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Factorization of the t-extension of the p-Fibonacci and the Pascal matrices

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Abstract. In this paper, we introduce the t-extension of the p-Fibonacci matrix and give a Factorization of the Pascal matrix involving the t-extension of the p-Fibonacci matrix. Also, we obtain some results on the relations between the Stirling matrix of the second kind and the 1-extension of the p-Fibonacci matrix.

Keywords: Pascal matrix, t-extension of the p-Fibonacci matrix, factorization of a matrix.
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1 Introduction

The Fibonacci sequence and generalized Fibonacci sequence are famous sequences in mathematics. Many authors have studied these sequences (see [1,5,10]). The Fibonacci sequence is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}, n \geq 3$, with the initial values $f_1 = f_2 = 1$. This sequence has been extended in many ways. Two such extensions that will be used in this paper are the p-Fibonacci sequence and the t-extension of the p-Fibonacci sequence (see [6, 12]). For $p \geq 0$, the p-Fibonacci sequence $f^p(n)$, defined by the following relation:

$$f^p(n) = f^p(n-1) + f^p(n-p-1), \quad n > p+1,$$

with initial terms

$$f^p(1) = f^p(2) = \cdots = f^p(p) = f^p(p+1) = 1.$$

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Definition 1. *For $t, p \geq 1$, the t-extension of the p-Fibonacci sequence $\{f^p(t,n)\}_{-\infty}^{\infty}$ is given by the following recurrence relation:*

$$f^p(t,n) = \begin{cases} 0, & n < 1, \\ 1, & n = 1, \\ tf^p(t,n-1) + f^p(t,n-p-1), & n > 1. \end{cases}$$

For example if $t = 1$ and $p = 2$, we have $f^2(1,n) = f^2(1,n-1) + f^2(1,n-3)$ and $\{f^2(1,n)\}_{-\infty}^{\infty} = \{\ldots,0,1,1,1,2,3,4,6,9,\ldots\}$.

The $n \times n$ lower triangular Pascal matrix, denoted by $P_n = [p_{ij}]$, is defined as follows [2]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we define the $n \times n$ t-extension of the p-Fibonacci matrix $(p \geq 2)$, denoted by $F^p_{(t,n)} = [f^p_{(t,ij)}]$, with $f^p_{(t,ij)} = f^p(t,i-j+1)$. For example,

$$F^2_{(1,8)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 1 & 1 & 0 & 0 \\ 6 & 4 & 3 & 2 & 1 & 1 & 1 & 0 \\ 9 & 6 & 4 & 3 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

The set of all $n \times n$ matrices with real entries is denoted by M_n. Any matrix $B \in M_n$ of the form $B = A^*A$, $A \in M_n$ may be written as $B = LL^*$ where $L \in M_n$ is a lower triangular matrix with nonnegative diagonal entries. The factorization of the matrix $B = LL^*$ is unique if A is nonsingular and A^* is the transpose of it. This is called the Cholesky factorization of B. In particular, a matrix B is positive definite if and only if there exists a nonsingular lower triangular matrix $L \in M_n$ with positive diagonal entries such that $B = LL^*$. If B is a real matrix, L may be taken to be real.

For $n,\ k \in \mathbb{N}$ and $n \geq k$, the Stirling number of the second kind $S(n,k)$ is defined as follows (see [3])

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n. \tag{1}$$

Definition 2. *The Stirling matrix of the second kind, denoted by $\zeta_n(2) = [\iota_{ij}]$, is defined by:*

$$\iota_{ij} = \begin{cases} S(i,j), & i \geq j, \\ 0, & \textit{otherwise.} \end{cases}$$

The Pascal matrix and its factorizations were studied by many authors (see [4, 7, 11]). Also, in [8, 9], the authors gave some results about the p-Fibonacci matrix for $p = 1$. Here, for $t = 1$, $p \geq 2$, we consider the t-extension of the p-Fibonacci matrix and define the Pascal matrix. Then, in Section 2, we obtain a Factorization of the Pascal matrix. In Section 3, using the product of the 1-extension of the p-Fibonacci matrix and its transpose, we give the Cholesky factorization of S_n^p. Section 4 is devoted to obtaining some results on the relations between the Stirling matrix of the second kind and 1-extension of the p-Fibonacci matrix. In Section 5, we will generalize the notion of the t-extension of the p-Fibonacci matrix $(t \geq 2)$ and study some properties of the t-extension of the p-Fibonacci matrix.

