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Numerical investigation of the time-fractional Black-Scholes equation with barrier choice of regulating European option

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Abstract. The price variance of the associated fractal transmission mechanism was used to estimate the Black-Scholes fractional model of which a time-fractional derivative is α. In the current paper, the time-fractional Black-Scholes equation (TFBSE) that the temporal derivative is the Caputo fractional derivative is known by regulating the European option. At first, linear interpolation with a temporally $\tau^{2-\alpha}$ order accuracy is used for constructing the semi-discrete. Then, the spatial derivative terms are approximated with the help of the collocation approach centered on the Chebyshev polynomials of the third form (CPTF). Finally, The unconditional stability and convergence order are analyzed by applying the energy method. To show the precision of the numerical system, we solved two instances of the TFBSE. Numerical results and comparisons indicate the proposed approach is very reliable and efficient.

Keywords: The fractional Black–Scholes equation, the linear interpolation, the Chebyshev polynomials of the third kind, the collocation method.
AMS Subject Classification 2010: 41A50, 91G80, 34K37, 97N50.

1 Introduction

One of the most prevalent and important of the financial derivatives is the selection of an option, hence in view of both theorization and practice, how to value an option is a striking topic. Therefore, a strong comprehension of the design is required where one has undertaken the duty of pricing options. As pricing options exist, Black and Scholes [3] and Merton [19] in 1973 developed a formula for explaining the estimated behavior of the underlying finance called the Black–Scholes model (BSM). This model has been commonly used by traders of options and ultimately contributes a significant rise in options trading due to the accuracy and efficacy of the model in forecasting options prices. On financial theory and the development of the fractal arrangement for the stochastic system and the financial region, fractional calculus and fractional partial differential equations were introduced by replacing the usual Brownian

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motion associated with fractional Brownian action in classical nature. Numerical methods represent a strong tool for solving them, due to the extreme non-locality of fractional integration and differentiation. For example, finite element methods [33], finite difference methods [31, 32], meshless methods [15], finite volume methods [16], and spectral methods [27, 30] was introduced in order to solve fractional differential equations.

In recent years, many researchers have extended the BSM [2, 17] because fractional integrals and derivatives are important instruments to describe the inherited and memory characteristics of various substances. For example, the pricing of the European call option was firstly carried out using a TF-BSE [24]. Liang et al. in [14] proposed a specific state of the bi-fractional BSM of the TFBSE. Cartea in [4] conducted another investigation into this model, presenting that a partial-integral-differential equation could describe the worth of European-style derivatives that contains a non-local time-to-maturity technician called the Caputo fractional derivative. In addition, Leonenko et al. provided powerful explicit solutions, implementing spectral methods in fractional Pearson diffusions relying on the correlating time-fractional of diffusion model which was actually applied to develop BSM [13]. The authors also have made use of a non-Markovian stable inverse time variance to give stochastic solutions. In this paper, we investigate the time fractional Black–Scholes model, as

$$\begin{gathered} {}_0\mathscr{D}_t^{\alpha} u(x,t) = p\frac{\partial^2 u(x,t)}{\partial x^2} + q\frac{\partial u(x,t)}{\partial x} - ru(x,t) + f(x,t), \\ 0 < x < 1, \quad 0 < t \leq T, \quad 0 < \alpha \leq 1, \end{gathered} \tag{1}$$

with the initial condition $u(x,0) = \varphi(x)$ and the following boundary conditions

$$u(0,t) = \psi_0(t), \quad u(1,t) = \psi_1(t), \quad 0 < t \leq T, \tag{2}$$

in which $p = \frac{1}{2}\sigma^2 > 0$, $q = r - p$. In addition, r is nonnegative and the term for the source is $f(x,t)$. In this model, the fractional derivative ${}_0\mathscr{D}_t^{\alpha}$ is the right Caputo fractional derivative of the order $0 < \alpha \leq 1$ that is

$${}_0\mathscr{D}_t^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, \;\; n-1 < \alpha < n.$$

However, taking $\alpha \to n$, we get

$$\lim_{\alpha \to n} {}_0\mathscr{D}_t^{\alpha} u(x,t) = \frac{\partial^n u(x,t)}{\partial t^n}.$$

