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Existence of positive solutions for a p-Laplacian equation with applications to Hematopoiesis

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Abstract. This paper is concerned with the existence of at least one positive solution for a boundary value problem (BVP), with p-Laplacian, of the form

$$(\Phi_p(x'))' + g(t)f(t,x) = 0, \quad t \in (0,1),$$
$$x(0) - ax'(0) = \alpha[x], \quad x(1) + bx'(1) = \beta[x],$$

where $\Phi_p(x) = |x|^{p-2}x$ is a one dimensional p-Laplacian operator with $p > 1, a, b$ are real constants and α, β are the Riemann-Stieltjes integrals

$$\alpha[x] = \int_0^1 x(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t),$$

with A and B are functions of bounded variation. A Homotopy version of Krasnosel'skii fixed point theorem is used to prove our results.

Keywords: Fixed point, positive solution, p-Laplacian, non-local boundary conditions, boundary value problem.
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1 Introduction

In this paper, we discuss the existence of at least one positive solution to the p-Laplacian nonlinear differential equation

$$(\Phi_p(x'))' + g(t)f(t,x) = 0, \quad t \in (0,1), \tag{1}$$

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together with the non-local boundary conditions (BCs)

$$\begin{aligned} x(0) - ax'(0) &= \alpha[x], \\ x(1) + bx'(1) &= \beta[x], \end{aligned} \tag{2}$$

where a, b are positive constants, α and β are the linear functionals on $C[0,1]$), defined by the Riemann-Stieltjes integrals

$$\alpha[x] = \int_0^1 x(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t), \tag{3}$$

with A and B are nondecreasing functions of bounded variation, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function, $g : [0,1] \to [0,\infty)$ and g does not vanish identically on any subinterval of $[0,\infty)$. In (1), the function $\Phi_p(x) = |x|^{p-2}x$ is a one-dimensional p-Laplacian operator with $p > 1$, and the inverse operator Φ_q is defined by $\Phi_q(x) = |x|^{q-2}x$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In order to obtain our existence results, we assume the following conditions throughout this paper.

(A1) $0 < \alpha[1] < 1$ and $0 < \beta[1] < 1$;

The Riemann-Stieltjes integral $\alpha[x]$ and $\beta[x]$, defined in (3) satisfying the conditions in (**A1**), can be reduced to simple and easily verifiable nonlocal conditions, such as:

(i) If

$$\alpha[x] = \sum_{i=1}^{l} \alpha_i x(\eta_i), \quad 0 < \eta_i < 1 \text{ and } \beta[x] = \sum_{j=1}^{m} \beta_j x(\mu_j),\ 0 < \mu_j < 1,$$

then the assumption (**A1**) reduces to $0 < \sum_{i=1}^{l} \alpha_i < 1$ and $0 < \sum_{j=1}^{m} \beta_j < 1$.

(ii) If

$$\alpha[x] = \frac{\alpha}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} tx(t)\,dt, \text{ and } \beta[x] = \frac{\beta}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} tx(t)\,dt,$$

with $0 < \eta_1 < \eta_2 < 1$ and $0 < \mu_1 < \mu_2 < 1$, α and β are positive constants, then the assumption (**A1**) reduces to $0 < \alpha(\eta_1 + \eta_2) < 2$ and $0 < \beta(\mu_1 + \mu_2) < 2$.

(iii) If

$$\alpha[x] = \alpha \int_0^1 t^m x(t)dt \text{ and } \beta[x] = \beta \int_0^1 t^n x(t)dt, \quad m, n > -1,$$

then the assumption (**A1**) reduces to $0 < \alpha < m+1$ and $0 < \beta < n+1$.

