{-1}.$$

Finally for $m = l$, $A_B(\varepsilon_0)$ is of full row and column ranks and Moore-Penrose inverse is reduced to the standard inverse, i.e., $A_B^{\dagger}(\varepsilon) = A_B^{-1}(\varepsilon)$, then

$$A_B^{\dagger}(\varepsilon) = A_B^{-1}(\varepsilon) = (I_m + (\varepsilon-\varepsilon_0)A_B^{-1}(\varepsilon_0)\triangle A_B)^{-1}A_B^{-1}(\varepsilon_0).$$

Without loss of generality let $m < l$, then by using (3), the left hand side of (2) can be rewritten as

$$1+(\varepsilon-\varepsilon_0)e_q^T(I_l + (\varepsilon-\varepsilon_0)A_B^{\dagger}(\varepsilon_0)\triangle A_B)^{-1}A_B^{\dagger}(\varepsilon_0)(b+\hat{\lambda}\triangle b).$$

Realization theory with

$$\begin{aligned} &\varepsilon := \varepsilon-\varepsilon_0, \;\; c^T := e_q^T, \\ &C := A_B^{\dagger}(\varepsilon_0)\triangle A_B, \;\; b := A_B^{\dagger}(\varepsilon_0)(b+\hat{\lambda}\triangle b), \\ &C^{\times} := C + bc^T := A_B^{\dagger}(\varepsilon_0)(\triangle A_B + (b+\hat{\lambda}\triangle b)e_q^T), \end{aligned}$$

implies

$$1+(\varepsilon-\varepsilon_0)e_q^T A_B^{\dagger}(\varepsilon)(b+\hat{\lambda}\triangle b) = \prod_{j=1}^{l} \frac{1+(\varepsilon-\varepsilon_0)\beta_{q,j}}{1+(\varepsilon-\varepsilon_0)\alpha_j},$$

where $\beta_{q,1},\dots,\beta_{q,l}$ and $\alpha_1,\dots,\alpha_l$ are defined in the assumption of this theorem. The proof is complete. □

Theorem 3. *Let $\hat{\lambda}$ and $\hat{\gamma}$ be fixed and $\pi = (B,N)$ be known for ε_0. Further, let $p \in \mathrm{Range}(N)$, $\gamma_{p,1},\ldots,\gamma_{p,l}$ and $\delta_{p,1},\ldots,\delta_{p,l}$ be eigenvalues of $A_B^{\dagger}(\varepsilon_0)(\triangle A_B + \triangle A_p(c_B+\hat{\gamma}\triangle c_B)^T)$ and*

$$A_B^{\dagger}(\varepsilon_0)(\triangle A_B + (A_p+\varepsilon_0\triangle A_p)(c_B+\hat{\gamma}\triangle c_B)^T),$$

respectively. Then, the strictly dual feasibility condition $s_N(\varepsilon,\hat{\gamma})>0$ is reduced to

$$\prod_{j=1}^{l}\frac{1+(\varepsilon-\varepsilon_0)\gamma_{p,j}}{1+(\varepsilon-\varepsilon_0)\alpha_j}+\frac{1}{\varepsilon-\varepsilon_0}\prod_{j=1}^{l}\frac{1+(\varepsilon-\varepsilon_0)\delta_{p,j}}{1+(\varepsilon-\varepsilon_0)\alpha_j}<1+\frac{1}{\varepsilon-\varepsilon_0}+c_p,$$

where α_j's are defined as in Theorem 2.

Proof. By substituting $c+\hat{\gamma}\triangle c$ for c, the proof goes similar to the argument given in the Condition 3 of [4]. □

In the next two theorems, for a fixed $\hat{\varepsilon}$, we adapt the presented method in [15] for determining the invariancy regions of problems $P(\hat{\varepsilon},\lambda,0)$ and $P(\hat{\varepsilon},0,\gamma)$, respectively. In more detail, using Theorems 4 and 5, the end points of invariancy intervals for the parameters λ and γ are determined.

