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Moment properties of lower record values from generalized inverse Weibull distribution and characterization

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Abstract. In this paper, the exact expressions as well as recurrence relations for single and product moments of the generalized lower record values from generalized inverse Weibull distribution are obtained. Further, the characterization of the given distribution is carried out through recurrence relations and conditional moment.

Keywords: Lower record values, generalized inverse Weibull distribution, single moments, product moments, recurrence relations, characterization.
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1 Introduction

For the past decade, many authors have studied the several distributions by adding the parameters as a combination (mixture) of distributions or make generalizations, modifications or extensions of the traditional two or three parameter distributions, because one or two parameters distribution lacks the ability in adequately modeling the given data set. New parameters are implemented to the Weibull distribution, all in order to extend its properties for better performances in fitting the data set. The mixtures of two inverse Weibull distribution and generalizations of Weibull distribution are defined in Gusmão *et al*. [13] which are more flexible than classical Weibull distribution. Several models of generalizations of Weibull distribution families are exponentiated Weibull [24, 25, 33], extended Weibull [21, 36], modified Weibull [16, 20, 27, 35], odd Weibull [6] and so on.

A random variale X is said to have a generalized inverse Weibull distribution [13], if its probability density function (pdf) is given by

$$f(x) = \gamma\beta\,\alpha^{\beta}x^{-(\beta+1)}\exp\Big[-\gamma\big(\frac{\alpha}{x}\big)^{\beta}\Big], \qquad x>0,\ \alpha>0,\ \beta>0,\ \gamma>0, \tag{1}$$

and the corresponding distribution function (df) is

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$$F(x)=\exp\Big[-\gamma(\frac{\alpha}{x})^{\beta}\Big], \qquad x>0, \alpha>0, \beta>0, \gamma>0. \tag{2}$$

The relation between pdf and df can be seen as

$$xf(x)=\gamma\beta\,\alpha^{\beta}\,F(x)=\beta\,[-\ln F(x)]F(x). \tag{3}$$

This new distribution is much more flexible than the inverse Weibull distribution and could have increasing, decreasing and unimodal hazard rates. Moreover, this distribution demonstrates a more reasonable fit than the other competitive distributions as Weibull and inverse Weibull distributions.

An observation is called a record if its value is greater (or lesser) than all the previous observations (Chandler [7]). Record values are used in several real life problems involving weather, economic studies, sports, industrial stress testing, meteorological analysis, hydrology, seismology, mining surveys. Prediction about future record values are very important such as intensity of the next earthquake, highest(record) level of water dam hold or discharge, extreme point of market index. For a survey on important results in this area one may refer to [1,2,4,15]. Dziubdziela and Kopcoinski [9] have generalized the concept of record values of Chandler [7] by random variables of a more generalized nature and called them the k-th record values. Later, Minimol and Thomas [22] called the record values defined by Dziubdziela and Kopcoinski [9] as the generalized record values, since the k-th member of the sequence of the ordinary record values is also known as the k-th record value. By setting $k=1$, we obtain ordinary record values.

Let $\{X_n, n\geq 1\}$ be a sequence of independent and identical distributed (iid) continuous random variables with df $F(x)$ and pdf $f(x)$. For a fixed positive integer k, we define the sequence $\{L_k(n), n\geq 1\}$ of k-th lower record times of $\{X_n, n\geq 1\}$ as follows:

$$\begin{aligned} L_k(1) &= 1,\\ L_k(n+1) &= \min\{j>L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}. \end{aligned}$$

The sequence $\{Z_n^{(k)}, n\geq 1\}$ with $Z_n^{(k)}=X_{k:L_k(n)+k-1}$, $n=1,2,\ldots$, is called the sequence of k-th lower record values of $\{X_n, n\geq 1\}$. Suppose $Z_0^{(k)}=0$, then for $k=1$, we have $Z_n{}^{(1)}=X_{L(n)}$, $n\geq 1$, i.e. the lower record values of $\{X_n, n\geq 1\}$. The pdf of $Z_n^{(k)}$ and the joint pdf of $Z_m^{(k)}$ and $Z_n^{(k)}$ are defined as

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1}f(x), \quad n\geq 1, \tag{4}$$

$$\begin{aligned} f_{Z_m^{(k)},Z_n^{(k)}}(x,y) &= \frac{k^n}{(m-1)!\,(n-m-1)!}[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}\\ &\quad\times[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y), \quad y<x,\ 1\leq m<n,\ n\geq 2, \end{aligned} \tag{5}$$

respectively by Pawlas and Szynal [29].