In a recent paper, Padhi and Jaffar [**?**] used the fixed point index approach to study the positive solutions of the BVP (1)–(2). Yang and Wang [11] used the Avery-Peterson fixed point theorem to study the existence of at least three positive solutions of the p-Laplacian equation (1) together with the integral BCs of type

$$\begin{aligned} x(0) - ax'(0) &= \int_0^1 g_1(s)x(s)ds, \\ x(1) + bx'(1) &= \int_0^1 g_2(s)x(s)ds, \end{aligned} \tag{4}$$

where $a,b \geq 0,\ p>1$, and the inverse operator $\Phi_q(x)$ defined by $\Phi_q(x)=\Phi_p^{-1}(x)=|x|^{q-2}x$ with $\frac{1}{p}+\frac{1}{q}=1$. For boundary value problems with p-Laplacians, one may refer to [1, 2, 5, 8–10, 12–14] and the references cited therein. The main tools used in the above-cited paper are upper-lower solution method, Krasnosel'skii fixed point theorem, Avery-Peterson fixed point theorem, Leggett-William fixed point theorem and the fixed point index approach. We note that the integral on the right hand side of (4) are particular cases of the Riemann-Stieltjes integrals $\alpha[x]$ and $\beta[x]$, defined in (3). From the above discussion, it seems the no work is available in the literature on the existence of the positive solution of the problem (1) together with the BCs (2).

This work has been divided into three sections. Section 1 contains the basic information on the problem (1)–(2). Section 2 is Preliminary where all basic results are incorporated. Results concerning the existence of positive solutions of (1) are given in Section 3.

2 Preliminaries

In this section, we provide results similar to those obtained in [**?**]. The proof of Lemmas 1-4 are imported from [**?**], and their proofs are similar to the proofs in [**?**, 7] and [12].

Lemma 1. *([?]) For any $x\in C([0,1])$, let $F(t,x):[0,1]\times[0,\infty)\to[0,\infty)$ be a continuous function. Consider the problem*

$$(\Phi_p(x'))'+F(t,x)=0, \quad t\in(0,1),$$

together with the non-local BCs in (2). Then $x(t)\geq 0$ and concave on (0,1).

Throughout this work, we consider the Banach space $X=C([0,1])$ equipped with the norm $||x||=\max_{0\leq t\leq 1}|x(t)|$.

Lemma 2. *([?]) Suppose that*

$$1-\alpha[1]\neq 0 \text{ and } 1-\beta[1]\neq 0. \tag{5}$$

Then for any given $y\in X$, the equation

$$-(\Phi_p(x'))'=y(t) \ for\ a.e.\ t\in(0,1), \tag{6}$$

together with the BCs (2), has the solution

$$x(t)=\frac{a\Phi_q(\bar{\phi}_0)+\int_0^1\int_0^t\Phi_q\left(\bar{\phi}_0-\int_0^s y(r)dr\right)ds\,dA(t)}{1-\alpha[1]}+\int_0^t\Phi_q\left(\bar{\phi}_0-\int_0^s y(r)dr\right)ds, \tag{7}$$

where $\bar{\phi}_0$ satisfies the integral equation

$$\begin{aligned} a\Phi_q(\bar{\phi}_0)=&\int_0^1\int_t^1\Phi_q\left(\bar{\phi}_0-\int_0^s y(r)dr\right)ds\,dA(t)-\int_0^1\Phi_q\left(\bar{\phi}_0-\int_0^s y(r)dr\right)ds\\ &-\left(\frac{1-\alpha[1]}{1-\beta[1]}\right)\left[b\Phi_q\left(\bar{\phi}_0-\int_0^1 y(r)dr\right)+\int_0^1\int_t^1\Phi_q\left(\bar{\phi}_0-\int_0^s y(r)dr\right)ds\,dB(t)\right]. \end{aligned} \tag{8}$$

Our next lemma provides the existence of a real $\rho \in (0,1)$ satisfying $\tilde{\phi}_0 = \int_0^\rho y(r)dr$.