Theorem 4. *For a fixed value $\hat{\varepsilon}$, let $\hat{\lambda}$ be arbitrary and $(y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},\underline{0}))$ be any optimal solution of problem $D(\hat{\varepsilon},\hat{\lambda},0)$. Then, the linearity interval is between two points $(\hat{\varepsilon},\overline{\lambda},0)$ and $(\hat{\varepsilon},\underline{\lambda},0)$ where*

$$\underline{\lambda} = \min_{\lambda,x}\{\lambda : (A+\hat{\varepsilon}\triangle A)x = b+\lambda\triangle b,\ x\geq 0,\ x^T s^*(\hat{\varepsilon},0)=0\}, \tag{4}$$

$$\overline{\lambda} = \max_{\lambda,x}\{\lambda : (A+\hat{\varepsilon}\triangle A)x = b+\lambda\triangle b,\ x\geq 0,\ x^T s^*(\hat{\varepsilon},0)=0\}. \tag{5}$$

Proof. We prove (4) and then (5) can be proved similarly. Consider the optimization problem

$$\min_{\lambda,x}\{\lambda :\ (A+\hat{\varepsilon}\triangle A)x = b+\lambda\triangle b,\ x\geq 0,\ x^T s^*(\hat{\varepsilon},0)=0\}. \tag{6}$$

Since problem $D(\hat{\varepsilon},\hat{\lambda},0)$ has optimal solution, then, its dual $P(\hat{\varepsilon},\hat{\lambda},0)$ has an optimal solution, too. If $x^*\geq 0$ and $(A+\hat{\varepsilon}\triangle A)x^* = b+\lambda\triangle b$, then $x^{*T}s^*(\hat{\varepsilon},0)=0$. Thus, $(\hat{\lambda},x^*)$ is a feasible solution of (6). If (6) is unbounded, then there exists a vector x such that $(A+\hat{\varepsilon}\triangle A)x = b+\lambda\triangle b,\ x\geq 0,\ x^T s^*(\hat{\varepsilon},0)=0$ for any $\lambda\leq\hat{\lambda}$. Here, $(y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},0))$ is feasible for $D(\hat{\varepsilon},\lambda,0)$ and x is feasible for $P(\hat{\varepsilon},\lambda,0)$. Since $x^T s^*(\hat{\varepsilon},0)=0$, then x is optimal for $P(\hat{\varepsilon},\lambda,0)$ and $(y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},0))$ is optimal for $D(\hat{\varepsilon},\lambda,0)$. The optimal value of these problems is $b^T y^*(\hat{\varepsilon},0)+\lambda\triangle b^T y^*(\hat{\varepsilon},0)$ showing that the slope of the objective value function does not change for any such λ. Thus, λ belongs to the linearity interval containing $\hat{\lambda}$. Therefore, the left end of this linearly interval is $-\infty$, as it should be.
Now, let (6) be bounded. Then, it has optimal solution, say (λ^*,x^*), and thus equality $(A+\hat{\varepsilon}\triangle A)x^* = b+\lambda^*\triangle b$ shows that x^* is feasible for $P(\hat{\varepsilon},\lambda^*,0)$. Since $(y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},0))$ is feasible for $D(\hat{\varepsilon},\lambda^*,0)$, $x^{*T}s^*(\hat{\varepsilon},0)=0$ indicating that $(x^*,y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},0))$ is optimal for $P(\hat{\varepsilon},\lambda^*,0)$ and $D(\hat{\varepsilon},\lambda^*,0)$. The optimal value of these problems is $b^T y^*(\hat{\varepsilon},0)+\lambda^*\triangle b^T y^*(\hat{\varepsilon},0)$; that is λ^* belongs to the linearity interval containing $\hat{\lambda}$ that leads to $\lambda^*\geq\underline{\lambda}$. On the other hand, $(y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},0))$ is optimal for $D(\hat{\varepsilon},\underline{\lambda},0)$. For x^* as an optimal solution of $P(\hat{\varepsilon},\underline{\lambda},0)$, we have

$$(A+\hat{\varepsilon}\triangle A)x^* = b+\underline{\lambda}\triangle b,\ x\geq 0, x^{*T}s^*(\hat{\varepsilon},0)=0,$$

showing that $(\underline{\lambda},x^*)$ is feasible for (6). Hence, $\lambda^*\leq\underline{\lambda}$ and then, $\lambda^*=\underline{\lambda}$ which completes the proof. □

Corollary 1. *If $(\hat{\varepsilon},\hat{\lambda},0)$ is an extreme point of $\Lambda_{\varepsilon\lambda 0}$ and $(y^*(\hat{\varepsilon},0),s^*(\hat{\varepsilon},0))$ is a strictly complementary optimal solution of $D(\hat{\varepsilon},\hat{\lambda},0)$, then $\underline{\lambda}=\overline{\lambda}=\hat{\lambda}$.*

