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Numerical solution of singularly perturbed differential difference equations with mixed parameters

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Abstract. In this paper, numerical solution of the singularly perturbed differential equations with mixed parameters are considered. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, two model problems are considered for numerical experimentation and solved for different values of the perturbation parameter, ε and mesh size, h. The numerical results are tabulated and it is observed that the present method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$, where the classical numerical methods fails to give good result.

Keywords: Singular perturbation, ε-uniformly convergent, large delay, small delay.
AMS Subject Classification 2010: 65L11, 65L12, 65L20.

1 Introduction

The solution of singular perturbation problems possesses boundary and/or interior layer(s). As a result, many of classical numerical methods have severe restrictions on the step-size to preserve the stability properties when the perturbation parameter(s) is(are) very small. To avoid the restrictions, there are two approaches to design method that gives small truncation errors inside these layers. The first approach is the class of fitted operator methods whereas the second one is the fitted mesh methods. However, both approaches have their own merits and demerits. Fitted operator methods work well on a uniform mesh but are difficult to extend for higher dimensional problems except the fitted operator methods of non-standard type that are under consideration in this work. Fitted mesh methods are easy to extend for higher dimensional and nonlinear problems but require the priori knowledge of the location and width of the layer(s).

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Singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small perturbation parameter and consist of at least one term involving delay argument. Such types of differential equations play very important role in the mathematical models of science and engineering such as the human pupil light reflex with mixed delay type [11], variational problems in control theory with small state problem [9], models of HIV infection [4], and signal transition [6].

Finding the solution of singularly perturbed delay differential equations, whose applications were mentioned above is a challenging task. In response to these, in recent years, there has been a growing interest in numerical treatment of singularly perturbed delay differential equations. The authors of [7,16,17] have developed various numerical schemes on uniform meshes for singularly perturbed second order differential equations having small delay on the convection term. The authors of [3,5,8,10,18] have presented second order differential equations with large delay.

In this work, motivated by the works of [8,10,18], we develop a non-standard finite difference scheme on uniform mesh for the numerical solution of the problem under consideration. The simplicity in the construction and possibility of their extensions allowed researchers to apply NSFD (nonstandard finite difference) schemes to solve several complex differential equations that arise in the interface of engineering and natural sciences. Historically, for the first time, Mickens set following five rules [13] for the construction of discrete models that have the capability to replicate the properties of the exact solution.
Rule 1: The orders of the discrete derivatives must be exactly equal to the orders of the corresponding derivatives of the differential equations.
Rule 2: Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step sizes than those conventionally used.
Rule 3: Nonlinear terms should be approximated in a nonlocal way.
Rule 4: Special solutions of differential equations should also be special discrete solutions of the finite difference models.
Rule 5: The finite-difference equations should not have solutions that do not correspond exactly to solutions of the differential equations.

Following to this, Anguelov and Lubuma [1] reworded these rules and presented them as a definition for the NSFD schemes. This definition was further generalized in Lubuma and Patidar [12]. Attracted by a large readership of his earlier books on the topic published in 1994 and 2000 [13,14], Mickens came up with another book [15] that he edited and was published in 2005. Many relevant topics ranging from natural sciences to biomedical and engineering domains were covered in this book.

As far as the researchers' knowledge is concerned numerical solution of singularly perturbed boundary value problems containing both large and small delay is new and it is not well developed so far.

Throughout our analysis, C is a generic positive constant that is independent of the parameter ε and the number of mesh points is $2N$. We assume that $\bar{\Omega} = [0, 2]$, $\Omega = (0, 2)$, $\Omega_1 = (0, 1)$, $\Omega_2 = (1, 2)$, $\Omega* = \Omega_1 \cup \Omega_2$, $\overline{\Omega}^{2N}$ is denoted by $\{0, 1, 2, \ldots, 2N\}$, Ω_1^{2N} is denoted by $\{1, 2, \ldots, N\}$ and Ω_2^{2N} is denoted by $\{N+1, N+2, \ldots, 2N-1\}$. K_1 and K_2 are the linear operators associated to the domain Ω_1 and Ω_2, respectively.