Lemma 3. *([?]) Let (A1) holds and $y \in C[0,1]$ with $y \geq 0$. Let $x(t)$, given in (7), be a solution of (6) together with the BCs (2). Then there exist constants $l \in \left(0, \int_0^1 y(s)ds\right)$ and $\rho \in (0,1)$ such that (8) is satisfied for $\tilde{\phi}_0 = l := \int_0^\rho y(r)dr$. Hence we can rewrite the solution $x(t)$, given in (7), as*

$$x(t) = \frac{a\Phi_q\left(\int_0^\rho y(r)dr\right) + \int_0^1\int_0^t \Phi_q\left(\int_s^\rho y(r)dr\right)dsdA(t)}{1-\alpha[1]} + \int_0^t \Phi_q\left(\int_s^\rho y(r)dr\right)ds.$$

In this paper, we define a cone K on X by $K = \left\{x \in X : \ x(t) \geq 0, \, t \in [0,1]\right\}$, and an operator $T : X \to X$ by

$$Tx(t) = \frac{a\Phi_q\left(\int_0^\rho g(r)f(r,x(r))dr\right) + \int_0^1\int_0^t \Phi_q\left(\int_s^\rho g(r)f(r,x(r))dr\right)dsdA(t)}{1-\alpha[1]} + \int_0^t \Phi_q\left(\int_s^\rho g(r)f(r,x(r))dr\right)ds. \tag{9}$$

Lemma 4. *([?]) Assume that there exist a positive constants ρ with $\rho \in (0,1)$ such that*

$$\lambda\left(\int_0^\rho g(s)f(s,x(s))\,ds\right) = 0,$$

where

$$\begin{aligned}\lambda\left(\int_0^\rho g(s)f(s,x(s))\,ds\right) =& a\Phi_q\left(\int_0^\rho g(s)f(s,x(s))\,ds\right) + \int_0^1\int_0^t \Phi_q\left(\int_s^\rho g(s)f(s,x(s))ds\right)dsdA(t)\\ &+(1-\alpha[1])\int_0^1 \Phi_q\left(\int_s^\rho g(s)f(s,x(s))ds\right)ds\\ &+\frac{(1-\alpha[1])}{(1-\beta[1])}\left[b\Phi_q\left(-\int_\rho^1 g(s)f(s,x(s))ds\right)\right.\\ &\left.+\int_0^1\int_t^1 \Phi_q\left(\int_s^\rho g(s)f(s,x(s))ds\right)ds\,dB(t)\right].\end{aligned}$$

Then, a function x is a solution of problem (1) – (2) if and only if x is a fixed point of the operator Tx, given in (9).

The next lemma follows from the concavity property of a continuous function.

Lemma 5. *([6]) Let $x(t)$ be a solution of problem (1) – (2). Then for any $\delta \in (0,1/2)$, we have*

$$\min_{t\in[\delta,1-\delta]} x(t) \geq \delta \max_{0\leq t\leq 1} x(t) = \delta ||x||.$$

In this paper, we shall use a homotopy version of the Krasnosel'skii fixed point theorem to prove the main results in Section 3. First, we define the some notations for our use. Let X be a Banach space and $K \subset X$ a cone and r, R two numbers with $0 < r < R$. Denote $\Omega_r = \{x \in K : \|x\| < r\}$, $\partial\Omega_r = \{x \in K : \|x\| = r\}$, and consider the conical shell $\Omega_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$. Let $T : \Omega_{r,R} \to K$ be a continuous and compact mapping and consider the fixed point equation $x = Tx$, $x \in \Omega_{r,R}$. Now we provide the homotopy version of the Krasnosel'skii fixed point theorem [3,4] for our use in the sequel.

Theorem 1 (Krasnosel'skii fixed point theorem [3, 4])**.** *The mapping T has a fixed point in $\Omega_{r,R}$ if it satisfies one of the following conditions:*

(i) $Tx \neq \mu x$ for $x \in \partial\Omega_r$, $\mu < 1$, and $Tx \neq \mu x$, for $x \in \partial\Omega_R$, $\mu > 1$ and $\inf_{x\in\Omega_r} \|Tx\| > 0$ (compression condition);

(ii) $Tx \neq \mu x$ for $x \in \partial\Omega_r$, $\mu > 1$, and $Tx \neq \mu x$, for $x \in \partial\Omega_R$, $\mu < 1$ and $\inf_{x\in\Omega_R} \|Tx\| > 0$ (expansion condition).