The conditional pdf of $Z_n^{(k)}$ given $Z_m^{(k)}=x$, is

$$f_{Z_n^{(k)}|Z_m^{(k)}}(y|x)=\frac{k^{n-m}}{(n-m-1)!}[\ln F(x)-\ln F(y)]^{n-m-1}\times\Big[\frac{F(y)}{F(x)}\Big]^{k-1}\frac{f(y)}{F(x)}, \quad y<x. \tag{6}$$

For some recent developments on generalized record values with reference to those arising from the different distributions out by [3,5,11,12,17,18,23,28–32,34].

The paper is organized as follow: Section 2 contains some auxiliary results in the form of lemmas, which are subsequently needed for studying the moment properties of generalized lower record values from the generalized inverse Weibull distribution. In Section 3, the exact expressions as well as the recurrence relations for single moment are obtained whereas Section 4 deals with product moments of generalized lower record values. Section 5 is devoted to characterization of the said distribution by the recurrence relations and conditional expectation. Finally, the whole study concluded in Section 6.

2 Auxiliary results

Here, we shall present some auxiliary results which are needed in the following sections:

Lemma 1. *For a fixed positive integer $k \geq 1$ and distribution as given in (1)*

$$E(Z_1^{(k)})^j = (k\alpha^\beta \gamma)^{j/\beta}\,\Gamma(1-\frac{j}{\beta}). \tag{7}$$

Proof. In view of (4), for $n=1$, we have

$$E(Z_1^{(k)})^j = k\int_0^\infty x^j\,[F(x)]^{k-1} f(x)dx. \tag{8}$$

Now from (1) and (2)

$$E(Z_1^{(k)})^j = k\alpha^\beta \gamma \int_0^\infty x^{j-\beta-1}\,exp\Big[-k\gamma\Big(\frac{\alpha}{x}\Big)^\beta\Big]dx.$$

Consider $t = x^{-\beta}$, we get the result given in (7). □

Lemma 2. *Fix a positive integer $k \geq 1$, for $1 \leq m \leq n-2$ and $i, j = 0, 1, \ldots,$*

$$\begin{aligned} E[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_n^{(k)})^j] &= -\frac{i\,k^n}{m!\,(n-m-1)!}\int_\alpha^\beta \int_y^\beta x^{i-1} y^j \\ &\quad \times [-\ln F(x)]^m [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)dxdy. \end{aligned} \tag{9}$$

Proof. From (5), we have

$$\begin{aligned} &E[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_n^{(k)})^j] \\ &\quad = \frac{k^n}{(m-1)!\,(n-m-1)!} \times \int_\alpha^\beta \int_y^\beta x^i y^j [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-2} \\ &\qquad \times [F(y)]^{k-1} f(y) \Big\{ [-\ln F(x)] \frac{(n-m-1)}{m} - [\ln F(x) - \ln F(y)] \Big\} dxdy. \end{aligned} \tag{10}$$

Let

$$\begin{aligned} h(x,y) &= \frac{1}{m}[\ln F(x) - \ln F(y)]^{n-m-1} [-\ln F(x)]^m, \\ \frac{\partial}{\partial x} h(x,y) &= [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-2} \\ &\quad \times \Big\{ [-\ln F(x)] \frac{(n-m-1)}{m} - [\ln F(x) - \ln F(y)] \Big\}. \end{aligned} \tag{11}$$

Taking into account the value of (11) in (10) and simplifying, we get the required result. □

Lemma 3. *For the generalized inverse Weibull distribution given in (1). Fix a positive integer $k \geq 1$, for $1 \leq m \leq n-1$ and $i, j = 0, 1, \ldots,$*

$$E[(Z_1^{(k)})^i (Z_3^{(k)})^j] = (k\alpha^\beta \gamma)^{\frac{i+j}{\beta}} \sum_{a=0}^{1} \frac{(-1)^a}{1+a-\frac{i}{\beta}} \binom{1}{a} \Gamma(3-(i+j)/\beta). \tag{12}$$