2 Definition of the problem

Consider the following singularly perturbed problem

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) + d(x)y'(x-\delta) = f(x), \quad x \in \Omega, \quad (1)$$
$$y(x) = \phi(x), \quad x \in [-1,0], \quad y(2) = l, \quad l \in \mathbb{R}. \quad (2)$$

where δ is small, that is $\delta = O(\varepsilon)$, $0 < \varepsilon \ll 1$, $\phi(x)$ is sufficiently smooth on $[-1,0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$, $c(x)$, $d(x)$ and $f(x)$ satisfy

$$a(x) \geq a_1 \geq a > 0, \quad b(x) > b \geq 0, \quad c_1 \leq c(x) \leq c < 0, \quad d(x) \geq d \geq 0,$$
$$2(a+d) + 5b + 5c_1 \geq \eta > 0, \quad a(a_1 - a) + 2c_1 > 0.$$

The above assumptions ensure that $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$. The BVP (1)-(2) exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$. By expanding $y'(x-\delta)$ around x using the Taylor's expansion and discarding higher order terms, the above problem can be approximated by

$$Ky(x) = -c_{\varepsilon,\delta}(x)y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad (3)$$

where $c_{\varepsilon,\delta}(x) = \varepsilon + \delta d(x)$ and $p(x) = a(x) + d(x)$ with

$$y(x) = \phi(x), \quad x \in [-1,0], \quad y(2) = l. \quad (4)$$

For small δ, Eqs. (1) and (3) are asymptotically equivalent, because the difference between the two equations is of order $O(\delta^2)$. Now we assume again $0 < c_{\varepsilon,\delta}(x) = \varepsilon + \delta d(x) = c_\varepsilon$ and $p(x) = a(x) + d(x) \geq p > 0$.

As we observed from Eqs. (3) and (4), the values of $y(x-1)$ are known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it impossible to treat the problem by the same approach throughout the domain $\bar{\Omega}$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately.

Eqs. (3)-(4) are equivalent to

$$Ky(x) = R(x), \quad (5)$$

where

$$Ky(x) = \begin{cases} K_1y(x) = -c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x), & x \in \Omega_1, \\ K_2y(x) = -c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1), & x \in \Omega_2. \end{cases} \quad (6)$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega_1, \\ f(x), & x \in \Omega_2, \end{cases} \quad (7)$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), & x \in [-1,0], \\ y(1^-) = y(1^+), & y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \quad (8)$$

where $y(1^-)$ and $y(1^+)$ denote the left and right limits of y at $x = 1$, respectively.

3 Properties of continuous solution

Lemma 1. (Maximum Principle) *Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0$, $\psi(2) \geq 0$, $K_1\psi(x) \geq 0$, $\forall x \in \Omega_1$, $K_2\psi(x) \geq 0$, $\forall x \in \Omega_2$ and $\psi'(1^+) - \psi'(1^-) = [\psi'](1) \leq 0$. Then $\psi(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. Define the test function

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, & x \in [1,2]. \end{cases} \tag{9}$$

Note that $s(x) > 0$, $\forall x \in \bar{\Omega}$, $K^N s(x) > 0$, $\forall x \in \Omega_1 \cup \Omega_2$, $s(0) > 0$, $s(2) > 0$ and $[s'](1) < 0$. Let $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\}$. Then there exists $x_0 \in \bar{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0$, $\forall x \in \bar{\Omega}$. Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_0$. Suppose the theorem does not hold true, then $\mu > 0$.
Case (i): $x_0 = 0$,

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0,$$

which is a contradiction.
Case (ii): $x_0 \in \Omega_1$,

$$0 < K(\psi + \mu s)(x_0) = -c_\varepsilon(\psi + \mu s)''(x_0) + p(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \leq 0,$$

which is a contradiction.
Case (iii): $x_0 = 1$,

$$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0,$$

it is a contradiction.
Case (iv): $x_0 \in \Omega_2$,

$$\begin{aligned} 0 &< K(\psi + \mu s)(x_0) \\ &= -c_\varepsilon(\psi + \mu s)''(x_0) + p(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \\ &\quad + c(x_0)(\psi + \mu s)(x_0 - 1) \leq 0, \end{aligned}$$

Case (iv): $x_0 = 2$,

$$0 < (\psi + \mu s)(2) = (\psi + \mu s)(2) \leq 0,$$

which is a contradiction. Hence, the proof of the theorem is completed. □

Lemma 2. (Stability Result) *The solution y(x) of the problem (3)-(4), satisfies the bound*

$$|y(x)| \leq C \max\{|y(0)|, |y(2)|, \sup_{x \in \Omega*} |Ky(x)|\}, \quad x \in \overline{\Omega}.$$

Proof. This theorem can be proved by using Lemma 1 and the barrier functions $\theta^{\pm}(x) = CMs(x) \pm y(x)$, $x \in \overline{\Omega}$, where $M = \max\{|y(0)|, |Ky(2)|, \sup_{x \in \Omega*} |Ky(x)|\}$ and $s(x)$ is the test function as in Lemma 1. □

The uniqueness of the solution is implied by this minimum principle (Lemma 1). Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of Eqs. (3)–(4) is bounded. The following lemma shows the bound for the derivatives of the solution.