3 Main results: existence of positive solutions

In this section, we apply Krasnosel'skii fixed point theorem, that is, Theorem 1 to obtain the existence of positive solutions of (1). Throughout this section, we denote constants L, η and M by $L = \Phi_q\left(\int_0^1 g(r)\,dr\right)$, $\eta = \frac{\delta(1-\alpha[1])}{(1+a)L}$ and $M = \min\{\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3\}$, where

$$\mathscr{L}_1 = (1-\delta)\Phi_q\left(\int_\delta^{1-\delta} g(r)dr\right),$$
$$\mathscr{L}_2 = \int_0^\delta \Phi_q\left(\int_s^\delta g(r)dr\right)ds,$$
$$\mathscr{L}_3 = \int_{1-\delta}^1 \Phi_q\left(\int_{1-\delta}^s g(r)dr\right)ds.$$

Theorem 2. *Suppose that there exist constants r_i, $i = 1,2$ with $0 < r_1 < r_1/\delta < r_2$ such that*

$$f(t,x) \leq \Phi_p(\eta r_2) \text{ for } 0 \leq x \leq r_2,\ 0 \leq t \leq 1, \tag{10}$$

and

$$f(t,x) > \Phi_p\left(\frac{r_1}{\delta M}\right) \text{ for } r_1 \leq x \leq \frac{r_1}{\delta} \text{ and } 0 \leq t \leq 1, \tag{11}$$

hold. Then the problem (1) has at least one positive solution $x(t)$ with $r_1 \leq \|x\| \leq r_2$.

Proof. Let $x \in K$. Clearly

$$\alpha[Tx] = \int_0^1 Tx(t)\,dA(t) = \frac{1}{(1-\alpha[1])}\left[a\alpha[1]\Phi_q\left(\int_0^\rho g(r)f(r,x(r))dr\right)\right. \tag{12}$$
$$\left.+\int_0^1\int_0^s \Phi_q\left(\int_\theta^\rho g(r)f(r,x(r))dr\right)d\theta dA(s)\right]$$
$$= Tx(0) - a(Tx)'(0),$$

holds. Similarly, we can show that $Tx(1)+b(Tx)'(1) = \beta[Tx]$. Differentiating Tx with respect to t, we see that Tx satisfies the equation

$$((\Phi_p(Tx)'))' + g(t)f(t,x) = 0, \quad t \in (0,1),$$

which implies that Tx satisfies the BVP

$$((\Phi_p(Tx)'))' + g(t)f(t,x) = 0, \quad t \in (0,1),$$
$$Tx(0) - a(Tx)'(1) = \alpha[Tx] \text{ and } Tx(1)+b(Tx)'(1) = \beta[Tx].$$

Then by Lemma 1, we have $Tx \geq 0$ on $[0,1]$. Hence $T(K) \subset K$.

We shall use Theorem 1(i) to prove this theorem. Set

$$\Omega_{r_i} = \{x \in K : ||x|| < r_i\}, \quad i = 1,2.$$

Then for any $x \in \partial\Omega_{r_i}$, $i = 1,2$, we have $0 \leq x(t) \leq ||x|| = r_i$, $t \in [0,1]$, and Ω_{r_2} is an open bounded set in K. We prove that $T(\Omega_{r_2}) \subset \Omega_{r_2}$ is completely continuous. The verification of continuity of T is straightforward, and hence we omit it. For any $x \in \Omega_{r_2}$ and $t \in [0,1]$, we have

$$||Tx|| = \max_{0\leq t\leq 1} Tx(t) = Tx(\rho) \leq \frac{(1+a)}{(1-\alpha[1])}\Phi_q\left(\int_0^1 g(r)f(r,x(r))dr\right) \leq r_2.$$

