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Mathematical modeling of the effect of an insulating stiffener on a nonlinear thermo-elastic plate

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Abstract. This paper deals with the mathematical modeling of the behavior of a reinforced rectangular thermo-elastic plate with a thin insulating stiffener. We use a variational asymptotic analysis, with respect to the thickness of the inserted body, in order to identify limit models that reflect its effect on the plate. We carry out a mathematical modeling for a stiffener of high rigidity and a stiffener of moderate rigidity.

Keywords: Asymptotic analysis, rectangular plate, thin insulating stiffener, nonlinear partial differential equations, approximate boundary conditions.
AMS Subject Classification 2010: 35A15, 35B40, 35G50, 35Q74, 74K20, 74K35.

1 Introduction

Mathematical modeling is a powerful tool, widely used in science and engineering. It consists in developing mathematical descriptions of various real-world phenomena and enable investigating, understanding analyzing and predicting the behavior of systems in a large variety of areas. In this context, we focus in this paper on the mathematical modeling of physical phenomena arising from elasticity and mechanical structures fields. More precisely, we deal with the asymptotic modeling of the behavior of a Thermo-Elastic rectangular plate reinforced by a thin insulating stiffener on a part of its boundary. Such structures are widely used in many branches of mechanics, civil and structural engineering such as bridges or storage tanks, and a growing attention was paid to the modeling of their behavior in the last decades. Indeed, the addition of stiffeners to structures aims to increase their strength and prevent from damages or buckling. Besides, the proprieties of the constitutive material of the stiffener is of a great importance and has a significant impact on the behavior of the reinforced structure. In several situations, especially

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in the design of insulating structures, thermally non-conductive materials are usually chosen in order to conserve the thermal energy by reducing heat loss or heat gain.

In this direction, the motivation of this study is twofold. It aims to:

- Provide simplified mathematical models, more suitable for numerical computations. Indeed, the discretization (by FEM method) inside the domain occupied by the thin stiffener needs very thin meshes and may increase the computational cost. That's why, an appropriate modeling of the behavior of such structures becomes necessary.
- Predict the behavior of elastic multi-structures according to the physical characteristics of their constitutive materials. We will be led to make the assumption that the physical coefficients of the stiffener vary as functions of its thickness, and identify the limit model describing the behavior of the structure, as this thickness approaches zero. The nature of the dependence of these coefficients on the thickness influences on the effects of the stiffener obtained at the limit. This analysis can also be seen as a way to optimize the choice of the thickness of the stiffener so as to have the desired effect on the behavior of the structure. Indeed, in several situations dealing with the design of the reinforcement of structures, the engineers would like to have just a mechanical effect inducing an additional resistance and strength to the structure, without obtaining a thermal effect which could generate possible damages related to the propagation of heat.

The idea of asymptotic modeling of the effect of thin layers has been investigated in several papers in electromagnetic (see [2–4,6]) or in structural mechanics (see [1,7,9–15]). The strategy consists in making use of the asymptotic methods (by considering the thickness of the layer as a small parameter) and modeling the behavior of the solution by eliminating the layer "geometrically" and reproducing its effect by new conditions on the region on which it is joined. More precisely, we seek an asymptotic model by reducing the layer to a boundary and approximating its effect by new conditions on this boundary. In [13], one of the authors of the present paper considered the thermo-elastic Von Karman model for a reinforced plate with a thin stiffener of high thermal conductivity and modelled its effect in the case where the thermal conductivity and the rigidity vary as δ^{-1} (δ being the thickness of the stiffener). In the present work, we investigate the case of an insulating stiffener, where the thermal conductivity vary as δ and the coefficient of thermal expansion as δ^2. We first consider the case where the stiffener is very rigid (with Young's modulus varying as δ^{-1}), which leads to an approximate model into which the mechanical effect of the stiffener appears. By contrast, the thermal effect of the stiffener is not considered. After, we deal with a stiffener of less rigidity, by letting the young's modulus varying as δ^{-a}, where a is any real number satisfying $0 < a < 1$. In this situation, neither the thermal nor the mechanical effects are incorporated in the limit model. Here, the layer even stiff, its rigidity is not sufficiently enough to induce mechanical effects on the plate and since it is insulating, the thermal effect is also neglected. The structure behaves as if the stiffener was not there.

It should be pointed out that the main difficulty in carrying out the mathematical modeling described above is related to the nonlinear aspect of the boundary value problem considered. The other major novelty in this paper consists in considering thermal coefficients varying as positive power of δ (for the thermal coefficients) and also, for the second part, the rigidity varying as

δ^{-a}, where $0 < a < 1$. This assumptions induces a lack of boundedness of the components of the solution in the appropriate spaces. All this, required then the introduction of auxiliary statements and other arguments in order to carry out the asymptotic analysis of the related problem and identify the limit models.

The paper is organized as follows: The second section is devoted to the description of the boundary value problem and its variational formulation. In the third section, following the ideas of Ciarlet (see [5]), we first transform the problem into a problem posed over a domain that doesn't depend on the small parameter δ. Thus, we establish a priori estimates that permit to pass through the limit in the variational problem. In the fourth section we use an asymptotic variational approach to identify limit models for the reinforced rectangular plate. We model both the effect of an insulating stiffener of high rigidity and moderate rigidity.

2 The Nonlinear model

2.1 Statement of the problem

We consider a bi-dimensional rectangular plate occupying the set $\bar{\Omega}_+ = [0,1]\times[0,1]$ of boundary $\partial\Omega_+ = \bar{\Sigma}\cup\bar{\Gamma}_+$, where $\Sigma =]0,1[\times\{0\}$. The plate is clamped on the portion Γ_+ of its boundary and is reinforced by a thin rigid layer on the other part Σ. The thin stiffener occupies the set $\bar{\Omega}^{\delta}_- = [0,1]\times[-\delta,0]$ of boundary $\partial\Omega^{\delta}_- = \bar{\Sigma}^{\delta}_-\cup\bar{\Sigma}\cup\bar{\Gamma}^{\delta}_-$, where $\Sigma^{\delta}_- =]0,1[\times\{-\delta\}$. These two elastic bodies form together an heterogeneous elastic multi-structure, viewed as a rectangular plate occupying the set $\bar{\Omega}^{\delta} = [0,1]\times[-\delta,1]$, where $\Omega^{\delta} = \Omega_+\cup\Sigma\cup\Omega^{\delta}_-$. For our forthcoming study, we consider the following nonlinear thermo-elastic model for the structure $\bar{\Omega}^{\delta}$ (see [8,13] for example):