Because of the memory trait of fractional derivatives, it is rather hard to obtain an accurate solution to this issue. Thus, various investigators have therefore sought methods to estimate these problems [9, 20, 21, 23, 25, 26]. The more method used to solve the analytical form [6, 8, 10, 12]. Generally, the solutions obtained through the mentioned schemes take in the form of an infinite series, which makes them hard to solve. For this purpose, greater attention is being paid to developing efficient numerical methods for the solution of fractional BSM. Some of those strategies will be described below. The option pricing under a time-fractional BSM is achieved in [29] and [22], respectively, by a second-order effective θ finite-difference scheme and an implicit finite difference structure with the first-order accuracy. In 2014, in order to expand the partial integrodifferential equation leading to the option pricing assumption [1], Bhowmik used an explicit-implicit numerical procedure [1]. In 2015, to approach American options pricing, a predictor-corrector was used in [5]. Besides that, for this option, the authors of [28] provide a discrete implicit modeling method. In 2019, the RBF meshless methods was used to solve time fractional BSM.

The rest of the current paper is formed as below: in Section 2, the time and spatial discretization scheme are constructed which is based on a quadratic interpolation for the time variable and a third-kind Chebyshev collocation approach for estimating the spatial fractional derivative. Finally, to illustrate the effectiveness of the existing method, we study two numerical examples during the last section.

2 The time and spatial discretization scheme

Let $t_j = j\tau,\ j = 0,1,\ldots,M$ be temporal nodes that $\tau = T/M$ is the step size. Then, the linear structure can discretize the Caputo derivative for $0 < \alpha \leq 1$ that is defined in [11] as

$$ {}_0\mathscr{D}_t^{\alpha} u(x,t) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{M} \mathscr{S}_{M,j} u(x,t_j) + \mathscr{O}(\tau^{2-\alpha}), \tag{3} $$

in which

$$ \mathscr{S}_{M,j} = \begin{cases} 1, & j = M, \\ (M-j-1)^{1-\beta} - 2(M-j)^{1-\beta} + (M-j+1)^{1-\beta}, & 1 \leq j < M, \\ (M-1)^{1-\beta} - (M)^{1-\beta}, & j = 0. \end{cases} \tag{4} $$

Denoting $u^j = u(x,t_j),\ \mathfrak{p} = \Gamma(2-\alpha)p,\ \mathfrak{q} = \Gamma(2-\alpha),\ \mathfrak{r} = r\Gamma(2-\alpha),\ F^M = \Gamma(2-\alpha)f(x,t_M)$ and $\overline{\mathscr{S}}_{M,j} = -\mathscr{S}_{M,j}$, then the following semi-discrete procedure of Eq. (1) is obtained as

$$ (\mathscr{S}_{M,M} + \mathfrak{r}\tau^{\alpha})u^M - \mathfrak{p}\tau^{\alpha}\frac{\partial^2 u^M}{\partial x^2} - \mathfrak{q}\tau^{\alpha}\frac{\partial u^M}{\partial x} = \sum_{j=0}^{M-1} \overline{\mathscr{S}}_{M,j} u^j + \tau^{\alpha} F^M + \tau^{\alpha}\mathscr{R}^M, \tag{5} $$

in which there exists a positive constant C such that $\mathscr{R}^M \leq C\mathscr{O}(\tau^{2-\alpha})$. The subsequent semi-discrete structure is produced by ignoring $\mathscr{R}^M$ in the above relationship, as

$$ \begin{cases} (\mathscr{S}_{M,M} + \mathfrak{r}\tau^{\alpha})U^M - \mathfrak{p}\tau^{\alpha}\frac{\partial^2 U^M}{\partial x^2} - \mathfrak{q}\tau^{\alpha}\frac{\partial U^M}{\partial x} = \sum_{j=0}^{M-1} \overline{\mathscr{S}}_{M,j} U^j + \tau^{\alpha} F^M, \\ U^0(x) = \varphi(x), \quad 0 < x < 1, \\ U^j(0) = \psi_0(t_j), \quad U^j(1) = \psi_1(t_j), \quad j = 0,1,\ldots,M. \end{cases} \tag{6} $$

Here for $j = 0,1,\ldots,M$, the approximate solution of Eq. (5) is U^j. The numerical scheme's stability and convergence analysis for semi discrete is introduced in [18] as following theorems.