Thus, we show that $T(\Omega_{r_2}) \subset \Omega_{r_2}$, and $T(\Omega_{r_2})$ is uniformly bounded. Next, we prove that $T : \Omega_{r_2} \to \Omega_{r_2}$ is equicontinuous in $[0,1]$, that is, for any $x \in \Omega_{r_2}, t_1, t_2 \in [0,1]$, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that when $|t_1 - t_2| < \delta$, then $|(Tx)(t_1) - (Tx)(t_2)| < \varepsilon$. Set $f^M = \max\{f(t,x); 0 \leq x \leq r_2,\ 0 \leq t \leq 1\}$; then for every $\varepsilon > 0$, there exists a $\delta \in \left(0, \frac{\varepsilon}{\Phi_q\left(f^M\int_0^1 g(s)\,ds\right)}\right)$ such that for ant $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$|Tx(t_2) - Tx(t_1)| \leq \left|\int_{t_1}^{t_2}\Phi_q\left(\int_0^1 g(r)f(r,x(r))dr\right)ds\right|$$
$$\leq |t_1 - t_2|\Phi_q\left(\int_0^1 g(r)f(r,x(r))dr\right)$$
$$< \delta\Phi_q\left(f^M\int_0^1 g(r)dr\right) < \varepsilon.$$

Hence $T(\Omega_{r_2}) \subset \Omega_{r_2}$ is equicontinuous. Consequently, $T(\Omega_{r_2}) \subset \Omega_{r_2}$ is completely continuous.

Let $x \in \partial\Omega_{r_2}$. We claim that $Tx \neq \mu x$ for $x \in \partial\Omega_{r_2}$, $\mu > 1$. If not, then there exists a $x^* \in \partial\Omega_{r_2}$ such that $Tx^* = \mu x^*$ and $\mu > 1$. Thus,

$$\delta r_2 = \delta\|x^*\| \leq \min_{t\in[\delta,1-\delta]} x^*(t) < \mu \min_{t\in[\delta,1-\delta]} x^*(t) = \min_{t\in[\delta,1-\delta]} Tx^*(t) \leq \|Tx^*\|.$$

Consequently, we have

$$\begin{aligned}
\delta r_2 < ||Tx^*|| &= \max_{0\leq t\leq 1} Tx^*(t) = Tx^*(\rho) \\
&= \frac{a\Phi_q\left(\int_0^\rho g(r)f(r,x^*(r))dr\right)}{1-\alpha[1]} + \frac{\int_0^1\int_0^t \Phi_q\left(\int_s^\rho g(r)f(r,x^*(r))dr\right)dsdA(t)}{1-\alpha[1]} \\
&\quad + \int_0^1 \Phi_q\left(\int_s^\rho g(r)f(r,x^*(r))dr\right)ds \\
&\leq \frac{(1+a)}{(1-\alpha[1])}\Phi_q\left(\int_0^1 g(r)f(r,x^*(r))dr\right) \leq \delta r_2,
\end{aligned}$$

a contradiction. Hence $Tx \neq \mu x$ for $x \in \partial\Omega_{r_2}$, $\mu > 1$.

Next, set $V_{r_1} = \{x \in K : \min_{t\in[\delta,1-\delta]} x(t) < r_1\}$; then $\Omega_{r_1} \subset V_{r_1} \subset \Omega_{r_1/\delta}$ and $\min_{t\in[\delta,1-\delta]} x(t) = r_1$ for $x \in K \cap \partial V_{r_1}$. By Lemma 5, for any $x \in \partial V_{r_1}$ we have $\max_{0\leq t\leq 1} x(t) \leq \frac{1}{\delta}\min_{t\in[\delta,1-\delta]} x(t) = \frac{r_1}{\delta}$. Therefore, for all $\delta \leq t \leq 1-\delta$, we have

$$r_1 = \min_{t\in[\delta,1-\delta]} x(t) \leq x(t) \leq \max_{0\leq t\leq 1} x(t) \leq \frac{r_1}{\delta}.$$

Differentiating the operator Tx with respect to t, we obtain

$$(Tx)'(t) = \Phi_q\left(\int_t^\rho g(r)f(r,x(r))dr\right) \geq 0 \text{ for } t \leq \rho,$$

and

$$(Tx)'(t) = -\Phi_q\left(\int_\rho^t g(r)f(r,x(r))dr\right) \leq 0 \text{ for } t \geq \rho.$$

Hence $\max_{0\leq t\leq 1} Tx(t) = Tx(\rho)$, and Tx can be expressed as

$$Tx(t) = Tx(0) + \int_0^t \Phi_q\left(\int_s^\rho g(r)f(r,x(r))dr\right)ds \text{ for } t \leq \rho, \tag{13}$$

and

$$Tx(t) = Tx(1) + \int_t^1 \Phi_q\left(\int_\rho^s g(r)f(r,x(r))dr\right)ds \text{ for } t \geq \rho. \tag{14}$$