Lemma 3. *Let $y(x)$ be the solution of Eqs. (3)-(4). Then, we have the following bounds*

$$|y^{(k)}(x)|_{\Omega^*} \leq C\varepsilon^{-k}, \quad for \quad k = 1, 2, 3.$$

Proof. To bound $y'(x)$ on the interval Ω_1, we consider

$$K_1 y(x) = -c_\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x).$$

By integrating the above equation on both sides, we get

$$\begin{aligned} -c_\varepsilon\big(y'(x) - y'(0)\big) \;=\; & -[a(x)y(x) - a(0)y(0)] + \int_0^x a'(t)y(t)dt - \int_0^x b(t)y(t)dt \\ & + \int_0^x [f(t) - c(t)\phi(t-1)]dt. \end{aligned}$$

Therefore,

$$\begin{aligned} c_\varepsilon y'(0) \;=\; & c_\varepsilon y'(x) - [a(x)y(x) - a(0)y(0)] + \int_0^x a'(t)y(t)dt - \int_0^x b(t)y(t)dt \\ & + \int_0^x [f(t) - c(t)\phi(t-1)]dt. \end{aligned}$$

Then by the mean value theorem, there exits $z \in (0, c_\varepsilon)$ such that

$$|c_\varepsilon y'(z)| \leq C(|y(x)|, |f(x)|, |\phi(x)|_{[-1,0]}) \quad \text{and} \quad |c_\varepsilon y'(0)| \leq C(|y(x)| + |f(x)| + |\phi(x)|).$$

Hence,

$$|c_\varepsilon y'(x)| \leq C \max(|y(x)|, |f(x)|, |\phi(x)|).$$

By a similar argument we can bound $y'(x)$ on Ω_2, as $|c_\varepsilon y'(x)| \leq C$. From Eqs. (6) and (7), we have

$$|y^{(k)}(x)|_{\Omega^*} \leq C c_\varepsilon^{-k}, \quad k = 2, 3, 4.$$

Hence, the proof is completed. □

Lemma 4. *The bound for derivative of the solution $y(x)$ of the problem (3)-(4) when $x \in \Omega_1 = (0, 1)$ is given by*

$$|y^{(k)}(x)| \leq C\left(1 + c_\varepsilon^{-k} \exp\left(\frac{-p(1 - x_j)}{c_\varepsilon}\right)\right), \quad for \quad k = 0 \leq k \leq 4, \quad j = 1, 2, 3, \ldots, N-1.$$

Proof. For the proof, we refer to [2]. □

4 Formulation of the method

The theoretical basis of non-standard discrete numerical method is based on the development of exact finite difference method. Mickens [15] presented techniques and rules for developing non-standard finite difference methods for different problem types. In Mickens' rules, to develop a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedures. These complicated functions constitute a general property of the schemes, which is useful while designing reliable schemes for such problems.

For the problem of the form (3)–(4), in order to construct exact finite difference scheme we follow the procedures used in [2]. Let us consider the following singularly perturbed differential equation of the form

$$-c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) = f(x). \tag{10}$$

The constant coefficient sub-equations can be given as

$$-c_\varepsilon y''(x) + py'(x) + by(x) = 0, \tag{11}$$

$$-c_\varepsilon y''(x) + py'(x) = 0, \tag{12}$$

where $p(x) \geq p$ and $b(x) \geq b$. Thus, Eq. (11) has two independent solutions namely $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ with

$$\lambda_{1,2} = \frac{-p \pm \sqrt{p^2 - 4c_\varepsilon b}}{2c_\varepsilon}. \tag{13}$$

We discretize the domain $[0,1]$ using uniform mesh length $\Delta x = h$ such that, $\Omega_1^N = \{x_i = x_0 + ih, 1, 2, \ldots, N, x_0 = 0, x_N = 1, h = \frac{1}{N}\}$, where N is the number of mesh points. We denote the approximate solution of $y(x)$ at x's by Y_i. Now our objective is to calculate a difference equation which has the same general solution as Eq. (11) at the grid point x_i given by

$$Y_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i).$$

Using the procedures used in [2], we have

$$\det \begin{bmatrix} Y_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ Y_i & \exp(\lambda_1 x_i)) & \exp(\lambda_2 x_i) \\ Y_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{bmatrix} = 0. \tag{14}$$