- Equations of motion in $\Omega^{\delta}\times(0,T)$:

$$\begin{aligned}
&\rho u'' - div\Big\{\mathcal{C}\left[\epsilon(u) + f(\nabla w)\right]\Big\} + \lambda\nabla\phi = 0,\\
&\rho\left[I-\Delta\right]w'' + D\Delta^2 w - div\Big\{\mathcal{C}\left[\epsilon(u)+f(\nabla w)\right]\nabla w\Big\} + \lambda\Delta\theta = 0,\\
&\rho\phi' - k\Delta\phi + \lambda div u' = 0,\\
&\rho\theta' - k\Delta\theta - \lambda\Delta w' = 0.
\end{aligned} \tag{1}$$

- Free boundary conditions on $\Sigma^{\delta}_-\times(0,T)$:

$$\begin{aligned}
&\mathcal{C}\left[\epsilon(u)+f(\nabla w)\right]n = 0, \qquad D\left[\Delta w + (1-\nu)B_1 w\right] = 0,\\
&D\left[\partial_n\Delta w + (1-\nu)\partial_\tau B_2 w\right] - \rho\partial_n w'' - \mathcal{C}\left[\epsilon(u)+f(\nabla w)\right]n.\nabla w + \lambda\partial_n\theta = 0,\\
&k\partial_n\theta + \lambda\partial_n w' = 0, \qquad k\partial_n\phi - \lambda u' n = 0.
\end{aligned} \tag{2}$$

- Clamped boundary conditions on $\Gamma_+\cup\Gamma^{\delta}_-\times(0,T)$:

$$u = 0,\; w = \partial_n w = 0,\; \theta = 0,\; \phi = 0. \tag{3}$$

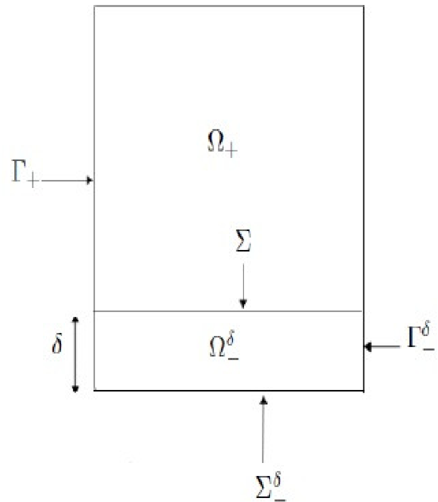

Figure 1: A plate reinforced with a thin stiffener.

- Transmission conditions on $\Sigma \times (0,T)$:

$$\begin{aligned}
&[[u]] = 0, \qquad [[w]] = [[\partial_n w]] = 0, \qquad [[\theta]] = [[\phi]] = 0,\\
&[[\mathcal{C}\left[\epsilon\left(u\right) + f\left(\nabla w\right)\right] n]] = 0, \qquad [[D\left[\Delta w + \left(1-\nu\right) B_1 w\right]]] = 0,\\
&\left[\left[k\partial_n\theta + \lambda\partial_n w'\right]\right] = 0, \qquad \left[\left[k\partial_n\phi - \lambda u' n\right]\right] = 0,\\
&\left[\left[D\left[\partial_n\Delta w + \left(1-\nu\right)\partial_\tau B_2 w\right] - \rho\partial_n w'' - \mathcal{C}\left[\epsilon\left(u\right) + f\left(\nabla w\right)\right] n.\nabla w + \lambda\partial_n\theta\right]\right] = 0.
\end{aligned} \tag{4}$$

We associate with the equations (1)-(4) the initial conditions in Ω^δ:

$$u\left(0\right) = u^0, \quad u'\left(0\right) = u^1, \quad w\left(0\right) = w^0, \quad w'\left(0\right) = w^1, \quad \theta\left(0\right) = \theta^0, \quad \phi\left(0\right) = \phi^0, \tag{5}$$

where $u^0 \in \left[H^1(\Omega^\delta)\right]^2$, $u^1 \in \left[L^2(\Omega^\delta)\right]^2$, $w^0 \in H^2(\Omega^\delta)$, $w^1 \in H^1(\Omega^\delta)$, θ^0, $\phi^0 \in H^1(\Omega^\delta)$. The variables w and (u_1, u_2) represent respectively the vertical (bending) and in-plane displacement of the structure (plate-layer). θ and ϕ describe the average temperature affecting the vertical and horizontal displacement, respectively. $\epsilon\left(u\right)$ is the linearised strain tensor defined by the formulae $\epsilon\left(u\right) = \left(\nabla u + \nabla^T u\right)/2$ and $\mathcal{C}$ is a fourth order tensor that belongs to S, the space of 2×2 symmetric matrices, given by:

$$\mathcal{C}\left(\xi\right) = D\Big[\nu\left(tr\xi\right) I_S + \left(1-\nu\right)\xi\Big], \quad \xi \in S,$$

where I_S is the identity matrix. The function f is defined by $f\left(s\right) = (1/2)\, s \otimes s, \; s \in \mathcal{R}^2$ and the trace operators B_1 and B_2 are given by:

$$B_1 w \equiv 2n_1 n_2 \partial^2_{xy} w - n_1^2 \partial^2_y w - n_2^2 \partial^2_x w, \quad B_2 w \equiv \left(n_1^2 - n_2^2\right)\partial^2_{xy} w + n_1 n_2 \left(\partial^2_y w - \partial^2_x w\right),$$

where $n = (n_1, n_2)$ is the unit normal to Σ oriented outwardly of Ω_+ and $\tau = (-n_2, n_1)$ stands for the tangential unit vector. We denote by $D = E/(1-\nu^2)$ the flexural rigidity of the structure,

E being the Young modulus, ν the Poisson's ratio, ρ is the mass density, k the coefficient of thermal conductivity and $\lambda = \alpha D(1+\nu)/2$, where α denotes the coefficient of thermal expansion. We assume that $0 < \nu < \frac{1}{2}$ and that all these physical and thermal coefficients are piecewise positive constants: E, ρ, k, α, ν are equal to E_+, ρ_+, k_+, α_+, ν_+ (resp. E_-^δ, ρ_-^δ, k_-^δ, α_-^δ, ν_-) in Ω_+ (in Ω_-^δ resp.). Accordingly, $D = D_+$, $\lambda = \lambda_+$ in Ω_+ and D_-^δ, λ_-^δ in Ω_-^δ. All over this study, we will make the assumption that these coefficients are independent of δ in the plate Ω_+ (the Poisson's ratio ν is supposed independent of δ in both the plate and the layer). Finally, we denote by $[[\cdot]]$ the jump through the interface Σ and g', g'' stand for the time derivatives of a function g.