Theorem 1. *The obtained semi-discrete scheme by Eq. (6) is unconditionally stable.*

Theorem 2. $\mathscr{O}(\tau^{2-\alpha})$ *is the convergence order of the obtained semi-discrete scheme by Eq. (6).*

At present, we use CPTF $\mathscr{V}_i(x),\ i = 0,1,\ldots,N$ to arrive at the full-discrete structure of Eq. (6). First of all, the analytical form of the Jacobi polynomials, i.e.,

$$ \mathscr{V}_i(x) = \frac{(i!2^i)^2}{(2i)!} J_i^{(\frac{-1}{2},\frac{1}{2})}(x), $$

describes CPTK. Then, we use the shifted Chebyshev polynomials of the third form (SCPTF), $\mathscr{V}_i^*(x)$, $i = 0,1,\ldots,N$, for the approximate expansion of $u(x,t_j)$ in the interval $[0,1]$. Now, the $u(x,t_j)$ expansion across the space variable is described by using only SCPTF's first $N+1$ terms as below relation at interval $[0,1]$:

$$u(x,t_j) = \sum_{i=0}^{N} \upsilon_i(t_j)\mathscr{V}_i^*(x), \tag{7}$$

where $\upsilon_i(t_j)$ is the unknown coefficients and $\mathscr{V}_i^*(x)$ rewrites as the following scheme:

$$\begin{aligned} &\mathscr{V}_i^*(x) = \mathbf{\Lambda}_i \sum_{k=0}^{i}\sum_{l=0}^{k} \mathbf{\Upsilon}_{i,k,l} \times x^{k-l}, \\ &\mathbf{\Lambda}_i = \frac{4^i\Gamma(i+0.5)}{(i+1)(2i)!}, \quad \mathbf{\Upsilon}_{i,k,l} = \frac{(-1)^l\Gamma(i+1+k)(i+1)!}{(i-k)!\Gamma(k+0.5)(k-l)!l!k!}. \end{aligned} \tag{8}$$

The unknown coefficients $\mathfrak{u}_i(t_j)$ in the Eq. (7) are defined as below:

$$\upsilon_i(t_j) = \frac{2}{\pi}\int_0^1 \sqrt{\frac{x}{1-x}}u(x,t_j)\mathscr{V}_i^*(x)dx, \quad j = 0,1,\ldots,M. \tag{9}$$

To obtain a full-discrete scheme Eq. (6), we approximate the first and second order space derivative, $\frac{\partial^l u^M}{\partial x^l}$, $l = 1,2$, based on SCPTF. By employing Eqs. (7) and (8) and using the linearity properties of the derivative, we have

$$\frac{\partial^\xi(u(x,t_j))}{\partial x^\xi} = \sum_{i=\xi}^{N}\sum_{k=0}^{i-\xi}\sum_{l=0}^{k} \upsilon_i(t_j)N_{i,k,l}^{\xi}x^{k-l}, \quad \xi \in \mathbb{N}, \tag{10}$$

where $N_{i,k,l}^{\xi}$ is given by

$$N_{i,k,l}^{\xi} = \frac{(-1)^l 2^{2i}(i)!\Gamma(i+0.5)\Gamma(i+k+\xi+1)\Gamma(k-l+\xi+1)}{l!(2i)!(i-k-\xi)!(k+\xi-l)!\Gamma(k+\xi+0.5)\Gamma(k-l+1)}.$$

With taking the collocation points $\{x_s\}_{s=1}^{N+1-\xi}$ that are the roots of SCPTF $\mathscr{V}_{N+1-\xi}^*(x)$ and substituing Eq. (10) in Eq. (6), we arrive in a point (x_s,t_j) at

$$\begin{aligned} &(\mathscr{S}_{j,j} + \mathfrak{r}\tau^\alpha)\sum_{i=0}^{N}\upsilon_i^j\mathscr{V}_i^*(x_s) - \mathfrak{p}\tau^\alpha\sum_{i=2}^{N}\sum_{k=0}^{i-2}\sum_{l=0}^{k}\upsilon_i^jN_{i,k,l}^2x_s^{k-l} - \mathfrak{q}\tau^\alpha\sum_{i=1}^{N}\sum_{k=0}^{i-1}\sum_{l=0}^{k}\upsilon_i^jN_{i,k,l}^1x_s^{k-l} \\ &= \sum_{m=0}^{j-1}\sum_{i=0}^{N}\overline{\mathscr{S}}_{j,m}\upsilon_i^m\mathscr{V}_i^*(x_s) + \tau^\alpha F(x_s,t_j), \quad j = 1,2,\ldots,M, \quad s = 0,1,\ldots,N, \end{aligned} \tag{11}$$