By simplifying Eq. (14), we obtain that

$$-\exp(\frac{ph}{2c_\varepsilon})Y_{i-1} + 2\cosh(\frac{h\sqrt{p^2 + 4c_{\varepsilon,\delta}(x)b}}{2c_\varepsilon})Y_i - \exp(\frac{-ph}{2c_\varepsilon})Y_{i+1} = 0, \tag{15}$$

is an exact difference scheme for Eq. (11). After doing the arithmetic manipulation and rearrangement on Eq. (15) we obtain

$$c_\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\Psi^2} + p\frac{Y_{i+1} - Y_i}{h} = 0, \tag{16}$$

where

$$\Psi^2 = \frac{hc_\varepsilon}{p}\left(\exp\left(\frac{hp}{c_\varepsilon}\right) - 1\right).$$

Adopting this function for the variable coefficient problem, we write it as

$$\Psi_i^2 = \frac{hc_\varepsilon}{p_i}\left(\exp\left(\frac{hp_i}{c_\varepsilon}\right) - 1\right), \tag{17}$$

where Ψ_i^2 is the function of c_ε, p_i.

Assume that $\bar{\Omega}^{2N}$ denote partition of [0,2] in to $2N$ subintervals such that

$$0 = x_0 < x_1 < \ldots < x_N = 1, \quad \text{and} \quad 1 < x_{N+1} < x_{N+2} < \ldots < x_{2N} = 2$$

with $x_i = ih$, $h = \frac{2}{2N} = \frac{1}{N}$, $i = 0, 1, 2, \ldots, 2N$.

Case 1: Consider Eq. (5) on the domain $\Omega_1 = (0, 1)$ which is given by

$$\begin{cases} -c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x-1), \\ y_0 = y(0) = \phi(0), y(1) = \theta. \end{cases} \tag{18}$$

Undertaking the notation $y_i = y(x_i)$ and using the nonstandard finite difference methodology of [15], for right layer in the domain Ω_1 the scheme to solve Eq. (18) is given by

$$-c_\varepsilon\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{\psi_i^2}\right) + p_i\left(\frac{y_i - y_{i-1}}{h}\right) + b_i y_i + \tau_1 = f_i - c_i\phi(x_i - 1), \tag{19}$$

where

$$\Psi_i^2 = \frac{hc_\varepsilon}{p_i}\left(\exp\left(\frac{hp_i}{c_\varepsilon}\right) - 1\right) = h^2 + O\left(\frac{h^4}{\varepsilon}\right),$$

with the local truncation term $\tau_1 = h\frac{p_i}{2}y_i'' + O(h^2) = O(h)$. Eq. (19) can be written as three term recurrence relation as

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{20}$$

where $E_i = \frac{-c_\varepsilon}{\psi_i^2} - \frac{p_i}{h}$, $F_i = \frac{2c_\varepsilon}{\psi_i^2} + \frac{p_i}{h} + b_i$, $G_i = \frac{-c_\varepsilon}{\psi_i^2}$ and $H_i = f_i - c_i\phi(x_i - 1)$.

Case 2: Consider Eq. (5) on the domain $\Omega_2 = (1, 2)$, for right layer in the domain Ω_2 using the nonstandard finite difference method which is given by

$$-c_\varepsilon\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{\psi_i^2}\right) + p_i\left(\frac{y_i - y_{i-1}}{h}\right) + b_i y_i + \tau_1 = f_i - c_i\phi(x_i - 1).$$

Similarly, this equation can be written as

$$c_i y_j + E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = N+1, N+2, \ldots, 2N-1, \tag{21}$$

where $y_j = y(x_i - 1), j = 1, 2, \ldots, N$, $E_i = \frac{-c_\varepsilon}{\psi_i^2} - \frac{p_i}{h}$, $F_i = \frac{2c_\varepsilon}{\psi_i^2} + \frac{p_i}{h} + b_i$, $G_i = \frac{-c_\varepsilon}{\psi_i^2}$ and $H_i = f_i - c_i\phi(x_i - 1)$. Therefore, on the whole domain $\overline{\Omega} = [0, 2]$, the basic schemes to solve Eqs. (1)-(2) are the schemes given in Eq. (20) and Eq. (21) together with the local truncation error of τ_1 .