Remark 1. *Throughout this paper, we set* $f^p(n) := f^p(1,n)$ *and* $F_n^p := F_{(1,n)}^p$.

2 Factorization of the Pascal matrix

In this section, we obtain the inverse of the 1-extension of the p-Fibonacci matrix F_n^p. Also, we give a factorization of the 1-extension of the p-Fibonacci matrix. We first get the inverse of the 1-extension of the p-Fibonacci matrix. For this, we need to define the matrix U_n^p. The $n \times n$ matrix $U_n^p = [u_{ij}^p]$ is defined by:

$$u_{ij}^p = \begin{cases} f^p(i), & \text{if } j = 1, \\ 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

that is

$$U_n^p = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ f^P(2) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ f^P(n) & 0 & 0 & \dots & 1 \end{bmatrix}.$$

By using a simple calculation, we get

$$F_n^p = U_n^p \times (I_1 \oplus U_{n-1}^p) \times (I_2 \oplus U_{n-2}^p) \times \cdots \times (I_{n-2} \oplus U_2^p),$$

where I_j is an $j \times j$ identity matrix. For example,

$$\begin{aligned} F_4^2 &= U_4^2 \times (I_1 \oplus U_3^2) \times (I_2 \oplus U_2^2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Hence, we have

$$(U_n^p)^{-1} = \begin{bmatrix} f^p(1) & 0 & 0 & \dots & 0 \\ -f^p(2) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ -f^p(n) & 0 & 0 & \dots & 1 \end{bmatrix},$$

and

$$(I_k \oplus U_{n-k}^p)^{-1} = I_k \oplus (U_{n-k}^p)^{-1}.$$

So, we get

$$(F_n^p)^{-1} = (I_{n-2} \oplus (U_2^p)^{-1}) \times \cdots \times (I_1 \oplus (U_{n-1}^p)^{-1}) \times (U_n^p)^{-1}. \tag{2}$$

From (2), we have $(F_n^p)^{-1} = [f_{ij}^{\prime p}]_{n\times n}$, where

$$f_{ij}^{\prime p} = \begin{cases} 1, & \text{if } \ i = j, \\ -1, & \text{if } \ j = i-1 \text{ or } j = i-(p+1), \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

For example

$$(F_7^3)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Here, we give a factorization of the 1-extension of the p-Fibonacci matrix. First, we introduce the matrix L_n^p.

Definition 3. *Entries of the $n\times n$ matrix $L_n^p = [l_{ij}^p]$ are defined as*

$$l_{ij}^p = \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-(p+2)}{j-1}. \tag{4}$$

For $i, j \geq 2$, using relation (4), we can write $l_{ij}^p = l_{i-1,j-1}^p + l_{i-1,j}^p$, where $l_{11}^p = 1$, $l_{1j}^p = 0$, $j \geq 2$.

For $p = 2$ and $n = 5$, we have

$$L_5^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 3 & 1 \end{bmatrix}.$$

By the above information, we prove the following theorem.