Theorem 5. *For any $\hat{\gamma}$ and any optimal solution $x^*(\hat{\varepsilon},0)$ of problem $P(\hat{\varepsilon},0,\hat{\gamma})$, the linearity interval is between two points $(\hat{\varepsilon},0,\overline{\gamma})$ and $(\hat{\varepsilon},0,\underline{\gamma})$ where*

$$\underline{\gamma}=\min_{\gamma,y,s}\{\gamma : (A+\hat{\varepsilon}\triangle A)^T y+s=c+\gamma\triangle c,\ s\geq 0,\ s^T x^*(\hat{\varepsilon},0)=0\},$$

$$\overline{\gamma}=\max_{\gamma,y,s}\{\gamma : (A+\hat{\varepsilon}\triangle A)^T y+s=c+\gamma\triangle c,\ s\geq 0,\ s^T x^*(\hat{\varepsilon},0)=0\}.$$

Proof. The proof is similar to the proof of Theorem 4. □

Corollary 2. *If $(\hat{\varepsilon},0,\hat{\gamma})$ is an extreme point of $\Lambda_{\varepsilon 0\gamma}$ and $x^*(\hat{\varepsilon},0)$ is a strictly complementary optimal solution of problem $P(\hat{\varepsilon},0,\hat{\gamma})$, then $\underline{\gamma}=\overline{\gamma}=\hat{\gamma}$.*

Optimal partition invariancy analysis of an LO with different parameters on the right-hand side and the objective function was studied in [5]. It was shown that the region is a Cartesian product of two intervals for each uni-parametric problem. The following theorem adapts this result for $P(\hat{\varepsilon},\lambda,\gamma)$.

Theorem 6. *Let $\hat{\varepsilon}$ be fixed, and $\Lambda_{\hat{\varepsilon}\lambda 0}$ and $\Lambda_{\hat{\varepsilon}0\gamma}$ denote the corresponding optimal partition invariancy intervals of problems $P(\hat{\varepsilon},\lambda,0)$ and $P(\hat{\varepsilon},0,\gamma)$. Then, we have $\Lambda_{\hat{\varepsilon}\lambda\gamma}=\Lambda_{\hat{\varepsilon}\lambda 0}\times\Lambda_{\hat{\varepsilon}0\gamma}$.*

Proof. The proof is similar to the proof of Theorem 2.6 of [5]. One only needs to replace A with $A+\hat{\varepsilon}\triangle A$ in the argument. □

Remark 1. *One may start with $\lambda=\gamma=0$ and identify the interval $(\underline{\varepsilon},\overline{\varepsilon})$ using Theorems 2 and 3. First assume that $(\underline{\varepsilon},\overline{\varepsilon})$ contains values other than $\varepsilon=0$. For a fixed $\hat{\varepsilon}\in(\underline{\varepsilon},\overline{\varepsilon})$, if two regions $\Lambda_{\hat{\varepsilon}\lambda 0}$ and $\Lambda_{\hat{\varepsilon}0\gamma}$ are not singletons, then $\Lambda_{\hat{\varepsilon}\lambda\gamma}$ is an open rectangle. This rectangle might be unbounded provided that at least one of $\Lambda_{\hat{\varepsilon}\lambda 0}$ and $\Lambda_{\hat{\varepsilon}0\gamma}$ is unbounded. Whenever either $\Lambda_{\hat{\varepsilon}\lambda 0}$ or $\Lambda_{\hat{\varepsilon}0\gamma}$ is singleton, $\Lambda_{\hat{\varepsilon}\lambda\gamma}$ is a line passing through $(\hat{\varepsilon},0,0)$ either parallel to γ-axis or λ-axis. It is again an open region and possibly unbounded. When both $\Lambda_{\hat{\varepsilon}\lambda}$ and $\Lambda_{\hat{\varepsilon}\gamma}$ are singletons, and $\hat{\varepsilon}\neq 0$, then $(\hat{\varepsilon},0,0)$ is an extreme point of $\Lambda_{\varepsilon\lambda\gamma}$. If $\underline{\varepsilon}=\overline{\varepsilon}=0$ for $\lambda=\gamma=0$, then $\Lambda_{\varepsilon\lambda\gamma}$ is on the $\lambda\gamma$-plane. It would be of dimension 2, 1, or 0. When it is of dimension 2, it is an open rectangle containing the origin. When it is of dimension 0, $\Lambda_{\varepsilon\lambda\gamma}=\{(0,0,0)\}$. Otherwise, it is an open line segment passing through $(0,0,0)$ on either λ-axis or γ-axis.*

5 Procedure for detecting the invariancy region containing the origin

In this section, we devise a procedure to find the possible boundary curves of the invariancy region Λ for Problem $P(\varepsilon,\lambda,\gamma)$ in $\mathbb{R}^3$. This procedure is summarized in Algorithm 1. Having these boundary curves, one can identify the possible boundary faces of this region approximately. Identifying these faces are out of the scope of our procedure and it is just a numerical challenge.