5 Uniform convergence analysis

The discrete scheme corresponding to the original Eqs. (3)-(4) is as follows:

$$\begin{aligned} K_1^N Y_i &= f_i - c_i\phi_{i-N}, \quad & i = 1, 2, 3, \dots, N-1, \\ K_2^N Y_i &= f_i, & i = N+1, N+2, \dots, 2N-1 \end{aligned}$$

subject to the boundary conditions

$$\begin{aligned} Y_i &= \phi_i, \quad i = -N, -N+1, \dots, 0, \\ Y_{2N} &= l, \end{aligned}$$

where

$$\begin{cases} K_1^N y_i = -c_\varepsilon \delta^2 Y_i + p(x_i) D^- Y_i + b(x_i) Y_i, \\ K_2^N y_i = -c_\varepsilon \delta^2 Y_i + p(x_i) D^- Y_i + b(x_i) Y_i + c(x_i) Y_{i-N}. \end{cases}$$

Let us define the forward, backward and second order finite difference operators as

$$\begin{cases} D^+ Y_i = \frac{Y_{i+1} - Y_i}{h}, \\ D^- Y_i = \frac{Y_i - Y_{i-1}}{h}, \\ \delta^2 Y_i = D^+ D^- Y_i = \frac{D^+ Y_i - D^- Y_i}{h}. \end{cases}$$

Theorem 1. *Let the coefficients functions $a(x)$,$b(x)$ and the source function $R(x)$ in Eqs. (6)-(7) of the domain Ω_1 be sufficiently smooth, so that $y(x) \in C^4[0,1]$. Then, the discrete solution Y_i satisfies*

$$|K^N((y(x_i) - Y_i)| \le Ch \left(1 + \sup_{x_i \in (0,1)} \left(\frac{\exp(\frac{-p(l-x_i)}{\varepsilon})}{c_\varepsilon^3} \right) \right).$$

Proof. We consider the truncation error discretization as

$$\begin{aligned} |K^N((y(x_i) - Y_i)| =& |K^N y_i - K^N Y_i| \\ \le& C c_\varepsilon |y_i'' - D^+ D^- Y_i| + C c_\varepsilon |(\frac{h^2}{\Psi_i^2} - 1) D^+ D^- Y_i| + Ch|y_i''| \\ \le& C c_\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''| + Ch|y_i''| \\ \le& C c_\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|. \end{aligned}$$

The bound $c_\varepsilon|\frac{h^2}{\Psi^2} - 1| \le Ch$ used in above expression is based on the behavior of the denominator function Ψ^2 in non-standard finite difference. To illustrate the bound given there, let us define $\sigma =: \dfrac{a_i h}{c_\varepsilon}, \sigma \in (0, \infty)$. Then,

$$c_\varepsilon |\frac{h^2}{\Psi^2 - 1}| = a_i h |\frac{1}{\exp(\sigma) - 1} - \frac{1}{\sigma}| =: a_i h Q(\sigma).$$

By simplifying and writing explicitly we obtain

$$Q(\sigma) = \frac{\exp(\sigma) - \sigma - 1}{\sigma(\exp(\sigma) - 1)},$$

and we obtain the limit is bounded as

$$\lim_{\sigma\longrightarrow 0} Q(\sigma)=\frac{1}{2}, \quad \lim_{\sigma\longrightarrow\infty} Q(\sigma)=0.$$

Hence, for all $\sigma\in(0,\infty)$ we have $Q(\sigma)\leqslant C$. So, the error estimation in the discretization is bounded as

$$|K^N((y(x_i)-Y_i)|\leqslant Cc_\varepsilon h^2|y_i^{(4)}|+Ch|y_i''|. \tag{22}$$

From Eq. (22) and boundedness of derivatives of solution in Lemma 4 , we obtain

$$\begin{aligned}|K^N(y(x_i)-Y_i)|&\leq Cc_\varepsilon h^2\left|\left(1+c_\varepsilon^{-4}\exp\left(\frac{-p(1-x_i)}{c_\varepsilon}\right)\right)\right|\\&+Ch\left|\left(1+c_\varepsilon^{-2}\exp\left(\frac{-p(1-x_i)}{c_\varepsilon}\right)\right)\right|\\&\leq Ch^2\left|\left(1+c_\varepsilon^{-3}\exp\left(\frac{-p(1-x_i)}{c_\varepsilon}\right)\right)\right|\\&+Ch\left|\left(1+c_\varepsilon^{-2}\exp\left(\frac{-p(1-x_i)}{c_\varepsilon}\right)\right)\right|\\&\leq Ch\left(1+\sup_{x_i\in(0,1)}\left(\frac{\exp(\frac{-p(1-x_i)}{c_\varepsilon})}{c_\varepsilon^3}\right)\right),\end{aligned}$$

since $(c_\varepsilon)^{-4}\geq(c_\varepsilon)^{-3}$. □

By similar argument for Ω_2, we get

$$|K^N((y(x_i)-Y_i)|\leq Ch\left(1+\sup_{x_i\in(0,1)}\left(\frac{\exp(\frac{-p(l-x_i)}{c_\varepsilon})}{c_\varepsilon^3}\right)\right).$$

Most of the time during analysis, one encounters with exponential terms involving divided by the power function in c_ε which are always the main cause of worry. For their careful consideration while proving the ε-uniform convergence, we prove for the right layer case as follows.