Theorem 1. *For the Pascal matrix P_n, we have $P_n = F_n^p L_n^p$.*

Proof. The matrix F_n^p is invertible. If we get $(F_n^p)^{-1} P_n = L_n^p$, then Theorem is proved. Let $(F_n^p)^{-1} P_n = B_n$ where $B_n = (b_{i,j})_{1\leq i,j\leq n}$, i.e.,

$$b_{i,j} = \sum_{k=j}^{i} f_{i,k}^{\prime p} P_{k,j}.$$

Since $(F_n^p)^{-1}$ and P_n are lower triangular matrices, by the definition of $(F_n^p)^{-1}$, we have

$$\begin{aligned} b_{i,j} &= \sum_{k=j}^{i} f'^{p}_{i,k}\binom{k-1}{j-1} \\ &= f'^{p}_{i,i-(p+1)}\binom{i-(p+2)}{j-1} + f'^{p}_{i,i-1}\binom{i-2}{j-1} + f'^{p}_{i,i}\binom{i-1}{j-1} \\ &= \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-(p+2)}{j-1} = (l_{i,j})_{1\le i,j\le n}. \end{aligned}$$

□

Corollary 1. *For $s,n \in \mathbb{N}$,*

$$\binom{n-1}{s-1} = P_{n,s} = \sum_{k=s}^{n} f^p_{n,k} l_{k,s} = f^p_{n,1} l_{1,s} + f^p_{n,2} l_{2,s} + \cdots + f^p_{n,n-1} l_{n-1,s} + f^p_{n,n} l_{n,s}.$$

For $s = 1$, we have

$$P_{n,1} = \sum_{k=s}^{n} f^p_{n,k} l_{k,s} = f^p_{n,1} l_{1,1} + f^p_{n,2} l_{2,1} + \cdots + f^p_{n,n-1} l_{n-1,1} + f^p_{n,n} l_{n,1}.$$

Proof. From Theorem 1, we have $P_n = F_n^p L_n^p$. Hence,

$$P_n = l_{11} + f^p(n-1) l_{21} + \cdots + f^p(2) l_{n-1,1} + f^p(1) l_{n1}.$$

Let $s = 1$. Since

$$l_{i1} = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{if } i \le p+1, \\ -1, & \text{if } i \le p+1, \end{cases} \tag{5}$$

we have the result. □

Now, in the following theorem, we obtain the inverse of the matrix $L_n^p = [l'^{p}_{ij}]$.

Corollary 2. *Let $(L_n^p)^{-1} = [l'^{p}_{ij}]$. Then*

$$l'^{p}_{ij} = \sum_{s=j}^{i} \binom{i-1}{s-1} \times (-1)^{i+s} f^p(s).$$

Proof. Since $P_n^{-1} F_n^p = (L_n^p)^{-1}$, we have the result. □

Corollary 3. *For $p = 2$, we get*

$$f^2(n) = 1 + \sum_{j=4}^{n} (-1)^j \binom{n-1}{j-1} f^2(j-1).$$

Proof. We have $f^2(n) = \sum_{j=1}^{n} p_{nj} l'^2_{j1}$. Hence, from Corollary 2,

$$l'^2_{11} = 1,\ l'^2_{21} = l'^2_{31} = 0,\ l'^2_{41} = 1$$

and $l'^2_{i1} = (-1)^i f^2(i-1)$. Consequently,

$$f^2(n) = p_{n1} + \sum_{j=4}^{n} (-1)^j f^2(j-1) p_{nj} = 1 + \sum_{j=4}^{n} (-1)^j \binom{n-1}{j-1} f^2(j-1).$$

□

3 The Cholesky factorization of a symmetric 1-extension of the p-Fibonacci matrix

Here, we define a symmetric 1-extension of the p-Fibonacci matrix S_n^p. Then using the prouduct of the 1-extension of the p -Fibonacci matrix F_n^p and its transpose F_n^p, we get the Cholesky factorization of S_n^p. First, we need the following definition.

Definition 4. *A symmetric 1-extension of the p-Fibonacci matrix, denoted by $S_n^p = [s(p)_{ij}]$ for $i, j = 1, 2, \ldots, n$, is defined as follows:*

$$s(p)_{ij} = s(p)_{ji} = \begin{cases} \sum_{k=1}^{i} (f^p(n))^2, & \text{if } i = j, \\ s(p)_{i,j-1} + s(p)_{i,j-(p+1)}, & \text{if } i+1 \leq j, \end{cases} \tag{6}$$

where $j \leq p+1, s(p)_{i,j} = 0$.