Recall that in Step 4, when the invariancy region is unbounded, the slice extreme points might be less than four. Connecting a family of these extreme points that two parameters are at their ends, say $(\varepsilon_i,\underline{\lambda}_i,\underline{\gamma}_i)$ for $i=1,\ldots,k$, produces a piecewise linear curve as a rough approximate of (some part) one of the four possible boundaries of Λ. To smooth this boundary, one can use some fitting tool. Considering big k would result in a more accurate boundary of this region. Note that at the end of step 5, we could

Algorithm 1: Identifying invariancy region for Problem $P(\varepsilon,\lambda,\gamma)$.

Input: $A,b,c,\triangle A,\triangle b,\triangle c$; optimal partition $\pi(0,0,0)=(B,N)$;
Output: Invariancy region Λ;

1 Let $\lambda=\gamma=0$;
2 Find the invariancy interval $(\underline{\varepsilon},\overline{\varepsilon})$ for ε by using Theorems 2 and 3;
3 If $\underline{\varepsilon}=\overline{\varepsilon}=0$, use Theorem 6 to identify $\Lambda_{\varepsilon\lambda\gamma}$ as a possible rectangle (Degenerate region);
4 If $\underline{\varepsilon}<0<\overline{\varepsilon}$, for several values $\varepsilon_i\in(\underline{\varepsilon},\overline{\varepsilon})$ $(i=1,\dots,k)$, using Theorem 6, find k potential rectangles parallel to each other as cuts of $\Lambda_{\varepsilon\lambda\gamma}$ perpendicular to the ε-axis. Identify

$$(\varepsilon_i,\underline{\lambda}_i,\underline{\gamma}_i),(\varepsilon_i,\underline{\lambda}_i,\overline{\gamma}_i),(\varepsilon_i,\overline{\lambda}_i,\underline{\gamma}_i),(\varepsilon_i,\overline{\lambda}_i,\overline{\gamma}_i),$$

for $i=1,\dots,k$ as the potential extremes of ith rectangle;
5 By some appropriate fitting method, determine approximate boundary of the region using potential boundary points produced in Step 4 (Potentially four boundary curves);
6 Apply Theorem 4 or 5 to identify the corresponding invariancy interval for λ or γ at one of the endpoints of $(\underline{\varepsilon},\overline{\varepsilon})$ (Say at $(\overline{\varepsilon},0,0)$);
7 If problem $P(\overline{\varepsilon},0,0)$ has optimal solution, and the interval for λ is not a singleton, then the right side boundary of the region is detected;
8 If the determined interval in the Step 7 is singleton and the lower and upper boundaries of the region intersect each other on the ε-axis, then this point is an extreme point of the region;
9 If the lower and upper boundaries do not intersect each other, and the invariancy region extends to the right (for some $\varepsilon\geq\overline{\varepsilon}$), then either $\lambda=0$ or $\gamma=0$;
10 For $\varepsilon_{k_0}<\overline{\varepsilon}$ and enough close to $\overline{\varepsilon}$, find the invariancy interval for λ as $(\underline{\lambda}_{k_0},\overline{\lambda}_{k_0})$ with $\underline{\lambda}_{k_0}<0<\overline{\lambda}_{k_0}$ for Problem $P(\varepsilon_{k_0},\lambda,0)$;
11 Consider several points $(\overline{\varepsilon},\lambda_j,0)$ with $\underline{\lambda}_{k_0}<\lambda_j<0$ and identify optimal partitions $\pi_{\overline{\varepsilon}\lambda_j}$ for Problems $P(\overline{\varepsilon},\lambda_j,0)$;
12 If $\pi_{\overline{\varepsilon}\lambda_j}\neq\pi$ for every j, then the right boundary is the line passing through the points $(\overline{\varepsilon},\lambda_j,0)$ for such values of j's;
13 If for some values of j's, $\pi_{\overline{\varepsilon}\lambda_j}=\pi$, apply using Theorems 2 and 3 to find the left and right endpoints of the invariancy interval for ε containing $(\overline{\varepsilon},\lambda_j,0)$ for such j's.

detect all boundary curves of the invariancy region for $\varepsilon\in(\underline{\varepsilon},\overline{\varepsilon})$. Observe that considering big k would result in a more accurate boundary of the invariancy region, while additional computational complexities may arise in Step 5. Our experimental results revealed that the region would exist before $\underline{\varepsilon}$ or after $\overline{\varepsilon}$. This situation is investigated in Steps 6-12. In Step 7, the result is also valid when the problem is infeasible or unbounded at $(\overline{\varepsilon},0,0)$. In Step 8, no further action is necessary and the right boundary is just a point. Note that in Steps 10-12, it is assumed that $\gamma=0$. One may replace $\gamma=0$ with $\lambda=0$ and identify the region accordingly.