2.2 Variational setting

Let (w, u, ϕ, θ) be a classical solution of (1)-(5) and consider the following spaces:

$$W(\Omega^\delta) = \left\{ w \in H^2(\Omega^\delta), \quad w_{|\Gamma} = \partial_n w_{|\Gamma} = 0 \right\}, \; V(\Omega^\delta) = \left\{ w \in H^1(\Omega^\delta) \; ; \; w_{|\Gamma} = 0 \right\},$$
$$U(\Omega^\delta) = \left\{ u \in H^1(\Omega^\delta) \times H^1(\Omega^\delta) \; ; \; u_{|\Gamma} = 0 \right\}.$$

Denoting by $\langle \cdot, \cdot \rangle_D$ the scalar product in $\left[L^2(D)\right]^l$, $l \in \mathbb{N}$, it follows from Green's formulas that an appropriate variational formulation of (1)-(5) reads:

$$(\mathcal{P}) \begin{cases} \text{Find } u \in L^\infty\left(0,T; U(\Omega^\delta)\right), \; w \in L^\infty\left(0,T; W(\Omega^\delta)\right), \; w' \in L^\infty\left(0,T; V(\Omega^\delta)\right), \\ u' \in L^\infty\left(0,T; \left[L^2(\Omega^\delta)\right]^2\right), \; \phi \in L^\infty\left(0,T; L^2(\Omega^\delta)\right) \cap L^2\left(0,T; V(\Omega^\delta)\right), \\ \theta \in L^\infty\left(0,T; L^2(\Omega^\delta)\right) \cap L^2\left(0,T; V(\Omega^\delta)\right), \text{ such that:} \\ \rho\langle u', \hat{u}\rangle'_{\Omega^\delta} + \langle \mathcal{C}\left[\epsilon(u) + f(\nabla w)\right], \epsilon(\hat{u})\rangle_{\Omega^\delta} + \lambda\langle \nabla\phi, \hat{u}\rangle_{\Omega^\delta} + \rho\left[\langle w', \hat{w}\rangle_{\Omega^\delta} + \langle \nabla w', \nabla\hat{w}\rangle_{\Omega^\delta}\right]' \\ +a(w,\hat{w}) + \langle \mathcal{C}\left[\epsilon(u) + f(\nabla w)\right]\nabla w, \nabla\hat{w}\rangle_{\Omega^\delta} - \lambda\langle\nabla\theta, \nabla\hat{w}\rangle_{\Omega^\delta} + \rho\langle\phi, \hat{\phi}\rangle'_{\Omega^\delta} \\ +k\langle\nabla\phi, \nabla\hat{\phi}\rangle_{\Omega^\delta} - \lambda\langle u', \nabla\hat{\phi}\rangle_{\Omega^\delta} + \rho\langle\theta, \hat{\theta}\rangle'_{\Omega^\delta} + k\langle\nabla\theta, \nabla\hat{\theta}\rangle_{\Omega^\delta} + \lambda\langle\nabla w', \nabla\hat{\theta}\rangle_{\Omega^\delta} = 0, \end{cases}$$

$\forall(\hat{u}, \hat{w}, \hat{\phi}, \hat{\theta}) \in U(\Omega^\delta) \times W(\Omega^\delta) \times V(\Omega^\delta) \times V(\Omega^\delta)$, with the initial conditions (5), where:

$$a(w,\hat{w}) = \int_{\Omega^\delta} D\Big[\left(\partial_x^2 w + \nu\partial_y^2 w\right)\partial_x^2\hat{w} + 2(1-\nu)\,\partial_{xy}^2 w\,\partial_{xy}^2\hat{w} + \left(\partial_y^2 w + \nu\partial_x^2\right)\partial_y^2\hat{w}\Big]d\Omega^\delta.$$

We can refer to [8] for the well-posedness of the problem $(\mathcal{P})$.

3 The scaled problem

3.1 Scaling

In order to carry out an asymptotic analysis of the problem $(\mathcal{P})$ as the small parameter δ goes to zero, and because the unknowns are defined on the set Ω^δ which itself vary with δ, our first task naturally consists in transforming $(\mathcal{P})$ into a problem posed over a set that doesn't depend on δ. Accordingly, we let $\Omega_- =]0,1[\times]-1,0[$ and perform the scaling:

$$\begin{cases} \Omega_- \longrightarrow \Omega_-^\delta, \\ (x,z) \longrightarrow (x,y) = (x,\delta z). \end{cases} \tag{6}$$

We identify Σ with $\Sigma\times\{0\}$ and set $\Sigma_- = \Sigma\times\{1\}$, $\Gamma_- = \partial\Omega_-\backslash(\Sigma\cup\Sigma_-)$ and $\Omega = \Omega_+\cup\Sigma\cup\Omega_-$. For a function ζ and a vector field $v = (v_1, v_2)$ defined on Ω_-^δ, we associate ζ^δ and v^δ defined on Ω_- by: $\zeta^\delta(x,z) = \zeta(x,\delta z)$ and $v^\delta(x,z) = (v_1(x,\delta z),\ \delta v_2(x,\delta z))$. Clearly, we have $\partial_y = \delta^{-1}\partial_z$ and

$$\int_0^1\int_{-\delta}^0 \zeta dxdy = \delta\int_0^1\int_{-1}^0 \zeta^\delta dxdz.$$

We denote by u_-^δ, w_-^δ, θ_-^δ and ϕ_-^δ the functions obtained respectively from $u_{|\Omega_-^\delta}$, $w_{|\Omega_-^\delta}$, $\theta_{|\Omega_-^\delta}$ and $\phi_{|\Omega_-^\delta}$ through the scaling (6) and set $u^\delta = (u_+^\delta, u_-^\delta)$, $w^\delta = (w_+^\delta, w_-^\delta)$, $\theta^\delta = (\theta_+^\delta, \theta_-^\delta)$ and $\phi^\delta = (\phi_+^\delta, \phi_-^\delta)$, where $u_+^\delta = u_{|\Omega_+}$, $w_+^\delta = w_{|\Omega_+}$, $\theta_+^\delta = \theta_{|\Omega_+}$ and $\phi_+^\delta = \phi_{|\Omega_+}$.