We consider three cases depending on the location of ρ in $(0,1)$, and prove that $Tx \neq \mu x$, for $x \in \partial\Omega_{r_1/\delta}$ and $\mu < 1$ in each case. If possible, suppose that there exists a $x^* \in \partial\Omega_{r_1/\delta}$ such that $Tx^* = \mu x^*$ and $\mu < 1$. Then, for $x^* \in \partial\Omega_{r_1/\delta}$, we have $x^*(t) > \mu x^*(t) = Tx^*(t)$. Consequently, $r_1/\delta = \|x^*\| > \|Tx^*\|$ holds. First suppose that $\rho \in [\delta, 1-\delta]$. Then we have, either $\min_{t\in[\delta,1-\delta]} Tx^*(t) = Tx^*(\delta)$ or $\min_{t\in[\delta,1-\delta]} Tx^*(t) = Tx^*(1-\delta)$. If $\min_{t\in[\delta,1-\delta]} Tx^*(t) = Tx^*(\delta)$, then from (13) and the fact that $Tx^*(0) \geq 0$, we have

$$\begin{aligned} r_1/\delta > \|Tx^*\| &\geq Tx^*(\delta) \\ &= Tx^*(0) + \int_0^\delta \Phi_q\left(\int_s^\rho g(r)f(r,x^*(r))dr\right)ds \\ &\geq \int_0^\delta \Phi_q\left(\int_s^\delta g(r)f(r,x^*(r))dr\right)ds > r_1/\delta, \end{aligned}$$

a contradiction. If $\min_{t\in[\delta,1-\delta]} Tx^*(t) = Tx^*(1-\delta)$, then from (14) and the fact that $Tx^*(1) \geq 0$, we have

$$\begin{aligned} r_1/\delta > \|Tx^*\| &\geq Tx^*(1-\delta) \\ &\geq \int_{1-\delta}^1 \Phi_q\left(\int_\rho^s g(r)f(r,x^*(r))dr\right)ds \\ &\geq \int_{1-\delta}^1 \Phi_q\left(\int_{1-\delta}^s g(r)f(r,x^*(r))dr\right)ds > r_1/\delta, \end{aligned}$$

a contradiction. Next suppose that $\rho \in [1-\delta, 1)$. Then from (13) and $Tx^*(0) \geq 0$, we have

$$\begin{aligned} r_1/\delta > \|Tx^*\| &\geq Tx^*(1-\delta) \\ &\geq Tx^*(0) + \int_0^{1-\delta} \Phi_q\left(\int_s^\rho g(r)f(r,x^*(r))dr\right)ds \\ &\geq \int_0^{1-\delta} \Phi_q\left(\int_s^{1-\delta} g(r)f(r,x^*(r))dr\right)ds \\ &\geq \int_0^{1-\delta} \Phi_q\left(\int_\delta^{1-\delta} g(r)f(r,x^*(r))dr\right)ds, \quad (\because s \leq \delta) \\ &\geq (1-\delta)\Phi_q\left(\int_\delta^{1-\delta} g(r)f(r,x^*(r))dr\right) > r_1/\delta, \end{aligned}$$

a contradiction. Finally, suppose that $\rho \in (0,\delta)$. So $\rho \leq t \in [\delta, 1-\delta]$ and $\rho \leq t \in [\delta, 1-\delta]$. Hence from (14) and $Tx^*(1) \geq 0$, we have

$$\begin{aligned} r_1/\delta &> \|Tx^*\| \\ &\geq \int_{1-\delta}^{1} \Phi_q\left(\int_{\rho}^{s} g(r)f(r,x^*(r))dr\right) ds + \lambda^* \frac{r_2}{\delta} \\ &\geq \int_{1-\delta}^{1} \Phi_q\left(\int_{1-\delta}^{s} g(r)f(r,x^*(r))dr\right) ds > r_1/\delta, \end{aligned}$$

a contradiction. Hence, $Tx \neq \mu x$, for $x \in \partial\Omega_{r_1/\delta}$, $\mu < 1$.