Proof. In the view of (5), we have

$$\begin{aligned} E[(Z_1^{(k)})^i (Z_3^{(k)})^j] &= k^3 \int_0^\infty \int_y^\infty x^i y^j [\ln F(x) - \ln F(y)] \frac{f(x)}{F(x)} [F(y)]^{k-1} f(y) dx dy \\ &= k^3 \sum_{a=0}^{1} (-1)^a \binom{1}{a} \int_0^\infty y^j [-\ln F(y)]^{1-a} [F(y)]^{k-1} f(y) I(y) dy, \end{aligned} \tag{13}$$

where

$$I(y) = \int_y^\infty x^i [-\ln F(x)]^a \frac{f(x)}{F(x)} dx.$$

In view of (1) and (2), we get

$$I(y) = \frac{\alpha^i \gamma^{\frac{i}{\beta}}}{1+a-\frac{i}{\beta}} [-\ln F(y)]^{1+a-\frac{i}{\beta}}. \tag{14}$$

Now by putting (14) in (13), we get

$$E[(Z_1^{(k)})^i (Z_3^{(k)})^j] = k^3 \alpha^{(i+j)} \gamma^{\frac{i+j}{\beta}} \sum_{a=0}^{1} \frac{(-1)^a}{1+a-\frac{i}{\beta}} \binom{1}{a} \int_0^\infty [-\ln F(y)]^{2-\frac{(i+j)}{\beta}} [F(y)]^k \frac{f(y)}{F(y)} dy. \tag{15}$$

Setting $[-\ln F(y)] = z$ in (15), we get

$$E[(Z_1^{(k)})^i (Z_3^{(k)})^j] = k^3 \alpha^{(i+j)} \gamma^{\frac{i+j}{\beta}} \sum_{a=0}^{1} \frac{(-1)^a}{1+a-\frac{i}{\beta}} \binom{1}{a} \int_0^\infty z^{2-\frac{(i+j)}{\beta}} \exp(-kz) dz,$$

which leads to the result given in (12). □

3 Main results for single moments

Theorem 1. *For the generalized inverse Weibull distribution as given in (1). Fix a positive integer $k \geq 1$. Then for $n \geq 1$ and $j = 0, 1, \ldots$*

$$E(Z_{n+1}^{(k)})^j = \left(1 - \frac{j}{n\beta}\right) E(Z_n^{(k)})^j, \tag{16}$$

and subsequently

$$E(Z_n^{(k)})^j = (k\alpha^\beta \gamma)^{j/\beta} \frac{\Gamma(n-\frac{j}{\beta})}{\Gamma(n)}. \tag{17}$$

Proof. In view of (3) and (4), we have

$$E(Z_n^{(k)})^j = \frac{\beta\, k^n}{(n-1)!}\int_0^\infty x^{j-1}\,[-\ln F(x)]^n [F(x)]^k\, dx. \tag{18}$$

Integrating (18) by parts, treating x^{j-1} for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned} E(Z_n^{(k)})^j &= \frac{\beta\, k^n\, n}{(n-1)!\, j}\int_0^\infty x^j\,[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\, dx \\ &\quad - \frac{\beta\, k^{n+1}}{(n-1)!\, j}\int_0^\infty x^j\,[-\ln F(x)]^n [F(x)]^{k-1} f(x)\, dx \\ &= \frac{n\beta}{j}[E(Z_n^{(k)})^j - E(Z_{n+1}^{(k)})^j]. \end{aligned}$$

Now by arranging the above expression, we get the relation given in (16). The relation (17) can be obtained by writing (16) recursively and using Lemma 1. □

Remark 1.