After the scaling (6), $(\mathcal{P})$ is transformed into a new problem posed on the fixed domain Ω. We introduce the following functional spaces:

$$\begin{aligned} W^\delta(\Omega) = \Big\{&(\hat w_+, \hat w_-) \in H^2(\Omega_+)\times H^2(\Omega_-),\quad \hat w_{+|\Sigma} = \hat w_{-|\Sigma},\quad \partial_n \hat w_{+|\Sigma} = \delta^{-1}\partial_z \hat w_{-|\Sigma},\\ &\hat w_{+|\Gamma_+} = \partial_n \hat w_{+|\Gamma_+} = 0,\ \hat w_{-|\Gamma_-} = \partial_n \hat w_{-|\Gamma_-} = 0\Big\},\\ V^\delta(\Omega) = \Big\{&(\hat w_+, \hat w_-) \in H^1(\Omega_+)\times H^1(\Omega_-),\quad \hat w_{+|\Sigma} = \hat w_{-|\Sigma},\quad \hat w_{+|\Gamma_+} = \hat w_{-|\Gamma_-} = 0\Big\},\\ U^\delta(\Omega) = \Big\{&(\hat u_+, \hat u_-) \in \big[H^1(\Omega_+)\big]^2\times\big[H^1(\Omega_-)\big]^2,\ \hat u_{-1|\Sigma} = \hat u_{+1|\Sigma},\ \hat u_{-2|\Sigma} = \delta\hat u_{+2|\Sigma},\\ &\hat u_{+|\Gamma_+} = \hat u_{-|\Gamma_-} = 0\Big\}. \end{aligned}$$

Using the above notations, we reformulate the variational problem $(\mathcal{P})$ in the following form

$$(\mathcal{P}^\delta)\begin{cases} \text{Find } u^\delta \in L^\infty(0,T;U^\delta(\Omega)),\ w^\delta \in L^\infty(0,T;W^\delta(\Omega)),\ (w^\delta)' \in L^\infty(0,T;V^\delta(\Omega)),\\ \phi^\delta, \theta^\delta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V^\delta(\Omega)),\\ \text{such that:}\\ \rho_+\langle (u_+^\delta)', \hat u_+\rangle'_{\Omega_+} + \rho_-^\delta\delta\big[\langle (u_{-1}^\delta)', \hat u_{-1}\rangle'_{\Omega_-} + \delta^{-2}\langle (u_{-2}^\delta)', \hat u_{-2}\rangle'_{\Omega_-}\big]\\ \quad + \rho_+\langle (w_+^\delta)', \hat w_+\rangle'_{\Omega_+} + \rho_-^\delta\delta\langle (w_-^\delta)', \hat w_-\rangle'_{\Omega_-} + \rho_+ b_+\big[(w_+^\delta)', \hat w_+\big]'\\ \quad + \rho_-^\delta\delta b_-^\delta\big[(w_-^\delta)', \hat w_-\big]' + D_+ a_+(w_+^\delta, \hat w_+) + D_-^\delta\delta a_-^\delta(w_-^\delta, \hat w_-)\\ \quad + \lambda_+ c_+(\phi_+^\delta, \hat u_+) + \lambda_-^\delta\delta c_-^\delta(\phi_-^\delta, \hat u_-) - \lambda_+ b_+(\theta_+^\delta, \hat w_+) - \lambda_-^\delta\delta b_-^\delta(\theta_-^\delta, \hat w_-)\\ \quad + \rho_+\langle \phi_+^\delta, \hat\phi_+\rangle'_{\Omega_+} + \rho_-^\delta\delta\langle \phi_-^\delta, \hat\phi_-\rangle'_{\Omega_-} + k_+ b_+(\phi_+^\delta, \hat\phi_+) + k_-^\delta\delta b_-^\delta(\phi_-^\delta, \hat\phi_-)\\ \quad - \lambda_+ d_+\big[(u_+^\delta)', \hat\phi_+\big] - \lambda_-^\delta\delta d_-^\delta\big[(u_-^\delta)', \hat\phi_-\big] + \rho_+\langle \theta_+^\delta, \hat\theta_+\rangle'_{\Omega_+} + \rho_-^\delta\delta\langle \theta_-^\delta, \hat\theta_-\rangle'_{\Omega_-}\\ \quad + \lambda_+ b_+\big[(w_+^\delta)', \hat\theta_+\big] + \lambda_-^\delta\delta b_-^\delta\big[(w_-^\delta)', \hat\theta_-\big] + k_+ b_+(\theta_+^\delta, \hat\theta_+) + k_-^\delta\delta b_-^\delta(\theta_-^\delta, \hat\theta_-)\\ \quad + N_+(u_+^\delta, w_+^\delta, \hat u_+, \hat w_+) + D_-^\delta\delta N_-^\delta(u_-^\delta, w_-^\delta, \hat u_-, \hat w_-) = 0, \end{cases} \tag{7}$$

for all $(\hat{u}, \hat{w}, \hat{\phi}, \hat{\theta}) \in U^\delta(\Omega) \times W^\delta(\Omega) \times V^\delta(\Omega) \times V^\delta(\Omega)$, where:

$$a_+(w_+^\delta, \hat{w}_+) = \int_{\Omega_+} \Big[\Big(\partial_x^2 w_+^\delta + \nu_+ \partial_y^2 w_+^\delta \Big) \partial_x^2 \hat{w}_+ + 2\,(1-\nu_+)\, \partial_{xy}^2 w_+^\delta \partial_{xy}^2 \hat{w}_+ + \Big(\partial_y^2 w_+^\delta + \nu_+ \partial_x^2 w_+^\delta \Big) \partial_y^2 \hat{w}_+ \Big] \, dxdy,$$

$$a_-^\delta(w_-^\delta, \hat{w}_-) = \int_{\Omega_-} \Big[\Big(\partial_x^2 w_-^\delta + \delta^{-2} \nu_- \partial_z^2 w_-^\delta \Big) \partial_x^2 \hat{w}_- + 2\,(1-\nu_-)\, \delta^{-1} \partial_{xz}^2 w_-^\delta \delta^{-1} \partial_{xz}^2 \hat{w}_- + \Big(\delta^{-2} \partial_z^2 w_-^\delta + \nu_- \partial_x^2 w_-^\delta \Big) \delta^{-2} \partial_z^2 \hat{w}_- \Big] \, dxdz,$$

$$b_+(w_+^\delta, \hat{w}_+) = \int_{\Omega_+} \nabla w_+^\delta \nabla \hat{w}_+ dxdy, \quad b_-^\delta(w_-^\delta, \hat{w}_-) = \int_{\Omega_-} \Big[\partial_x w_-^\delta \partial_x \hat{w}_- + \delta^{-2} \partial_z w_-^\delta \partial_z \hat{w}_- \Big] \, dxdz,$$

$$c_+(\phi_+^\delta, \hat{u}_+) = \int_{\Omega_+} \nabla \phi_+^\delta \hat{u}_+ dxdy, \quad c_-^\delta(\phi_-^\delta, \hat{u}_-) = \int_{\Omega_-} \Big[\partial_x \phi_-^\delta \hat{u}_{-1} + \delta^{-2} \partial_z \phi_-^\delta \hat{u}_{-2} \Big] \, dxdz,$$

$$d_+((u_+^\delta)', \hat{\phi}_+) = \int_{\Omega_+} (u_+^\delta)' \nabla \hat{\phi}_+ dxdy, \quad d_-^\delta((u_-^\delta)', \hat{\phi}_-) = \int_{\Omega_-} \Big[(u_{-1}^\delta)' \partial_x \hat{\phi}_- + \delta^{-2} (u_{-2}^\delta)' \partial_z \hat{\phi}_- \Big] \, dxdz,$$

$$N_+(u_+^\delta, w_+^\delta, \hat{u}_+, \hat{w}_+) = \langle \mathcal{C}[\epsilon(u_+^\delta) + f(\nabla w_+^\delta)], \epsilon(\hat{u}_+) \rangle_{\Omega_+} + \langle \mathcal{C} \Big[\epsilon(u_+^\delta) + f(\nabla w_+^\delta) \Big] \nabla w_+^\delta, \nabla \hat{w}_+ \rangle_{\Omega_+},$$

$$\begin{aligned} N_-^\delta(u^\delta, w^\delta, \hat{u}, \hat{w}) = \int_{\Omega_-} \Big\{ &[\partial_x u_{-1}^\delta + \tfrac{1}{2}(\partial_x w_-^\delta)^2 + \tfrac{\nu_-}{\delta^2}(\partial_z u_{-2}^\delta + \tfrac{1}{2}(\partial_z w_-^\delta)^2)][\partial_x \hat{u}_{-1} + \partial_x w_-^\delta \partial_x \hat{w}_-] \\ &+ \tfrac{1-\nu_-}{2\delta^2}[\partial_z u_{-1}^\delta + \partial_x u_{-2}^\delta + \partial_x w_-^\delta \partial_z w_-^\delta][\partial_z \hat{u}_{-1} + \partial_x \hat{u}_{-2} + \partial_x w_-^\delta \partial_z \hat{w}_- + \partial_z w_-^\delta \partial_x \hat{w}_-] \\ &+ \tfrac{1}{\delta^2}[\tfrac{1}{\delta^2}(\partial_z u_{-2}^\delta + \tfrac{1}{2}(\partial_z w_-^\delta)^2) + \nu_-(\partial_x u_{-1}^\delta + \tfrac{1}{2}(\partial_x w_-^\delta)^2)][\partial_z \hat{u}_{-2} + \partial_z w_-^\delta \partial_z \hat{w}_-] \Big\} dxdz. \end{aligned}$$

We associate with $(\mathcal{P}^\delta)$ the scaled initial data obtained by performing the scaling (6) in (5).

3.2 A priori estimates

Here, we must establish a priori estimates that allow us to pass through the limit in the scaled variational problem. Let $(u^\delta, w^\delta, \phi^\delta, \theta^\delta)$ the solution of $(\mathcal{P}^\delta)$. We denote:

$$\begin{aligned} E^\delta(t) = \frac{1}{2} \Big\{ &\rho_+ \big\| (u_+^\delta)'(t) \big\|_{L^2(\Omega_+)}^2 + \rho_-^\delta \delta \big\| (u_-^\delta)'(t) \big\|_{L^2(\Omega_-)}^2 + \rho_+ \big\| (w_+^\delta)'(t) \big\|_{L^2(\Omega_+)}^2 \\ &+ \rho_-^\delta \delta \big\| (w_-^\delta)'(t) \big\|_{L^2(\Omega_-)}^2 + \rho_+ \big\| \phi_+^\delta(t) \big\|_{L^2(\Omega_+)}^2 + \rho_-^\delta \delta \big\| \phi_-^\delta(t) \big\|_{L^2(\Omega_-)}^2 + \rho_+ \big\| \theta_+^\delta(t) \big\|_{L^2(\Omega_+)}^2 \\ &+ \rho_-^\delta \delta \big\| \theta_-^\delta(t) \big\|_{L^2(\Omega_-)}^2 + \rho_+ b_+((w_+^\delta)'(t), (w_+^\delta)'(t)) + \rho_-^\delta \delta b_-^\delta((w_-^\delta)'(t), (w_-^\delta)'(t)) \\ &+ D_+ a_+(w_+^\delta(t), w_+^\delta(t)) + D_-^\delta \delta a_-^\delta(w_-^\delta(t), w_-^\delta(t)) + N_+(w_+^\delta(t), u_+^\delta(t), u_+^\delta(t), w_+^\delta(t)) \\ &+ D_-^\delta \delta N_-^\delta(w_-^\delta(t), u_-^\delta(t), u_-^\delta(t), w_-^\delta(t)) \Big\}. \end{aligned}$$