For example,

$$S_{10}^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 7 & 10 \\ 1 & 2 & 2 & 2 & 3 & 5 & 7 & 9 & 12 & 17 \\ 1 & 2 & 3 & 3 & 4 & 6 & 9 & 12 & 16 & 22 \\ 1 & 2 & 3 & 4 & 5 & 7 & 10 & 14 & 19 & 26 \\ 2 & 3 & 4 & 5 & 8 & 11 & 15 & 20 & 28 & 39 \\ 3 & 5 & 6 & 7 & 11 & 17 & 23 & 30 & 41 & 58 \\ 4 & 7 & 9 & 10 & 15 & 23 & 33 & 43 & 58 & 81 \\ 5 & 9 & 12 & 14 & 20 & 30 & 43 & 58 & 78 & 108 \\ 7 & 12 & 16 & 19 & 28 & 41 & 58 & 78 & 107 & 148 \\ 10 & 17 & 22 & 26 & 39 & 58 & 81 & 108 & 148 & 207 \end{bmatrix}.$$

Remark 2. *By Definition 4, we have* $\binom{n-1}{s-1} = p_{ns} = f^p(n)$,

$$s(p)_{1j} = f^p(j), \quad j \geq 1,$$
$$s(p)_{2j} = f^p(2)f^P(j) + f^p(1)f^p(j-1), \quad j \geq 2,$$
$$s(p)_{3j} = f^p(3)f^p(j) + f^p(2)f^p(j-1) + f^p(1)f^p(j-2), \quad j \geq 3.$$

So, for $j \geq i$, by an induction on i, we get

$$s(p)_{ij} = f^p(i)f^p(j) + f^p(i-1)f^p(j-1) + f^p(i-2)f^p(j-2) + \cdots + f^p(1)f^p(j-i+1). \quad (7)$$

Lemma 1. *For $j \geq i$, $s(p)_{i,j} = s(p)_{i-1,j} + s(p)_{i-(p+1),j} + f^p(j-i+1)$.*

Proof. By the relation (7), we have

$$s(p)_{i-1,j} = f^p(i-1)f^p(j) + f^p(i-2)f^p(j-1) + \cdots + f^p(1)f^p(j-i).$$

$$\begin{aligned} s(p)_{i-(p+1),j} = &f^p(i-(p+1))f^p(j) + f^p(i-(p+1)-1)f^p(j-1) + \cdots \\ &+ f^p(1)f^p(j-i+(p+2)). \end{aligned}$$

Hence,

$$\begin{aligned} s(p)_{i-1,j} + s(p)_{i-(p+1),j} &= f^p(i-1)f^p(j) + f^p(i-2)f^p(j-1) + \cdots + f^p(1)f^p(j-i) \\ &\quad + f^p(i-(p+1))f^p(j) + f^p(i-(p+1)-1)f^p(j-1) + \cdots \\ &\quad + f^p(1)f^p(j-i+(p+2)) \\ &= (f^p(i-1) + f^p(i-(p+1)))\, f^p(j) + (f^p(i-2) \\ &\quad + f^p(i-(p+1)-1))f^p(j-1) + \cdots + (f^p(j-i) \\ &\quad + f^p(j-i+(p+2)))f^p(1). \end{aligned}$$

By Definition 1, we have

$$f^p(n) = f^p(n-1) + f^p(n-p-1), \; f^p(1) = f^p(2) = \cdots = f^p(p) = f^p(p+1) = 1.$$

Therefore $s(p)_{i-1,j} + s(p)_{i-(p+1),j} + f^p(j-i+1) = s(p)_{ij}$. □

Theorem 2. *For $n \in \mathbb{N}$, the Cholesky factorization of S_n^p is given by $S_n^p = F_n^p (F_n^p)^T$.*