Observe that when only two sides of the constraints are perturbed, that is $\triangle c=0$, Theorem 4 identifies some points $(\varepsilon_i,\underline{\lambda}_i),(\varepsilon_i,\overline{\lambda}_i)$ $(i=1,\dots,k)$. Using some appropriate fitting procedures the upper and lower boundaries to the ε-axis for the region $\Lambda_{\varepsilon\lambda}$ can be approximated for any $\varepsilon\in(\underline{\varepsilon},\overline{\varepsilon})$ on the $\varepsilon\lambda$-plane. Analogous argument valid for the another special case $\triangle b=0$. It is enough to adapt the above-presented process of the general case for these special cases for determining the left and right boundaries of invariancy regions $\Lambda_{\varepsilon\lambda}$ and $\Lambda_{\varepsilon\gamma}$.

6 Illustrative examples

Here, we explain the proposed procedure by some concrete examples. The first example is an instance when parameters appear just in the constraints. The second example shows that the invariancy region is a degenerate nonconvex, and the boundaries are not straight lines in general. Final example is an instance that the invariancy region is not always degenerate. An interesting observation is that, when the boundary is a line, the optimal partition is invariant at all points of it. On the contrary, a boundary (e.g., the upper boundary in Example 2) may consist of several parts with different optimal partitions. Note that the numerical method for curve fitting could be different and it is problem-based. Here, we use the Curve Fitting Toolbox with Library Model Types for Curves as "polynomial" and Polynomial Model Names as "poly1".

Example 1. Consider the bi-parametric problem

$$\begin{array}{ll} \min & -x_1 - 2x_2 \\ s.t. & (1+2\varepsilon)x_2 + x_3 = 2 + \lambda \\ & x_1 - \varepsilon x_2 + x_4 = 2 - \dfrac{1}{2}\lambda \\ & x_1, x_2, x_3, x_4 \geq 0, \end{array} \tag{7}$$

where $\varepsilon, \lambda \in \mathbb{R}$ are the parameters. Optimal partition at $\varepsilon = \lambda = 0$ is $\pi = (B,N)$ with $B = \{1,2\}, N = \{3,4\}$. By using Theorems 2 and 3 for $\lambda = 0$, the optimal partition invariancy interval for parameter ε is $(-1/3, \infty)$. At $(\varepsilon, \lambda) = (-1/3, 0)$, Problem (7) has optimal solution with optimal partition $B = \{2\}, N = \{1,3,4\}$. By applying Theorem 4 for some points in the interval $(-1/3, \infty)$, extreme points of the linearly interval for λ are calculated. For instance, at $\varepsilon = 0$ this interval is $-2 < \lambda < 4$. The optimal partition is $B = \{1\}$ and $N = \{2,3,4\}$ at $(\varepsilon, \lambda) = (0, -2)$ and it is $B = \{2\}$ and $N = \{1,3,4\}$ at $(\varepsilon, \lambda) = (0, 4)$. Therefore, two more boundary points $(0,4), (0,-2)$ other than $(-1/3, 0)$ are determined. By applying the same calculations for the other selected points in $(-1/3, \infty)$ more boundary points will be determined.

With using fitting methods, we understood that upper boundary curve is the line $\lambda = 12\varepsilon + 4$ with the identical optimal partition for all points over this line as $B = \{2\}, N = \{1,3,4\}$. Similarly, the lower boundary curve is $\lambda = -2$ and the optimal partition for all points on this line is $B = \{1\}$ and $N = \{2,3,4\}$. These two boundaries intersect each other at $(\varepsilon, \lambda) = (-0.5, -2)$. Observe that -0.5 is less than $\underline{\varepsilon} = -1/3$ which suggests that $(-1/3, 0)$ is not an extreme point of the invariancy region. To verify whether the invariancy region extends to the left, we consider several points on the line segment between $(-1/3, 0)$ and $(-1/3, -2)$. It is observed that optimal partition at these points is π, too. By Theorems 2 and 3, the point $(-0.5, -2)$ is an extreme point of the invariancy region with optimal partition $B = \{2\}$ and $N = \{1,3,4\}$. It means that in an extreme point, optimal partition differs from the invariancy region but not necessarily different from the boundaries containing the partitions optimal. In another words, since the optimal partition of a parametric linear optimization problem could defers from one point to another, this would be reads as a nature of the optimal partition, and then one concludes that optimal partition is different from the interior of the invariancy region and its boundaries. This could be