Proposition 1. *We suppose that* $\mathrm{E}^\delta(0)$ *is bounded independently of* δ. *Then:*

- w_+^δ *and* $(w_+^\delta)'$ *are bounded independently of* δ *in* $L^\infty(0,T,H^2(\Omega_+))$ *and* $L^\infty(0,T,H^1(\Omega_+))$, *respectively.*

- u_+^δ *and* $(u_+^\delta)'$ *are bounded independently of* δ *in* $L^\infty(0,T,[H^1(\Omega_+)]^2)$ *and* $L^\infty(0,T,[L^2(\Omega_+)]^2)$, *respectively.*
- ϕ_+^δ, θ_+^δ *are bounded independently of* δ *in* $L^\infty(0,T,L^2(\Omega_+))$.
- $\sqrt{\rho_-^\delta\delta}(\delta^{-1}\partial_z w_-^\delta)'$, $\sqrt{\rho_-^\delta\delta}(\partial_x w_-^\delta)'$, $\sqrt{E_-^\delta\delta}\delta^{-2}\left(\partial_z u_{-2}^\delta+\frac{1}{2}(\partial_z w_-^\delta)^2\right)$, $\sqrt{E_-^\delta\delta}\delta^{-1}\partial_{xz}^2 w_-^\delta$, $\sqrt{E_-^\delta\delta}\left(\partial_x u_{-1}^\delta+\frac{1}{2}(\partial_x w_-^\delta)^2\right)$, $\sqrt{E_-^\delta\delta}\delta^{-2}\partial_z^2 w_-^\delta$, $\sqrt{E_-^\delta\delta}\delta^{-1}\left(\partial_x u_{-2}^\delta+\partial_z u_{-1}^\delta+\partial_x w_-^\delta\partial_z w_-^\delta\right)$ *and* $\sqrt{E_-^\delta\delta}\partial_x^2 w_-^\delta$ *are bounded independently of* δ *in* $L^\infty(0,T,L^2(\Omega_-))$.
- $\sqrt{\rho_-^\delta\delta}(u_{-1}^\delta)'$ *and* $\sqrt{\rho_-^\delta\delta}(\delta^{-1}u_{-2}^\delta)'$ *are bounded independently of* δ *in* $L^\infty(0,T,L^2(\Omega_-))$.
- $\sqrt{\rho_-^\delta\delta}\phi_-^\delta$, *and* $\sqrt{\rho_-^\delta\delta}\theta_-^\delta$ *are bounded independently of* δ *in* $L^\infty(0,T,L^2(\Omega_-))$.
- $\nabla\phi_+^\delta$ *and* $\nabla\theta_+^\delta$ *are bounded independently of* δ *in* $L^2(0,T,L^2(\Omega_+))$.
- $\sqrt{k_-^\delta\delta}\partial_x\phi_-^\delta$, $\sqrt{k_-^\delta\delta}\partial_x\theta_-^\delta$, $\sqrt{k_-^\delta\delta}\delta^{-1}\partial_z\phi_-^\delta$ *and* $\sqrt{k_-^\delta\delta}\delta^{-1}\partial_z\theta_-^\delta$ *are bounded independently of* δ *in* $L^2(0,T,L^2(\Omega_-))$.

Proof. Letting $\hat{u}=(u^\delta)'$, $\hat{w}=(w^\delta)'$, $\hat{\theta}=\theta^\delta$ and $\hat{\phi}=\phi^\delta$ in the variational formulation (7), integrating from 0 to t, we get

$$E^\delta(t)+k_+\int_0^t\left\|\nabla\theta_+^\delta(t)\right\|_{L^2(\Omega_+)}^2dt+k_+\int_0^t\left\|\nabla\phi_+^\delta(t)\right\|_{L^2(\Omega_+)}^2dt+\delta k_-^\delta\int_0^t b_-^\delta(\theta_-^\delta,\theta_-^\delta)dt$$
$$+\delta k_-^\delta\int_0^t b_-^\delta(\phi_-^\delta,\phi_-^\delta)dt=E^\delta(0),$$

first for smooth solutions which is then extended by density to all weak solutions. Using Poincaré and Korn's Inequalities, we obtain the previous a priori estimates. □

4 Asymptotic modeling of the stiffened plate

4.1 Case of a high rigid insulating layer

In this section we investigate the situation where the elastic plate Ω_+ is reinforced with an insulating thin layer with high rigidity. More precisely, we make the assumption that $E_-^\delta=\delta^{-1}E_-$, $\rho_-^\delta=\delta\rho_-$ and $k_-^\delta=\delta k_-$. We let $\alpha_-^\delta=\delta^2\alpha_-$.

Owing to the a priori estimates stated in Proposition 1, we can assert that, up to a subsequence, there exist $\tilde{w}_\pm$, $\tilde{u}_\pm$, $\tilde{\phi}_+$ and $\tilde{\theta}_+$ such that:

- $w_\pm^\delta\longrightarrow\tilde{w}_\pm$, $u_\pm^\delta\longrightarrow\tilde{u}_\pm$, weakly* in $L^\infty\left(0,T;H^2(\Omega_\pm)\right)$ and $L^\infty\left(0,T;(H^1(\Omega_\pm))^2\right)$, respectively.