Proof. By the relations (2) and (3), it is sufficient that to prove $(F_n^p)^{-1} S_n^p = (F_n^p)^T$. Let $X = [x(p)_{ij}] = (F_n^p)^{-1} S_n^p$. We have

$$x(p)_{ij} = \sum_{k=1}^{i} f'_{i,k} S_{k,j} = f'_{i,i-(p+1)} S_{i-(p+1),j} + f'_{i,i-1} S_{i-1,j} + f'_{i,i} S_{i,i},$$

and by (3)

$$x(p)_{ij} = -S_{i-(p+1),j} - S_{i-1,j} + S_{i,i} = \begin{cases} f^p(j), & \text{if } i = 1, \\ f^p(j-1), & \text{if } i = 2, \\ f^p(j+i-1), & \text{otherwise.} \end{cases}$$

Furthermore $X^T = F_n^p$. This completes the proof. □

By Theorem 2, the proof of the following corollary is trivial. So, we omit the proof.

Corollary 4. *For* $(S_n^p)^{-1} = [s'(p)_{ij}]$ *where* $(S_n^p)^{-1} = (F_n^P \times (F_n^p)^T)^{-1}$, *we have*
(i) if $i = j$, *then*

$$s'(p)_{ii} = \begin{cases} 1, & \text{if } i = n, \\ 2, & \text{if } n-p \le i \le n-1, \\ 3, & \text{otherwise.} \end{cases}$$

(ii) For $i \neq j$, *we get*

$$s'(p)_{ij} = s'(p)_{ji} = \begin{cases} -1, & \text{if } j = i+1 \text{ or } j = p+i+1, \\ 1, & \text{if } j = i+p. \end{cases}$$

For example,

$$(S_8^3)^{-1} = \begin{bmatrix} 3 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 3 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

4 The Stirling matrix of the second kind

We get some results on the relationships between the Stirling matrix of the second kind and the 1-extension of the p-Fibonacci matrix. For this, using the Stirling number of the second kind, we start with the following definition.

Definition 5. *A* $n \times n$ *matrix* $H_n^p = [h_{ij}^p]$ *is defined as follows:*

$$h_{ij}^p = S(i,j) - S(i-1,j) - S(i-(p+1),j),$$

where $S(m,k)$ *is the Stirling number of the second kind.*

By Definition 5, we see that $h_{11}^p = 1,\ h_{1j}^p = 0,\ j \ge 2,\ h_{2j}^p = 0,\ j \neq 2$ and

$$h_{i,j}^p = h_{i-1,j-1}^p + j h_{i-1,j}^p.$$

Theorem 3. *For the Stirling matrix* $\zeta_n(2)$ *and* F_n^p, *we have* $\zeta_n(2) = F_n^p H_n^p$.

Proof. The matrix F_n^p is invertible. If we get $(F_n^p)^{-1}\zeta_n(2) = H_n^p$, then Theorem is proved. Let $(F_n^p)^{-1}\zeta_n(2) = C_n$ where $C_n = (c_{i,j})_{1\le i,j\le n}$, i.e.,

$$c_{i,j} = \sum_{k=j}^{i} f'^{p}_{i,k} S(k,j).$$

Since $(F_n^p)^{-1}$ and $\zeta_n(2)$ are lower triangular matrices, by the defniton of $(F_n^p)^{-1}$, we have

$$\begin{aligned} c_{i,j} &= \sum_{k=j}^{i} f_{i,k}^{\prime p} S(k,j) \\ &= f_{i,i-(p+1)}^{\prime p} S(i-(p+1),j) + f_{i,i-1}^{\prime p} S(i-1,j) + f_{i,i}^{\prime p} S(i,j) \\ &= -S(i-(p+1),j) - S(i-1,j) + S(i,j) = (h_{i,j}^p)_{1\le i,j\le n}. \end{aligned}$$