- *At $k = 1$ in (16), we get the recurrence relation for the single moments of lower records from generalized inverse Weibull distribution obtained by Khan and Zia [19].*
- *By setting $k = 1$ and $\gamma = 1$ in (17), we get the expression for exact moments of lower records from the inverse Weibull distribution as obtained by Nigm and Khalil [26], with shape parameter $\alpha = 1$.*
- *By setting $k = 1$ in (17), we get the expression for exact moments of lower records from the generalized inverse Weibull distribution as obtained by Khan and Zia [19].*

3.1 Numerical computations

The distributional properties of generalized lower record values in terms of mean, variance, skewness and kurtosis with the different values of parameters of generalized the inverse Weibull distribution for different k are tabulated below in Table 1 to 4. All computations here we performed using Mathematica. Mathematica like other algebraic manipulation packages allow for arbitrary precisions, so the accuracy of the given values is not an issue. For any fixed j increase in sample size n leads to decrease in variance, which is universally true.

4 Main results for product moments

Theorem 2. *For $1 \le m \le n-2$ and $i, j = 0, 1, \ldots,$*

$$E[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j] = \Big(1 - \frac{i}{m\beta}\Big) E[(Z_m^{(k)})^i (Z_n^{(k)})^j]. \tag{19}$$

For $m \ge 1$ and $i, j = 0, 1, \ldots,$

$$E[(Z_m^{(k)})^i (Z_{m+1}^{(k)})^j] = \Big(\frac{m\beta}{m\beta - i}\Big) E[(Z_{m+1}^{(k)})^{i+j}], \tag{20}$$

Table 1: Distribution properties of lower record values

	$\alpha=0.50, \beta=4, \gamma=1, k=1$			
n	Mean	Variance	Skewness	Kurtosis
2	0.45953	0.010392179	1.086407821	-1.726798023
3	0.40209	0.004493632	1.045359486	-1.868264909
4	0.36858	0.002618784	1.030715185	-1.913629696
5	0.34555	0.001755197	1.023156112	-1.935948863
6	0.32827	0.001278807	1.018709985	-1.948671767
7	0.31459	0.000993332	1.016013857	-1.957170073
8	0.30335	0.000798978	1.013119971	-1.955381989
9	0.29388	0.000654546	1.011680388	-1.968633844
10	0.28571	0.000549796	1.010247424	-1.972392663

Table 2: Distribution properties of lower record values

	$\alpha=1, \beta=4, \gamma=2, k=1$			
n	Mean	Variance	Skewness	Kurtosis
2	1.09296	0.058748438	1.086439345	-1.726752049
3	0.95634	0.025403804	1.045416978	-1.868242104
4	0.87662	0.014857376	1.030679886	-1.913493117
5	0.82185	0.009972578	1.023125564	-1.935687577
6	0.78076	0.007283822	1.018562967	-1.948830644
7	0.74823	0.005611867	1.015499712	-1.957513056
8	0.72150	0.004507752	1.013313048	-1.963686799
9	0.69896	0.003704918	1.011653721	-1.968264062
10	0.67954	0.003135388	1.010360822	-1.971876307

and subsequently

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{(k\alpha^\beta \gamma)^{\frac{i+j}{\beta}}}{\Gamma(m)\Gamma(n-m)} \sum_{a=0}^{n-m-1} \frac{(-1)^a}{m+a-\frac{i}{\beta}} \binom{n-m-1}{a} \Gamma(n-(i+j)/\beta). \tag{21}$$

Proof. From Lemma 2, we have

$$\begin{aligned} E[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_n^{(k)})^j] &= -\frac{i\,k^n}{m!\,(n-m-1)!} \int_\alpha^\beta \int_y^\beta x^{i-1} y^j [-\ln F(x)]^m \\ &\quad \times [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\,dx\,dy. \end{aligned} \tag{22}$$

Now using the relation (3) in (22) and re-arranging the terms, we get the expression given in (19). Now putting $n=m+1$ in (19) and noting that $E[(Z_m^{(k)})^i (Z_m^{(k)})^j] = E[(Z_m^{(k)})^{i+j}]$, the recurrence relation given in (20) can be easily established. The relation (21) can be obtained by writing (19) recursively and using the Lemma 3. □

Table 3: Distribution properties of lower record values.