where $\upsilon_i^j = \upsilon_i(t_j)$ are the unknown coefficients. The above relation with the following boundary conditions gives $N+1$ linear algebraic equations which one can be determined the unknown coefficients υ_i^i, $i = 0,1,2,\ldots,N$ in each step of time j. Notice that we substitute Eq. (7) in (2) to establish the boundary conditions as

$$\sum_{i=0}^{N}(-1)^i(2i+1)\upsilon_i^j = \psi_0(t_j), \quad \sum_{i=0}^{N}\upsilon_i^j = \psi_1(t_j), \quad j = 1,\ 2,\ldots,\ M. \tag{12}$$

Moreover, the initial solution υ_i^0 is obtained with combining relation $u(x,0) = \varphi(x)$ in Eq. (9).

3 Numerical results

In this section, the present form of price barrier choice regulated by a time fractional BSM model has been manipulated which is one of the financial market's most fascinating models. Moreover, the efficiency and accuracy of the developed method are shown for the numerical scheme of TFBSE. The computational order (denoted by $\mathscr{C}_{\mathscr{O}}$) is calculated by the following relation:

$$\mathscr{C}_{\mathscr{O}} = \log_2\left(\frac{E_{i+1}}{E_i}\right),$$

in which E_{i+1} and E_i are errors corresponding to the mesh sizes $2M$ and M, respectively. The calculated results support the theoretical analysis.

Example 1. Consider the following TFBSM with homogeneous boundary conditions

$$\begin{cases} {}_0\mathscr{D}_t^{\alpha} u(x,t) = p\frac{\partial^2 u(x,t)}{\partial x^2} + q\frac{\partial u(x,t)}{\partial x} - ru(x,t) + f(x,t), \\ u(x,0) = x^2(1-x), \quad 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \end{cases}$$

with $\sigma = 0.25,\ p = \frac{1}{2}\sigma^2,\ q = r - p,\ r = 0.05,\ \alpha = 0.7$ and the source term

$$f(x,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)}\right)x^2(1-x) - (t+1)^2[p(2-6x) + q(2x-3x^2) - rx^2(1-x)].$$

The exact solution of the problem is $u(x,t) = (t+1)^2x^2(1-x)$.

In Table 1, the obtained error and order are shown at $T = 1$ with $N = 5$. From this table, we can see that the computational order is $\mathscr{O}(\tau^{2-\alpha})$ and it is close with time order of convergence (TOC). Moreover, the error decreases with increase of M. Based on the detailed comparisons in Table 2, one can derive that the outcomes are in comparatively good agreement with [7] and [9]. In addition, highly accurate results are given with very low space size.

The absolute error and the estimated solution in Figure 1 are seen at $T = 1$. According to this figure, we conclude that the numerical technique has an error close.

Example 2. By considering nonhomogeneous boundary conditions for the following TFBSE, we have

$$\begin{cases} {}_0\mathscr{D}_t^{0.7} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{2}\frac{\partial u(x,t)}{\partial x} - \frac{1}{2}u(x,t) + f(x,t), \\ u(x,0) = x^3 + x^2 + 1, \quad 0 < x < 1, \\ u(0,t) = (t+1)^2, \quad u(1,t) = 3(t+1)^2, \end{cases}$$

in which the source term $f(x,t)$ is obtained from $u(x,t) = (t+1)^2(x^3 + x^2 + 1)$.

Comparisons of the approach in [7] and the mentioned method in [9] with the current method are shown in Table 3 to provide good outcomes for our process. Moreover, the computational order is shown in Table 4 at $T = 1$ with $N = 5$ that the convergence order supports the theoretical results. Figure 2 demonstrates the numerical solution and the absolute error in which the approximation solution is closed to the exact solution.

Table 1: The achieved errors and computational order with $N=5$ at $T=1$ for Example 1.