So, we get the result. □

Corollary 5. *For* $1 \le t \le n$,

$$S(n,t) = \sum_{i=t}^{n} f^P(n-i+1)\left(\frac{1}{t!}\sum_{l=0}^{t-1}(-1)^l \binom{t}{l}\left((t-1)^i - (t-1)^{i-1} - (t-1)^{i-(p+1)}\right)\right).$$

Proof. For $i > p+1$, by Definition 5 and relation (1), we have

$$h_{it}^p = \frac{1}{t!}\sum_{l=0}^{t-1}(-1)^l \binom{t}{l}\left((t-1)^i - (t-1)^{i-1} - (t-1)^{i-(p+1)}\right).$$

On the other hand, $S(n,t) = \sum_{k=1}^{t} f_{nk}^p h_{kt}^p$, So, we get

$$S(n,t) = \sum_{i=t}^{n} f^p(n-i+1)\left(\frac{1}{t!}\sum_{l=0}^{t-1}(-1)^l \binom{t}{l}\left((t-1)^i - (t-1)^{i-1} - (t-1)^{i-(p+1)}\right)\right).$$

□

Lemma 2. *Let* $\zeta_{n-1}(2)$ *be the Stirling matrix. Then,*

$$H_n^p = L_n^p([1] \oplus \zeta_{n-1}(2)).$$

Proof. Suppose $C_n^p = [c_{ij}^p] = L_n^p([1] \oplus \zeta_{n-1}(2))$. We prove that $c_{ij}^p = h_{ij}^p$. For $i = 1$, we have

$$l_{11}^p = 1 = h_{11}^p,\ l_{21}^p = 0 = h_{21}^p,\ l_{22}^p = 1 = h_{22}^p.$$

So, for $i = 1$ and 2, we obtain $c_{ij}^p = h_{ij}^p$. For $i \ge 3$,

$$c_{ij}^p = \sum_{l=j-1}^{i-1}\left[\binom{i-1}{l} S(l,j-1) - \binom{i-2}{l} S(l,j-1) - \binom{i-(p+2)}{l} S(l,j-1)\right].$$

Hence, by the relation (1), we get

$$c_{ij}^p = S(i,j) - S(i-1,j) + S(i-(p+1),j).$$

Therefore, $c_{ij}^P = h_{ij}^p$. □

Using Lemma 2, we have the following corollary.

Corollary 6. *For* $n \ge 2$, $\zeta_n(2) = F_n^p L_n^p([1] \oplus \zeta_{n-1}(2))$.

5 Factorization of the t-extension of the p-Fibonacci matrix

In this section, for $t \geq 2$, first we obtain the inverse of the t-extension of the p-Fibonacci matrix $F^p_{(t,n)}$. Then, by this we give a factorization of it.

Theorem 4. *For the inverse of the t-extension of the p-Fibonacci matrix, denoted by* $(F^p_{(t,n)})^{-1} = [f'^p_{(t,ij)}]$, *we have*

$$f'^p_{(t,ij)} = \begin{cases} 1, & \text{if } i = j, \\ -t, & \text{if } j = i-1, \\ -1, & \text{if } j = i-(p+1), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. To find the inverse of the t-extension of the p-Fibonacci matrix, we define the $n \times n$ matrix $U^p_{(t,n)} = [u^p_{t,ij}]$ as follows:

$$u^p_{t,ij} = \begin{bmatrix} f^p(t,1) & 0 & 0 & \cdots & 0 \\ f^p(t,2) & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ f^p(t,n) & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Clearly, $U^p_{(t,n)}$ is invertible and

$$(U^p_{(t,n)})^{-1} = \begin{bmatrix} f^P(t,1) & 0 & 0 & \cdots & 0 \\ -f^P(t,2) & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ -f^P(t,n) & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Hence,