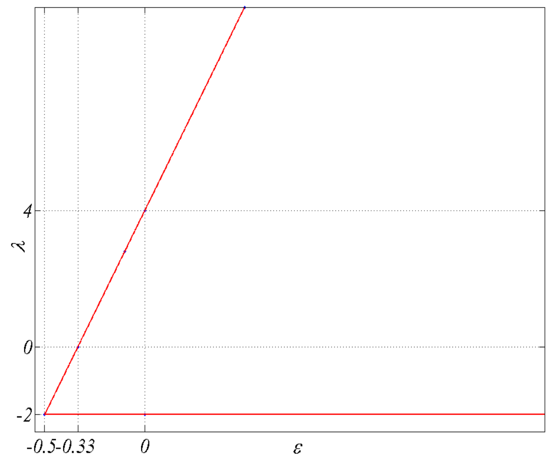

Figure 1: The invariancy region of Example 1.

considered as instability of the optimal partition in some sense. It is clear that the region is unbounded from the right. Clearly, this region is convex since it is the intersection two half-planes (See Figure 1). The main feature of this example is that, the problem is feasible at all boundary lines.

Example 2. Consider the multi-parametric problem

$$\begin{aligned} \min \quad & (-1-\gamma)x_1 - (1-\frac{1}{2}\gamma)x_2 \\ s.t. \quad & x_1 + x_2 + x_3 = 1 + \lambda \\ & \varepsilon x_1 + x_2 + x_4 = 1 - \frac{1}{2}\lambda \\ & x_1, x_2, x_3 \geq 0, \end{aligned} \tag{8}$$

where $\varepsilon, \lambda, \gamma \in \mathbb{R}$ are the parameters. The optimal partition for $\varepsilon = \lambda = \gamma = 0$ is $\pi = (B,N)$, where $B = \{1,2,4\}, N = \{3\}$. Moreover, when $\lambda = \gamma = 0$, the invariancy interval for ε is $(-\infty, 1)$. For every value of $\varepsilon \in (-\infty, 1)$, we have $\Lambda_{\varepsilon 0 \gamma} = \{0\}$ while $\Lambda_{\varepsilon \lambda 0}$ is not singleton. For instance at $\varepsilon = 0$, $\Lambda_{0 \lambda 0} = (-1, 2)$. This means that the invariancy region in this problem is degenerate, that is $\Lambda_{\varepsilon \lambda \gamma} = \Lambda_{\varepsilon \lambda 0}$.

The optimal partition is $B = \{4\}, N = \{1,2,3\}$ at $(\varepsilon, \lambda, \gamma) = (0, -1, 0)$. We also see that the lower bound of the invariancy interval for λ is -1 for every $\varepsilon \in (-\infty, 1)$, and optimal partitions are the same for all points on this boundary. Therefore, the lower boundary of the region is just the line $\lambda = -1$ and $\gamma = 0$.

On the other hand, problem (8) has optimal solution at $(\varepsilon, \lambda, \gamma) = (1, 0, 0)$ with the optimal partition $B = \{1,2\}, N = \{3,4\}$. The invariancy interval at this point for λ is just the singleton $\{0\}$. Thus, the upper boundary intersect the ε-axis at $(1,0,0)$, which is a boundary point. Since there is a gap between the lower bound of invariancy region for λ at $\varepsilon = 1$ and a point close to it, we may come to the point that the invariancy region might extend for $\varepsilon > 1$, too. Our computational results show that for all $\lambda \in (-1, 0)$ the invariancy region for ε include the whole real line. In addition, for every $\varepsilon > 1$ the invariancy region for λ is $(-1, 0)$. Moreover, the optimal partition is $B = \{2\}, N = \{1,3,4\}$ at $(\varepsilon, 0, 0)$ with $\varepsilon > 1$, that is different from π. Computational results denote that the upper boundary's representation for $\varepsilon < 1$ is $\lambda = (3.2\varepsilon^2 - \varepsilon + 5.7)/(5.3\varepsilon + 2.9)$ and the optimal partition is the same on this part of upper boundary

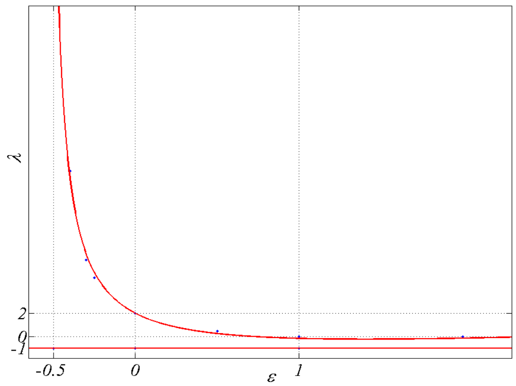

Figure 2: The invariancy region of Example 2.

as $B = \{1\}, N = \{2,3,4\}$. The invariancy region of this problem is pictured in Figure 2. It is clear that it is an unbounded and nonconvex 2-D set.