n	$\alpha = 0.50, \beta = 4, \gamma = 1, k = 3$			
	Mean	Variance	Skewness	Kurtosis
2	0.604777	0.017992781	1.086433362	-1.726763671
3	0.529180	0.007779528	1.045420181	-1.868234374
4	0.485082	0.004537453	1.030687867	-1.913500469
5	0.454764	0.003051704	1.023141997	-1.935676464
6	0.432026	0.002229535	1.018569016	-0.656299715
7	0.414025	0.001719299	1.015506098	-1.957501924
8	0.399238	0.001378019	1.013309340	-1.963669262
9	0.386762	0.001136155	1.011656738	-1.968275482
10	0.376019	0.000957712	1.010365124	-1.971852726

Table 4: Distribution properties of lower record values.

n	$\alpha = 1, \beta = 4, \gamma = 2, k = 3$			
	Mean	Variance	Skewness	Kurtosis
2	1.43841	0.101776672	1.086438364	-1.726756041
3	1.25861	0.044000868	1.045423396	-1.868227591
4	1.15373	0.025657087	1.030685139	-1.913498488
5	1.08162	0.017258176	1.023137048	-1.935678712
6	1.02754	0.012601548	1.018573961	-1.948810611
7	0.984722	0.009728583	1.015504895	-1.957503803
8	0.949554	0.007795201	1.013308124	-1.963671217
9	0.919880	0.006428786	1.011654531	-1.968278557
10	0.894328	0.005417428	1.010367731	-1.971845744

Identity 1. For $1 \leq m < n$,

$$\sum_{a=0}^{n-m-1} \frac{(-1)^a}{(m+a)} \binom{n-m-1}{a} = \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}. \tag{23}$$

Proof. Putting $i = j = 0$ in (21), we get the required result. □

Remark 2.

- *When $j = 0$, we get the result for single moment as given in Theorem 1.*
- *By setting $i = 0$ in (21) and using relation (23), we get the exact expression for single moment of the generalized inverse Weibull distribution as obtained in (17).*
- *When $k = 1$, the Theorem 2 reduces to the recurrence relations for product moments of lower record values from the generalized inverse Weibull distribution as obtained by Khan and Zia [19].*

Table 5: Product moments of lower record values.
($\alpha = 0.50$, $\gamma = 1$, $\beta = 4$, $k = 1$)

	n							
m	2	3	4	5	6	7	8	9
1	0.29541	0.25321	0.23019	0.21484	0.20353	0.19469	0.18747	0.18143
2		0.18991	0.17264	0.16113	0.15265	0.14601	0.14061	0.13607
3			0.15106	0.14099	0.13357	0.12776	0.12303	0.11906
4				0.12924	0.12244	0.11712	0.11278	0.10914
5					0.11479	0.10980	0.10573	0.10232
6						0.10431	0.10044	0.09720
7							0.09626	0.09315
8								0.08983

Table 6: Product moments of lower record values.
($\alpha = 1.0$, $\beta = 4$, $\gamma = 1$, $k = 1$)

	n							
m	2	3	4	5	6	7	8	9
1	1.67109	1.43236	1.30214	1.21533	1.15137	1.10131	1.06052	1.02631
2		1.07427	0.97661	0.91150	0.86353	0.82598	0.79539	0.76973
3			0.85453	0.79756	0.75559	0.72273	0.69597	0.67352
4				0.73110	0.69262	0.66251	0.63797	0.61739
5					0.64933	0.62110	0.59810	0.57880
6						0.59004	0.56819	0.54986
7							0.54452	0.52695
8								0.50813

4.1 Numerical computations

The product moments ($m < n$) of generalized lower record values with the different parameters of the generalized inverse Weibull distribution and various k are tabulated below in Table 5 to 8.