$$F^p_{(t,n)} = U^p_{(t,n)} \times (I_1 \oplus U^p_{(t,n-1)}) \times (I_2 \oplus U^p_{(t,n-2)}) \times \cdots \times (I_{n-2} \oplus U^p_{(2,2)}),$$

where I_j is an identity matrix. Since $(I_k \oplus U^p_{(t,n-k)})^{-1} = I_k \oplus (U^p_{(t,n-k)})^{-1}$, we have

$$(F^p_{(t,n)})^{-1} = (I_{n-2} \oplus (U^p_{(t,2)})^{-1}) \times \cdots \times (I_1 \oplus (U^p_{(t,n-1)})^{-1}) \times (U^p_{(t,n)})^{-1}.$$

Therefore,

$$f'^p_{(t,ij)} = \begin{cases} 1, & \text{if } i = j, \\ -t, & \text{if } j = i-1, \\ -1, & \text{if } j = i-(p+1), \\ 0, & \text{otherwise.} \end{cases}$$

□

Example 1. For $p=2$ and $n=4$, we have

$$F^2_{(t,4)}=\begin{bmatrix}1&0&0&0\\t&1&0&0\\t^2&t&1&0\\t^3&t^2&t&1\end{bmatrix},\qquad U^2_{(t,4)}=\begin{bmatrix}1&0&0&0\\t&1&0&0\\t^2&0&1&0\\t^3&0&0&1\end{bmatrix},$$

$$I_1\oplus U^2_{(t,3)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&t&1&0\\0&t^2&0&1\end{bmatrix},\quad I_2\oplus U^2_{(t,2)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&t&1\end{bmatrix}.$$

Then,

$$F^2_{(t,4)}=U^2_{(t,4)}(I_1\oplus U^2_{(t,3)})(I_2\oplus U^2_{(t,2)}).$$

So, for $t\geq 2$,

$$(F^2_{(t,4)})^{-1}=\begin{bmatrix}1&0&0&0\\-t&1&0&0\\0&-t&1&0\\-1&0&-t&1\end{bmatrix}.$$

In the following, we obtain a factorization of the t-extension of the p-Fibonacci matrix. First, we introduce the matrix $L^p_{(t,n)}$.

Definition 6. *The $n\times n$ matrix $L^p_{(t,n)}=[l^p_{t,ij}]$ is defined as:*

$$l^p_{t,ij}=\binom{i-1}{j-1}-t\binom{i-2}{j-1}-\binom{i-(p+2)}{j-1}.$$

By the above information, we prove the following theorem.

Theorem 5. *For the Pascal matrix P_n, we have $P_n=F^p_{(t,n)}L^p_{(t,n)}$.*

Proof. The matrix $F^p_{(t,n)}$ is invertible. If we get $(F^p_{(t,n)})^{-1}P_n=L^p_{(t,n)}$, then the theorem is proved. Let $(F^p_{(t,n)})^{-1}P_n=B_{(t,n)}$ where $B_{(t,n)}=(b_{i,j})_{1\leq i,j\leq n}$, i.e.,

$$b_{i,j}=\sum_{k=j}^{i}f'^{p}_{(t,ik)}P_{k,j}.$$

Since $(F^p_n)^{-1}$ and P_n are lower triangular matrices, by the definition of $(F^p_n)^{-1}$, we have

$$\begin{aligned}b_{i,j}&=\sum_{k=j}^{i}f'^{p}_{(t,ik)}\binom{k-1}{j-1}\\&=f'^{p}_{(t,ii-(p+1))}\binom{i-(p+2)}{j-1}+f'^{p}_{(t,ii-1)}\left(\binom{i-1}{j-1}\binom{i-2}{j-1}\right)+f'^{p}_{(t,ii)}\binom{i-1}{j-1}\\&=\binom{i-1}{j-1}-t\binom{i-2}{j-1}-\binom{i-(p+2)}{j-1}=(l_{t,ij})_{1\leq i,j\leq n}.\end{aligned}$$

□

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