- $\phi_+^\delta \longrightarrow \tilde\phi_+$, $\theta_+^\delta \longrightarrow \tilde\theta_+$, weakly* in $L^\infty(0,T,L^2(\Omega_+))$ and weakly in $L^2(0,T,H^1(\Omega_+))$. In order to give more information about this weak limit, we introduce the spaces:

$$W(\Omega_+) = \Big\{ w \in H^2(\Omega_+),\ w_{|\Gamma_+} = \partial_n w_{|\Gamma_+} = 0,\ w_{|\Sigma} \in H_0^2(\Sigma),\ \partial_n w \in H_0^1(\Sigma) \Big\},$$
$$U(\Omega_+) = \Big\{ u \in (H^1(\Omega_+))^2,\ u_{|\Gamma_+} = 0,\ u_{1|\Sigma} \in H_0^1(\Sigma) \Big\}. \tag{8}$$

Proposition 2. *The limit* $(\tilde u, \tilde w, \tilde\phi, \tilde\theta)$ *is characterized as follows:*

- $\tilde w_+ \in L^\infty(0,T,W(\Omega_+))$ *and* $(\tilde w_+)' \in L^\infty(0,T,H^1_{\Gamma_+}(\Omega_+))$,
- $\tilde u_+ \in L^\infty(0,T,U(\Omega_+))$ *and* $(\tilde u_+)' \in L^\infty(0,T,(L^2(\Omega_+))^2)$,
- $\tilde\phi_+, \tilde\theta_+ \in L^\infty(0,T,L^2(\Omega_+)) \cap L^2(0,T,H^1_{\Gamma_+}(\Omega_+))$.
- *Moreover, we have:*

$$\tilde w_- = \tilde w_{+|\Sigma}, \quad \tilde u_{-1} = \tilde u_{+1|\Sigma}, \quad \tilde u_{-2} = 0.$$

Proof. The proof follows from the a priori estimates of Proposition 1 and the transmission conditions. For w^δ, these estimates lead to the fact that $\partial_z^2 \tilde w_- = 0$, which combined with the transmission condition $\partial_n w^\delta_{+|\Sigma} = \delta^{-1}\partial_z w^\delta_{-|\Sigma}$ gives, at the limit $\partial_z \tilde w_- = 0$. Using once again the transmission conditions, we get $\tilde w_- = \tilde w_{+|\Sigma}$. In addition, we can easily show that

$$\int_{-1}^{0} \delta^{-1}\partial_z w_-^\delta\, dz \to \partial_n \tilde w_{+|\Sigma}, \tag{9}$$

weakly* in $L^\infty(0,T;H^1(\Sigma))$. Likewise, same arguments lead to the proprieties stated for $\tilde u$, $\tilde\phi$ and $\tilde\theta$. □

Proposition 3. *The following convergences hold true, weakly* in* $L^\infty(0,T,L^2(\Sigma))$*:*

- $$\int_{-1}^{0} \partial_x^2 w_-^\delta\, dz \overset{*}{\rightharpoonup} \partial_x^2 \tilde w_{+|\Sigma}, \qquad \int_{-1}^{0} \delta^{-1}\partial_{xz}^2 w_-^\delta\, dz \overset{*}{\rightharpoonup} \partial_x\partial_n \tilde w_{+|\Sigma}, \tag{10}$$
- $$\int_{-1}^{0} \delta^{-2}\partial_z^2 w_-^\delta\, dz \overset{*}{\rightharpoonup} -\nu_-\partial_x^2 \tilde w_{+|\Sigma}, \tag{11}$$
- $$\int_{-1}^{0} \big(\partial_x u_{-1}^\delta + \frac{1}{2}(\partial_x w_-^\delta)^2\big) dz \overset{*}{\rightharpoonup} \partial_x \tilde u_{+1|\Sigma} + \frac{1}{2}(\partial_x \tilde w_{+|\Sigma})^2, \tag{12}$$
- $$\int_{-1}^{0} \delta^{-2}\big(\partial_z u_{-2}^\delta + \frac{1}{2}(\partial_z w_-^\delta)^2\big) dz \overset{*}{\rightharpoonup} -\nu_-\big(\partial_x \tilde u_{+1|\Sigma} + \frac{1}{2}(\partial_x \tilde w_{+|\Sigma})^2\big), \tag{13}$$
- $$\int_{-1}^{0} \delta^{-1}\big(\partial_x u_{-2}^\delta + \partial_z u_{-1}^\delta + \partial_x w_-^\delta \partial_z w_-^\delta\big) dz \overset{*}{\rightharpoonup} 0. \tag{14}$$

Proof. The limits (10) follow from the a priori estimates of Proposition 1 and the transmission conditions. The limit (11) is obtained by applying the variational problem (7) with the test functions: $\hat u_+ = \hat u_- = 0$, $\hat\phi_+ = \hat\phi_- = 0$, $\hat\theta_+ = \hat\theta_- = 0$, $\hat w_+ = 0$ and $\hat w_- = \delta^2 \varrho(x)\frac{z^2}{2}$, where ϱ

is a smooth enough function, and passing through the limit, as δ goes to zero. It remains now to prove the three last limits, which require more investigation because of the nonlinear terms involved in their expressions. Recalling the fact that $w_-^\delta \longrightarrow \tilde{w}_-$ weakly* in $L^\infty(0,T,H^2(\Omega_-))$, it follows, by a compactness argument that

$$w_-^\delta \longrightarrow \tilde{w}_- \ \text{ strongly in } L^\infty(0,T,H^{2-\varepsilon}(\Omega_-)),$$

as δ goes to zero, for all $\varepsilon > 0$. Consequently, we get

$$\partial_x w_-^\delta \longrightarrow \partial_x \tilde{w}_- \ \text{ strongly in } L^\infty(0,T,H^{1-\varepsilon}(\Omega_-)).$$

Using the Sobolev embedding $H^{1-\varepsilon}(\Omega_-) \hookrightarrow L^{\frac{2}{\varepsilon}}(\Omega_-)$, we deduce that $\partial_x w_-^\delta \longrightarrow \partial_x \tilde{w}_-$ strongly in $L^\infty(0,T,L^4(\Omega_-))$, which implies the convergence of $(\partial_x w_-^\delta)^2$ towards $(\partial_x \tilde{w}_-)^2$ in $L^\infty(0,T,L^2(\Omega_-))$. Accordingly, owing to the fact that $\tilde{w}_- = \tilde{w}_{+|\Sigma}$ and $\tilde{u}_{-1} = \tilde{u}_{+1|\Sigma}$, we obtain (12).