Example 3. Consider the multi-parametric problem

$$\begin{aligned} \min \quad & (-1-\gamma)x_1 - (2-\frac{1}{2}\gamma)x_2 \\ s.t. \quad & (1+\varepsilon)x_2 + x_3 = 2+\lambda \\ & x_1 + \varepsilon x_2 + x_4 = 2 - \frac{1}{2}\lambda \\ & x_1, x_2, x_3 \geq 0, \end{aligned} \tag{9}$$

where $\varepsilon, \lambda, \gamma \in \mathbb{R}$ are the parameters. For $\varepsilon = \lambda = \gamma = 0$, the optimal partition is $B = \{1,2\}, N = \{3,4\}$, and when $\lambda = \gamma = 0$, the invariancy interval is $(-1,2)$ for the ε. It is seen that, this interval is bounded contrary to the previous examples. Invariancy regions $\Lambda_{\varepsilon\lambda 0}$ and $\Lambda_{\varepsilon 0\gamma}$ are shown in Figures 3 and 4, respectively, which are nonconvex 2-D regions. Moreover, problem (9) is unbounded at the left borders in both cases (shown by colored dots). Detailed information on boundaries of these regions is given in Tables 1 and 2.

Table 1: Optimal partitions on boundaries of the invariancy region $\Lambda_{\varepsilon\lambda 0}$ in Example 3.

Boundary	B	N	Approximate formula	ε	λ
Upper	$\{2\}$	$\{1,3,4\}$	$\lambda = \frac{0.018\varepsilon+3}{3\varepsilon+1}$	$(-\frac{1}{3},2)$	$(0.5739,\infty)$
Lower	$\{1\}$	$\{2,3,4\}$	$\lambda = -2$	$(-1,2]$	-2
Right	$\{1,2,3\}$	$\{4\}$	$\varepsilon = 2$	2	$(-2,0.5739)$
Left	NA	NA	$\varepsilon = -1$	-1	$(-2,\infty)$

Note that in Figure 3, neither the upper boundary nor the lower boundary intersects the ε-axis at $(\varepsilon,\lambda,0) = (2,0,0)$. In addition, the problem has optimal solution at this point with the optimal partition $B = \{1,2,3\}, N = \{4\}$, which is different from π. Thus, the vertical line passing through this point is the

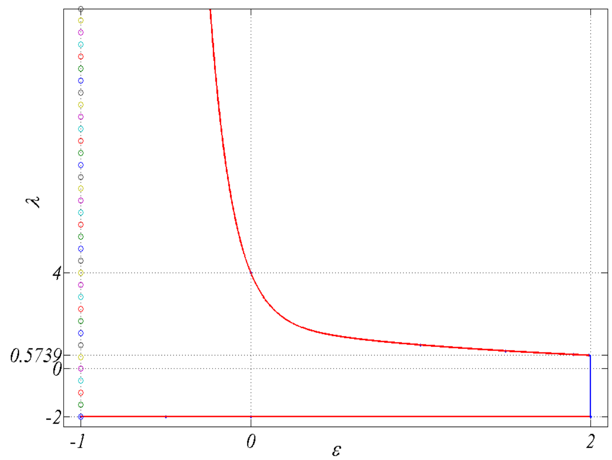

Figure 3: The invariancy region $\Lambda_{\varepsilon\lambda 0}$ of Example 3.