Lemma 5. *For a fixed mesh and for $c_\varepsilon\to 0$, it holds*

$$\lim_{c_\varepsilon\to 0}\max_{1\leq i\leq N-1}\left(\frac{\exp(\frac{-p(l-x_i)}{c_\varepsilon})}{c_\varepsilon^m}\right)=0, \quad m=1,2,3,\ldots, \tag{23}$$

where $x_i=ih, h=\frac{1}{N}, i=1,2,\ldots,N-1$.

Proof. Consider the partition $[0,1]:=\{0=x_0<x_1<\ldots.<x_{N-1}<x_N=1\}$ for the interior grid points. We have

$$\max_{1\leq i\leq N-1}\frac{\exp\left(\frac{-px_i}{c_\varepsilon}\right)}{c_\varepsilon^m}\leq\frac{\exp\left(\frac{-px_1}{c_\varepsilon}\right)}{c_\varepsilon^m}=\frac{\exp\left(\frac{-ph}{c_\varepsilon}\right)}{c_\varepsilon^m},$$

and

$$\max_{1\leq i\leq N-1}\frac{\exp\left(\dfrac{-p(1-x_i)}{c_\varepsilon}\right)}{c_\varepsilon^m}\leq\frac{\exp\left(\dfrac{-p(1-x_{N-1})}{c_\varepsilon}\right)}{c_\varepsilon^m}=\frac{\exp(\dfrac{(-\alpha h}{c_\varepsilon})}{c_\varepsilon^m},$$

as $x_1 = 1 - x_{N-1} = h$. The repeated application of L'Hospital's rule gives

$$\lim_{c_\varepsilon\longrightarrow 0}\frac{\exp\left(\dfrac{-ch}{c_\varepsilon}\right)}{c_\varepsilon^m}=\lim_{\sigma=\frac{1}{c_\varepsilon}\longrightarrow\infty}\frac{\sigma^m}{\exp(ch\sigma)}=\lim_{\sigma=\frac{1}{c_\varepsilon}\longrightarrow\infty}\frac{m!}{(ch)^m\exp(ch\sigma)}=0.$$

This complete the proof. □

Theorem 2. *Under the hypothesis of boundness of discrete solution (i.e., it satisfies the discrete minimum principle), Lemma 5 and Theorem 1, the discrete solution satisfies the following bound:*

$$\sup_{0\leq\varepsilon\leq 1}\max_i|y(x_i)-Y_i|\leq CN^{-1}. \tag{24}$$

Proof. Results from boundedness of the solution, Lemma 5 and Theorem 1 give the required estimations. □

Consistence of the scheme can be described as follows. Local truncation errors refer to the differences between the original differential equation and its finite difference approximation at a mesh points. Finite difference scheme is called consistent if the limit of truncation error $(T_i(h))$ is equal to zero as the mesh size h goes to zero. Hence, the proposed method in Eq.(20) with local truncation error in Eq. (24) satisfies the definition of consistency as

$$\lim_{h\to 0}T_i(h)=\lim_{h\to 0}Ch=0. \tag{25}$$

Thus, the proposed scheme is consistent.

6 Numerical examples and results

In this section, we consider the following two examples to illustrate the numerical method discussed above:

Example 1. Consider the singularly perturbed boundary value problem

$$-\varepsilon y''(x)+10y'(x)-y(x-1)+y'(x-\varepsilon)=x,\quad x\in(0,1)\cup(1,2),$$

subject to the boundary conditions

$$y(x)=1,\quad x\in[-1,0],\quad y(2)=2.$$

Example 2. Consider the singularly perturbed boundary value problem

$$-\varepsilon y''(x)+(x+10)y'(x)-y(x-1)=x,\quad x\in(0,1)\cup(1,2),$$

subject to the boundary conditions

$$y(x)=x,\ \ x\in[-1,0],\quad y(2)=2.$$

The considered problems contain large delay parameter on the reaction term and small delay parameter on the convection term. The solutions of the problems exhibit interior layer due to the delay parameter and strong right boundary layer due to the small perturbation parameter ε (see Fig. 1). Fig. 2 shows, as the number of mesh points increases (as the mesh size decreases), the absolute error deceases which shows the convergence of the scheme and Fig. 3 and Table 1 show, the ε-uniform convergence of our scheme for $h \geq \varepsilon$ where the classical numerical method fails. The exact solutions of the test problems are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$E_\varepsilon^N = \max_{0 \leq i \leq 2N} |Y_i^N - Y_{2i}^{2N}| \tag{26}$$