In order to complete the proof of the theorem, we are required to show that $\inf_{x\in\partial\Omega_{r_2}} \|Tx\| > 0$. Since $\|Tx\| = Tx(\rho)$, $\rho \in (0,1)$, and $Tx(t) \geq 0$ for all $t \in [0,1]$, then from the concavity property of Tx, we have $\inf_{x\in\partial\Omega_{r_2}} \|Tx\| > 0$. Hence by Theorem 1(i), the operator T has one fixed point x, which is a positive solution of the problem (1)-(2) satisfying $r_1 \leq ||x_1|| \leq r_2/\delta$. This completes the proof of the theorem. □

Remark 1. *The assumption (10) in Theorem 2 can be replaced by the condition*

$$\lim_{x\to\infty} \max_{0\leq t\leq 1} \frac{f(t,x)}{\Phi_p(x/M)} = 0. \tag{15}$$

Indeed, by the condition (15) we can find a suitable r_2 with $r_2 > r_1/\delta$ such that (10) is satisfied.

Thus, we have the following theorem.

Theorem 3. *Let (15) be satisfied and assume that there exists a constant $r_1 > 0$ such that (11) holds. Then the problem (1) has at least one positive solution.*

As an application of Theorem 3, we consider the case where the nonlinear function f in (1) is a model of hematopoiesis (red blood production model), that is, we consider

$$(\Phi_p(x'))' + \frac{x^l}{1+x^m} = 0, \quad t \in (0,1), \tag{16}$$

together with the BCs in (2). We have the following theorem.

Theorem 4. *Suppose that $l > p-1 > l-m > 0$, and for any $\delta \in (0,1/2)$. Let*

$$\frac{(l-p+1)^{\frac{l-p+1}{m}}(p-1-l+m)^{\frac{(p-1-l+m)}{m}}}{(l-p+1)+\delta^m(p-1-l+m)} > \frac{1}{\delta^{m+p-1}\min\left\{(1-2\delta)^{\frac{1}{p-1}}, \frac{(p-1)}{p}\delta^{\frac{p}{p-1}}\right\}}. \tag{17}$$

Then the problem (16) together with the BCs (2) has at least one positive solution.

Proof. We shall apply Theorem 3 to prove our theorem. Set $f(t,x) = \frac{x^l}{1+x^m}$, then by the assumption $l > p-1 > l-m > 0$, we can see that (15) holds. Thus, it remains to find the existence of a positive constant r_1 such that (11) is satisfied. Since $g(t) \equiv 1$, then $M = \min\left\{(1-2\delta)^{\frac{1}{p-1}}, \frac{(p-1)}{p}\delta^{\frac{p}{p-1}}\right\}$. Since $\frac{x^l}{1+x^m} \geq \frac{\delta^m r_1^l}{\delta^m + r_1^m}$ for $r_1 \leq x \leq \frac{r_1}{\delta}$, then (11) is satisfied if

$$\frac{r_1^{l-p+1}}{\delta^m + r_1^m} > \frac{1}{\delta^{m+p-1}\min\left\{(1-2\delta)^{\frac{1}{p-1}}, \frac{(p-1)}{p}\delta^{\frac{p}{p-1}}\right\}}, \tag{18}$$

holds. Set $r_1 = \left(\frac{l-p+1}{p-l-1+m}\right)^{\frac{1}{m}}\delta$; then $\frac{r_1^{l-p+1}}{\delta^m + r_1^m}$ attains its minimum

$$\frac{(l-p+1)^{\frac{l-p+1}{m}}(p-1-l+m)^{\frac{(p-1-l+m)}{m}}}{(l-p+1)+\delta^m(p-1-l+m)},$$

for $r_1 \leq x \leq \frac{r_1}{\delta}$ at $\frac{r_1}{\delta} = \left(\frac{l-p+1}{p-l-1+m}\right)^{\frac{1}{m}}$. Thus (18) is satisfied if (17) holds. This completes the proof of the theorem. □

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