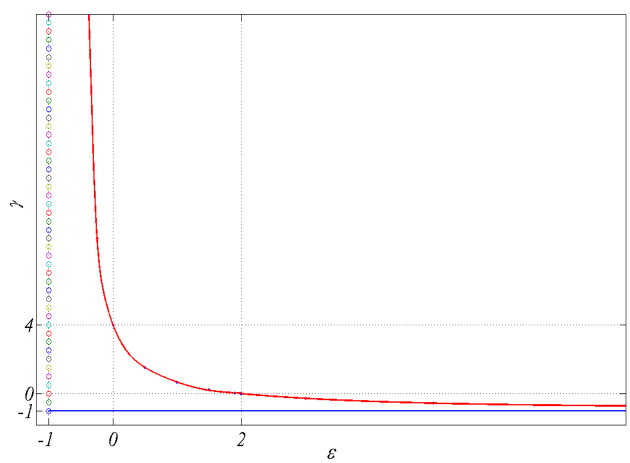

Figure 4: The invariancy region $\Lambda_{\varepsilon 0\gamma}$ of Example 3.

right boundary of $\Lambda_{\varepsilon\lambda 0}$. It is clear that the invariancy region $\Lambda_{\varepsilon 0\gamma}$ still exist for $\varepsilon > 2$ whereas it is not the general case as for $\Lambda_{\varepsilon\lambda 0}$. To determine the invariancy region $\Lambda_{\varepsilon\lambda\gamma}$, it is enough to detect $\Lambda_{\varepsilon\lambda 0} \times \Lambda_{\varepsilon 0\gamma}$. This region is shown roughly in Figure 5.

7 Conclusion

In this paper, a multi-parametric LO with three independent parameters each of them corresponding to the perturbation in both sides of constraints and in cost coefficients have been investigated. Then after, we determined the region where the optimal partition is not different one with the unperturbed problem. In general, it is shown that the corresponding optimal partition invariancy regions of the problems $P(\varepsilon,\lambda)$,

Table 2: Optimal partitions on boundaries of the invariancy region $\Lambda_{\varepsilon 0 \gamma}$ in Example 3.

Boundary	B	N	Approximate formula	ε	γ
Upper	$\{1,2,3\}$	$\{4\}$	$\gamma = \frac{-\varepsilon+2}{\varepsilon+0.5}$	$(-\frac{1}{2},\infty)$	$(-1,\infty)$
Lower	$\{1,2,4\}$	$\{3\}$	$\gamma=-1$	$(-1,\infty)$	-1
Right	NA	NA	NA	NA	NA
Left	NA	NA	$\varepsilon=-1$	-1	$(-1,\infty)$

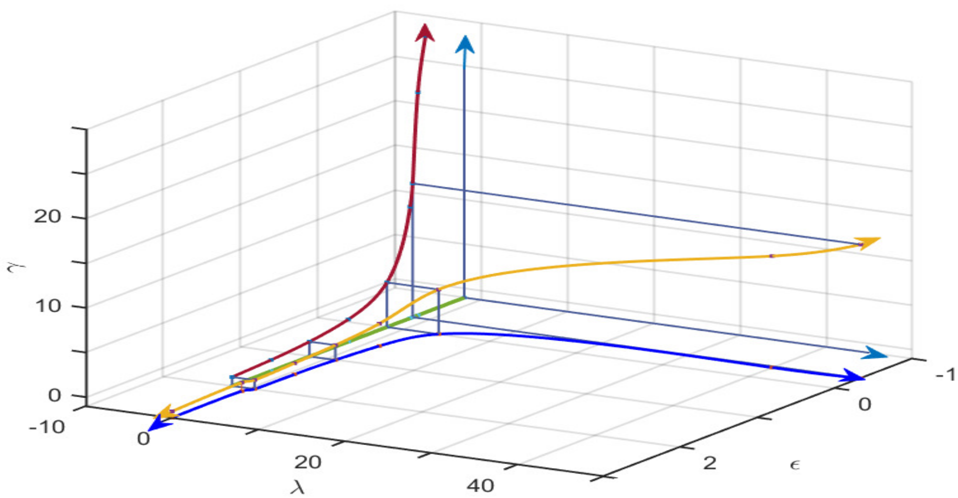

Figure 5: The invariancy region $\Lambda_{\varepsilon\lambda\gamma}$ of Example 3.

$P(\varepsilon,\gamma)$ and $P(\varepsilon,\lambda,\gamma)$ are not necessarily convex while this region is convex for uni parametric problems $P(\varepsilon)$, $P(\lambda)$, and $P(\gamma)$.

Lastly, we implemented the methodology on some small illustrative examples. While it is theoretically correct, identifying the region for large-scale problems is practically challenging, specially using efficient numerical methods for detecting the boundaries. Recall that any variation on matrix A is not under the control of user, and thus just we have to say that nothing can be done for this situation unless we hope the existing direction $\triangle A$ is admissible. Otherwise, one approach would be finding an admissible approximation to $\triangle A$ and applying the proposed algorithm. Finding such a direction was out of our study scope and could be another research direction.

This study could be directed to the case when there is a functional relationship between the parameters. As a special case, this functional relation would be linear.

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