where Y_i^N and Y_{2i}^{2N} are the i^{th} and $2i^{th}$ components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_{\varepsilon} E_\varepsilon^N, \quad \text{and} \quad R^N = \log_2\left(\frac{E^N}{E^{2N}}\right) \tag{27}$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{10^{-4}, 10^{-8}, \ldots, 10^{-20}\}$.

Table 1: Maximum absolute errors, rate of convergence and CPU (in seconds) for Example 1 for different values of N.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-4}	3.3848e-04	1.7340e-04	8.7738e-05	4.4129e-05	2.2130e-05
10^{-8}	3.3847e-04	1.7340e-04	8.7738e-05	4.4129e-05	2.2130e-05
10^{-12}	3.3847e-04	1.7340e-04	8.7738e-05	4.4129e-05	2.2130e-05
10^{-16}	3.3847e-04	1.7340e-04	8.7738e-05	4.4129e-05	2.2130e-05
10^{-20}	3.3847e-04	1.7340e-04	8.7738e-05	4.4129e-05	2.2130e-05
E^N	3.3847e-04	1.7340e-04	8.7738e-05	4.4129e-05	2.2130e-05
R^N	0.9828	0.9915	0.9957	0.9979	
CPU	0.3848	2.884361	23.718605	120.343685	994.0064301

7 Discussion and conclusion

This study has introduced non-standard fitted operator finite difference numerical method for solving singularly perturbed differential equations having both large and small delay. The behavior of the continuous solution of the problem has been studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme has been developed on uniform mesh using non-standard finite difference method in the given differential equation. The stability of the developed numerical method has been established and its uniform convergence has been proved. To validate the applicability of the method,

Table 2: Maximum absolute errors, rate of convergence and CPU(in second) for Example 2 for different values of N.

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
10^{-4}	7.4131e-04	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
10^{-8}	7.4131e-04	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
10^{-12}	7.4131e-04	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
10^{-16}	7.4131e-04	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
10^{-20}	7.4131e-04	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
E^N	7.4131e-04	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
R^N	0.9612	0.9808	0.9904	0.9949	
CPU	0.321228	2.893108	23.049239	117.15339	898.006433

Table 3: Comparision of maximum absolute errors and rate of convergence for Example 2 for different values of N.

ε	$N=32$	$N=64$	$N=128$	$N=256$
Present method	3.8075e-04	1.9293e-04	9.7108e-05	4.8727e-05
R^N	0.9808	0.9904	0.9949	
Method in [3]	2.7660e-03	1.4020e-03	7.0560e-04	3.5400e-04
R^N	0.9806	0.9904	0.9952	

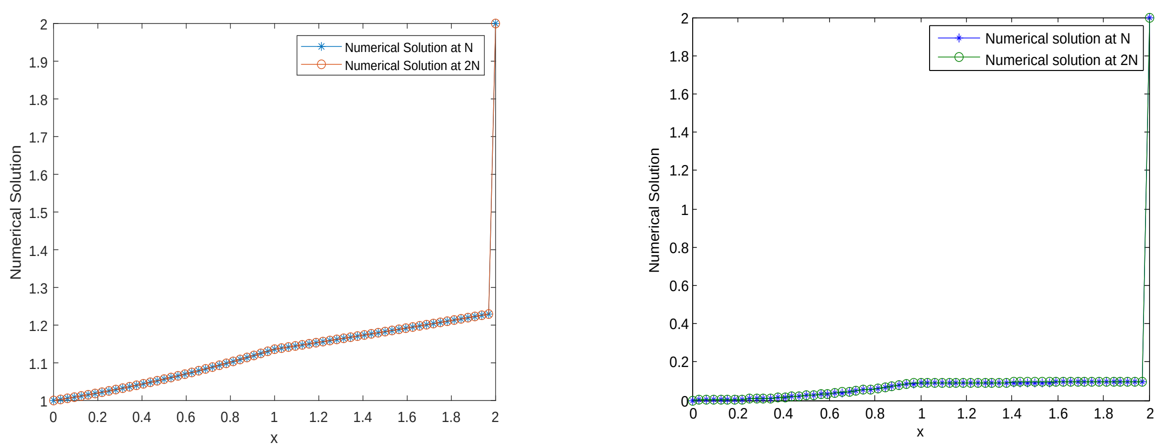

Figure 1: The behavior of the numerical solution for Example 1 and Example 2 at $\varepsilon = 10^{-12}$ and $N = 32$, respectively.