5 Characterization

Theorem 3. *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $0 < F(x) < 1$ for all $x > 0$, then*

$$E(Z_n^{(k)})^j = \frac{n\beta}{j}\Big[E(Z_n^{(k)})^j - E(Z_{n+1}^{(k)})^j\Big], \tag{24}$$

if and only if

$$F(x) = \exp\Big[-\gamma(\frac{\alpha}{x})^{\beta}\Big], \qquad x > 0, \alpha > 0, \beta > 0, \gamma > 0. \tag{25}$$

Table 7: Product moments of lower record values.
($\alpha = 0.50$, $\beta = 4$, $\gamma = 1$, $\beta = 4$, $k = 3$)

	n							
m	2	3	4	5	6	7	8	9
1	0.51166	0.43857	0.39870	0.37212	0.35253	0.33721	0.32472	0.31424
2		0.32893	0.29902	0.27909	0.26440	0.25290	0.24354	0.23568
3			0.26165	0.24420	0.23135	0.22129	0.21309	0.20622
4				0.22385	0.21207	0.20285	0.19534	0.18904
5					0.19882	0.19017	0.18313	0.17722
6						0.18066	0.17397	0.16836
7							0.16672	0.16134
8								0.15558

Table 8: Product moments of lower record values.
($\alpha = 1.0$, $\beta = 4$, $\gamma = 2$, $k = 3$)

	n							
m	2	3	4	5	6	7	8	9
1	2.89441	2.48092	2.25538	2.10502	1.99423	1.90753	1.83688	1.77762
2		1.86069	1.69154	1.57877	1.49567	1.43064	1.37766	1.33322
3			1.48009	1.38142	1.30871	1.25181	1.20545	1.16656
4				1.26630	1.19965	1.14750	1.10500	1.06935
5					1.12468	1.07578	1.03593	1.00252
6						1.02199	0.98414	0.95239
7							0.94313	0.91271
8								0.88011

Proof. The necessary part can be seen in view of (16). To prove sufficiency part, we have Bieniek and Szynal [5]

$$E(Z_n^{(k)})^j - E(Z_{n-1}^{(k)})^j = -\frac{jk^{n-1}}{(n-1)!}\int_{-\infty}^{\infty} x^{j-1}\,[-\ln F(x)]^{n-1}[F(x)]^k\,dx. \tag{26}$$

Therefore,

$$\frac{k^n}{(n-1)!}\int_0^{\infty} x^j\,[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\,dx = \frac{\beta\, k^n}{(n-1)!}\int_{-\infty}^{\infty} x^{j-1}\,[-\ln F(x)]^{n}[F(x)]^k\,dx,$$

which becomes

$$\frac{k^n}{(n-1)!}\int_0^{\infty} x^{j-1}\,[-\ln F(x)]^{n-1}[F(x)]^{k-1}\Big\{xf(x)-\beta[-\ln F(x)][F(x)]\Big\}dx = 0. \tag{27}$$

Now applying the generalization of Muntz-Szasz Theorem (see for example Hwang and Lin [14]) in (27), we get

$$xf(x) = \beta[-\ln F(x)][F(x)],$$

which proves that $f(x)$ has the form as given in (1). □

Theorem 4. *For a positive integer k, i and j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with pdf given by* (1)*, is that*

$$E[(Z_{m+1}^{(k)})^i(Z_n^{(k)})^j] = \Big(1-\frac{i}{m\beta}\Big)E[(Z_m^{(k)})^i(Z_n^{(k)})^j]. \tag{28}$$

Proof. The necessary part can be seen in view of Theorem 2. To prove sufficiency part, we have

$$E[(Z_{m+1}^{(k)})^i(Z_n^{(k)})^j] - E[(Z_m^{(k)})^i(Z_n^{(k)})^j] = -\frac{i}{m\beta}E[(Z_m^{(k)})^i(Z_n^{(k)})^j]. \tag{29}$$

Now in view of Lemma 2, we get

$$-\frac{i\,k^n}{m!\,(n-m-1)!}\int_0^\infty\int_y^\infty x^{i-1}y^j[-\ln F(x)]^m[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)dxdy$$
$$= -\frac{i\,k^n}{m!\,(n-m-1)!\,\beta}\int_0^\infty\int_y^\infty x^iy^j\,[-\ln F(x)]^{m-1}$$
$$\times\frac{f(x)}{F(x)}[\ln F(x)-\ln F(y)]^{n-m-1}\,[F(y)]^{k-1}f(y)dxdy. \tag{30}$$

This gives

$$\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_y^\infty x^{i-1}y^j[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}$$
$$\times[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)\Big\{\frac{F(x)}{f(x)}[-\ln F(x)]-\frac{x}{\beta}\Big\}dxdy = 0. \tag{31}$$