	$\alpha=0.2$				$\alpha=0.7$			
M	L_∞	$\mathscr{C}_{\mathscr{O}}$	L_2	$\mathscr{C}_{\mathscr{O}}$	L_∞	$\mathscr{C}_{\mathscr{O}}$	L_2	$\mathscr{C}_{\mathscr{O}}$
100	$5.5524E-6$		0.00001192		0.00019528		0.00041772	
200	$1.6558E-6$	1.74558	$3.5563E-6$	1.74547	0.00007960	1.29460	0.00017030	1.29450
400	$4.9081E-7$	1.75429	$1.0542E-6$	1.75420	0.00003240	1.29680	0.00006932	1.29674
800	$1.4477E-7$	1.76137	$3.1097E-6$	1.76130	0.00001318	1.29809	0.00002820	1.29805
1600	$4.2531E-8$	1.76721	$9.1361E-8$	1.76715	$5.3555E-6$	1.29886	0.00001146	1.29883
TOC		1.8		1.8		1.3		1.3

Table 2: The obtained errors and temporal convergence order at $T=1$ for Example 1.

	Method of [7] for $N=150$ and $\alpha=0.7$		Method of [9] for $N=150$ and $\alpha=0.7$		Present method for $N=5$ and $\alpha=0.7$		Present method for $N=5$ and $\alpha=0.2$	
M	L_∞	$\mathscr{C}_{\mathscr{O}}$	L_∞	$\mathscr{C}_{\mathscr{O}}$	L_∞	$\mathscr{C}_{\mathscr{O}}$	L_∞	$\mathscr{C}_{\mathscr{O}}$
10	$3.5000E-3$		$5.821E-3$		$3.72461E-3$		$2.88992E-4$	
20	$1.4400E-3$	1.3300	$2.304E-3$	1.3372	$1.54746E-3$	1.2672	$8.92113E-5$	1.6957
40	$5.9000E-4$	1.3150	$9.081E-4$	1.3421	$6.36723E-4$	1.2812	$2.71620E-5$	1.7156
80	$2.4000E-4$	1.3400	$3.572E-4$	1.3461	$2.60555E-4$	1.2891	$8.18432E-6$	1.7307
160	$9.5000E-5$	1.3600	$1.411E-4$	1.3400	$1.06288E-4$	1.2936	$2.44614E-6$	1.7424
320	$3.8000E-5$	1.3800	$5.387E-5$	1.3892	$4.32797E-5$	1.2962	$7.26393E-7$	1.7517
TOC		1.3		1.3		1.3		1.8

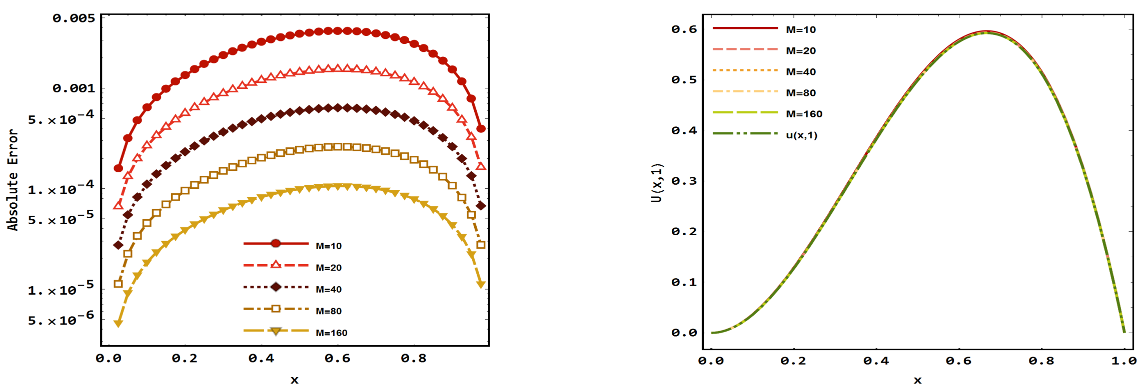

Figure 1: The absolute error (left-side) and approximate solution (right-side) for Example 1 at $T=1$.

4 Conclusion

The time fractional Black–Scholes model reflects a generalization of the classical BSM. The “globality” nature of the model’s fractional-order derivative leads to complicated precise and numerical solutions rather than the integer-order model. Accordingly, in the current study a numerical scheme was presented to solve TFBSE. The modified Riemann–Liouville fractional derivative was already replaced with the Caputo fractional derivative in the TFBSE by a variable transformation. First of all, the process of

Table 3: The obtained errors and temporal convergence order at $T = 1$ for Example 2.