two model problems have been considered for the numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results have been tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors and CPU

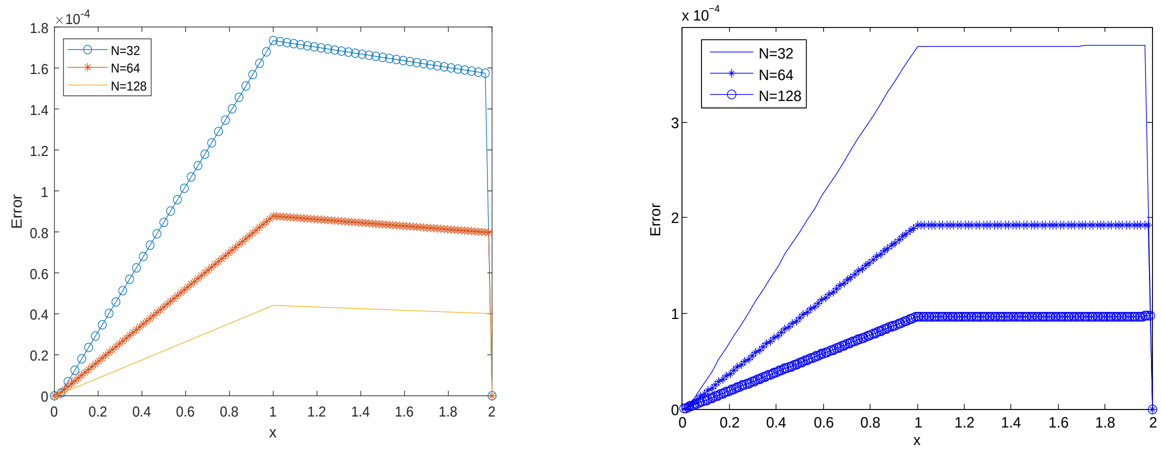

Figure 2: Point-wise absolute error of Example 1 and Example 2 at $\varepsilon = 10^{-12}$ for different values N, respectively.

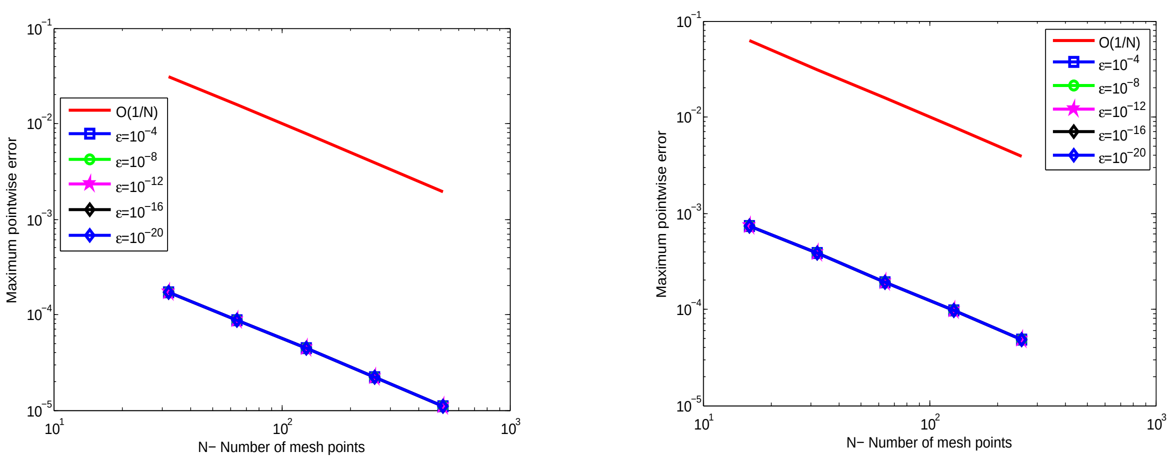

Figure 3: ε-uniform convergence with NSFDM in log-log scale for Example 1 and Example 2, respectively.

(in seconds) (see Table 1–Table 2) and compared with the results of the previously developed numerical methods existing in the literature (see Table 3). Further, behavior of the numerical solution (Fig 1), point-wise absolute errors (Fig 2) and the ε-uniform convergence of the method have been shown by the log-log plot (Fig 3). The method is shown to be ε-uniformly convergent with order of convergence $O(h)$. The proposed method gives more accurate, stable and ε-uniform numerical result.

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