Now applying the generalization of Müntz-Szász Theorem (see for example Hwang and Lin [14]) to (31), we get

$$xf(x) = \beta[F(x)][-\ln F(x)], \tag{32}$$

which leads to the form of pdf given in (1). □

Theorem 5. *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ and pdf $f(x)$ over the support $(0,\infty)$ and let $h(x)$ be a continuous and differentiable function of x, the for two consecutive value of m and $m+1$, then*

$$E[F(Z_n^{(k)})|(Z_l^{(k)}) = x] = g_{n|l}(x) = F(x)\Big(\frac{k}{k+1}\Big)^{n-l},\ l = m, m+1, m\geq k, \tag{33}$$

if and only if

$$F(x) = \exp\Big[-\gamma(\frac{\alpha}{x})^\beta\Big], \qquad x>0, \alpha>0, \beta>0, \gamma>0.$$

Proof. In view of (6), we have

$$E[F(Z_n^{(k)})|(Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\int_0^x F(y)[\ln F(x)-\ln F(y)]^{n-m-1}\Big[\frac{F(y)}{F(x)}\Big]^{k-1}\frac{f(y)}{F(x)}dy. \tag{34}$$

Setting $\frac{F(y)}{F(x)} = u$ in (34), we get

$$E[F(Z_n^{(k)})|(Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!} F(x) \int_0^1 u^k [-\ln u]^{n-m-1} du. \tag{35}$$

We have the Gradshteyn and Ryzhik [10]

$$\int_0^1 x^{\nu-1} [-\ln x]^{\mu-1} dx = \frac{\Gamma\mu}{\nu^{\mu}}. \tag{36}$$

Now on using (36) in (35) gives the result given in (33). To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!} \int_0^x F(y)[\ln F(x) - \ln F(y)]^{n-m-1} [F(x)]^{k-1} f(y) dy = [F(x)]^k g_{n|m}(x), \tag{37}$$

where

$$g_{n|m}(x) = F(x) \left(\frac{k}{k+1}\right)^{n-m}. \tag{38}$$

Differentiating (36) both sides with respect to x, we get

$$\frac{k^{n-m} f(x)}{(n-m-2)! F(x)} \int_0^x F(y)[\ln F(x) - \ln F(y)]^{n-m-2} [F(x)]^{k-1} f(y) dy$$

$$= g'_{n|m}(x) [F(x)]^k + k g_{n|m}(x) [F(x)]^{k-1} f(x)$$

or

$$k g_{n|m+1}(x) [F(x)]^{k-1} f(x) = g'_{n|m}(x) [F(x)]^k + k g_{n|m}(x) [F(x)]^{k-1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]},$$

where

$$g'_{n|m}(x) = \gamma\beta\,\alpha^{\beta} x^{-(\beta+1)} \exp\left[-\gamma\left(\frac{\alpha}{x}\right)^{\beta}\right] \left(\frac{k}{k+1}\right)^{n-m}$$

$$g_{n|m+1}(x) - g_{n|m}(x) = \exp\left[-\gamma\left(\frac{\alpha}{x}\right)^{\beta}\right] \frac{1}{k} \left(\frac{k}{k+1}\right)^{n-m}.$$

Thus,

$$\frac{f(x)}{F(x)} = \gamma\beta\,\alpha^{\beta} x^{-(\beta+1)},$$

which implying that

$$F(x) = \exp\left[-\gamma\left(\frac{\alpha}{x}\right)^{\beta}\right].$$

Hence the sufficiency part is proved. □

6 Conclusion and future work

The current study demonstrate the study of moments of lower generalized record values. The exact expressions for single and product moments are obtained. The recursive relations for single and product moments are also developed to make ease of computation of moments. The results obtained are used to study the distributional properties of lower generalized record values from the generalized inverse Weibull distribution in terms of mean, variance, skewness and kurtosis and covariance for different values of parameters for the generalized inverse Weibull distribution. Further, we characterized the generalized inverse Weibull distribution through recurrence relations and conditional moment. We can explore our study for dual generalized order statistics which contains several models of order random variates.

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