	Method of [7] for $N = 150$ and $\alpha = 0.7$		Method of [9] for $N = 150$ and $\alpha = 0.7$		Present method for $N = 5$ and $\alpha = 0.7$		Present method for $N = 5$ and $\alpha = 0.2$	
M	L_∞	$\mathscr{C}_\mathscr{O}$	L_∞	$\mathscr{C}_\mathscr{O}$	L_∞	$\mathscr{C}_\mathscr{O}$	L_∞	$\mathscr{C}_\mathscr{O}$
10	$5.2000E-3$		$6.345E-3$		$5.46782E-3$		$4.40574E-4$	
20	$2.0700E-3$	1.3300	$2.507E-3$	1.3372	$2.23846E-3$	1.2885	$1.35249E-4$	1.7038
40	$8.3000E-4$	1.3150	$9.957E-4$	1.3421	$9.13453E-4$	1.2931	$4.10200E-5$	1.7212
80	$3.3000E-4$	1.3400	$4.011E-4$	1.3461	$3.72052E-4$	1.2958	$1.23246E-4$	1.7348
160	$1.3000E-4$	1.3600	$1.591E-4$	1.3400	$1.51366E-4$	1.2975	$3.67548E-6$	1.7455
320	$5.0000E-4$	1.3800	$6.274E-5$	1.3892	$6.15399E-5$	1.2985	$1.08954E-4$	1.7542
TOC		1.3		1.3		1.3		1.8

Table 4: The temporal order, L_∞ and L_2 with $N = 5$ at $T = 1$ for Example 2.

	$\alpha = 0.2$				$\alpha = 0.9$			
M	L_∞	$\mathscr{C}_\mathscr{O}$	L_2	$\mathscr{C}_\mathscr{O}$	L_∞	$\mathscr{C}_\mathscr{O}$	L_2	$\mathscr{C}_\mathscr{O}$
15	$2.21166E-4$		$5.11645E-4$		$7.73898E-3$		$1.78839E-2$	
30	$6.73900E-5$	1.7145	$1.55898E-4$	1.7145	$3.61517E-3$	1.0981	$8.35420E-3$	1.0981
60	$2.03213E-5$	1.7295	$4.70103E-5$	1.7296	$1.68767E-3$	1.0990	$3.89997E-3$	1.0990
120	$6.07784E-6$	1.7414	$1.40601E-5$	1.7414	$7.87598E-4$	1.0995	$1.82003E-3$	1.0995
240	$1.80591E-6$	1.7508	$4.17765E-6$	1.7508	$3.67494E-4$	1.0997	$8.49227E-4$	1.0997
TOC		1.8		1.8		1.1		1.1

discretization through linear interpolation (precision order of $\mathscr{O}(\tau^{2-\alpha})$) was described in a temporal sense. Then, by applying the Chebyshev collocation methodology based on the third form, we have demonstrated how to achieve the numerical solutions. In addition, by using the energy process, the unconditional stability of the time-discrete structure as well as the convergence order of the time-discrete that is $\mathscr{O}(\tau^{2-\alpha})$ have been proven. Two numerical examples with analytical solution were selected to compare the consistency and convergence order of the numerical process, in which the numerical result has shown the validity of the proposed system.

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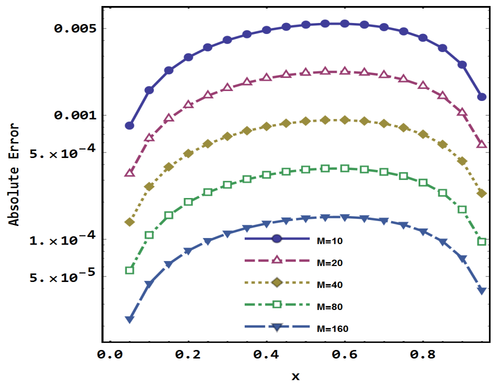

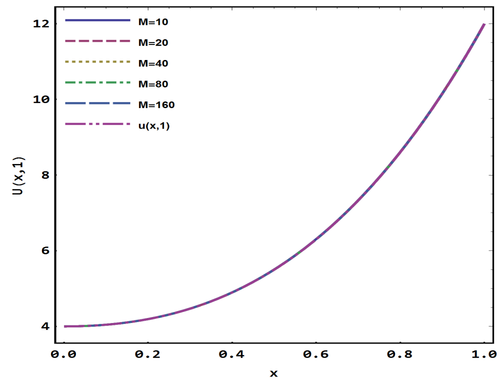

Figure 2: The absolute error (left-side) and approximate solution (right-side) at $T = 1$ for Example 2.

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