Similarly, the other limits follow by using same arguments. □

The following Theorem gives the main result of this section which consists in the identification of the limit model, posed only over the domain of the plate Ω_+. To this end, we first fix the limit behavior of the initial data and assume that there exist smooth enough functions w_+^*, w^{**}, ϕ^*, θ^*, u_+^*, u^{**} such that:

- $w_+^{0\delta}$ and $\displaystyle\int_{-1}^0 w_-^{0\delta} dz$ converge to w_+^* and $w_{+|\Sigma}^*$ weakly in $H^2(\Omega_+)$ and $H^2(\Sigma)$, respectively.
- $u_{+1}^{0\delta}$ and $u_{+2}^{0\delta}$ converge to u_{+1}^*, u_{+2}^* weakly in $H^1(\Omega_+)$.
- $\phi^{0\delta}$ and $\theta^{0\delta}$ converge to ϕ^*, θ^* weakly in $H^1(\Omega)$.
- $\displaystyle\int_{-1}^0 u_{-1}^{0\delta} dz$ and $\displaystyle\int_{-1}^0 u_{-2}^{0\delta} dz$ converge to $u_{+1|\Sigma}^*$, 0 weakly in $H^1(\Sigma)$.
- $w_+^{1\delta}$ and $u^{1\delta}$ converge to w^{**} and u^{**} weakly in $H^1(\Omega)$ and $[L^2(\Omega)]^2$ respectively.

Thus, the limit problem is identified in the following Theorem.

Theorem 1. *The weak limit $(\tilde{u}_+, \tilde{w}_+, \tilde{\phi}_+, \tilde{\theta}_+)$ verify:*

- $\tilde{w}_+ \in L^\infty(0,T,W(\Omega_+)), \quad (\tilde{w}_+)' \in L^\infty(0,T,H^1_{\Gamma_+}(\Omega_+)),$
- $\tilde{u}_+ \in L^\infty(0,T,U(\Omega_+)), \quad (\tilde{u}_+)' \in L^\infty(0,T,[L^2(\Omega_+)]^2),$
- $\tilde{\theta}_+$, $\tilde{\phi}_+ \in L^\infty(0,T,L^2(\Omega_+)) \cap L^2(0,T,H^1_{\Gamma_+}(\Omega_+))$, *and solve the variational problem*

$$(\mathcal{P}^0)\begin{cases} \rho_+\langle \tilde{u}'_+, \hat{u}_+\rangle'_{\Omega_+} + \rho_+\langle \tilde{w}'_+, \hat{w}_+\rangle'_{\Omega_+} + \rho_+ b_+(\tilde{w}'_+, \hat{w}_+)' + D_+ a_+(\tilde{w}_+, \hat{w}_+) \\ +a_\Sigma(\tilde{w}_+, \hat{w}_+) + \lambda_+ c_+(\tilde{\phi}_+, \hat{u}_+) - \lambda_+ b_+(\tilde{\theta}_+, \hat{w}_+) + \rho_+\langle \tilde{\phi}_+, \hat{\phi}_+\rangle'_{\Omega_+} + k_+ b_+(\tilde{\phi}_+, \hat{\phi}_+) \\ -\lambda_+ d_+(\tilde{u}'_+, \hat{\phi}_+) + \rho_+\langle \tilde{\theta}_+, \hat{\theta}_+\rangle'_{\Omega_+} + \lambda_+ b_+(\tilde{w}'_+, \hat{\theta}_+) + k_+ b_+(\tilde{\theta}_+, \hat{\theta}_+) \\ +N_+(\tilde{u}_+, \tilde{w}_+, \hat{u}_+, \hat{w}_+) + N_\Sigma(\tilde{u}_+, \tilde{w}_+, \hat{u}_+, \hat{w}_+) = 0, \end{cases}$$

$\forall(\hat{u}_+, \hat{w}_+, \hat{\phi}_+, \hat{\theta}_+) \in U(\Omega_+) \times W(\Omega_+) \times H^1_{\Gamma_+}(\Omega_+) \times H^1_{\Gamma_+}(\Omega_+)$, *with the initial conditions:*

$$\begin{aligned} &\tilde{u}_+(0) = u^*_+, \quad \tilde{u}'_+(0) = u^{**}_+, \quad \tilde{w}_+(0) = w^*_+, && \text{in} \quad \Omega_+, \\ &\tilde{w}'_+(0) = w^{**}_+, \quad \tilde{\phi}_+(0) = \phi^*_+, \quad \tilde{\theta}_+(0) = \theta^*_+ && \text{in} \quad \Omega_+, \\ &\tilde{w}_+(0) = w^*_{+|\Sigma}, \quad \tilde{u}_+(0) = u^*_{+|\Sigma} && \text{on} \quad \Sigma, \end{aligned} \tag{15}$$

where:

$$a_\Sigma(\tilde{w}_+, \hat{w}_+) = E_- \int_\Sigma \left[\left(\partial_x^2 \tilde{w}_{+|\Sigma}\right) \left(\partial_x^2 \hat{w}_{+|\Sigma}\right) + \frac{2}{1+\nu_-} \left(\partial_x \partial_n \tilde{w}_{+|\Sigma}\right) \left(\partial_x \partial_n \hat{w}_{+|\Sigma}\right) \right] dx,$$

$$N_\Sigma(\tilde{u}_+, \tilde{w}_+, \hat{u}_+, \hat{w}_+) = E_- \int_\Sigma \left[\partial_x \tilde{u}_{+1|\Sigma} + \frac{1}{2} \left(\partial_x \tilde{w}_{+|\Sigma}\right)^2 \right] \left[\partial_x \hat{u}_{+1|\Sigma} + \left(\partial_x \tilde{w}_{+|\Sigma}\right) \left(\partial_x \hat{w}_{+|\Sigma}\right) \right] dx.$$

Proof. The strategy consists in passing through the limit in the scaled variational problem (7) with some adequate test functions. Let $\hat{u}_+ \in D(\Omega_+) \cap U(\Omega_+)$, $\hat{w}_+ \in D(\Omega_+) \cap W(\Omega_+)$, $\hat{\phi}_+ \in D(\Omega_+) \cap H^1_{\Gamma_+}(\Omega_+)$ and $\hat{\theta}_+ \in D(\Omega_+) \cap H^1_{\Gamma_+}(\Omega_+)$, that we suppose independent of δ. (Next, the analysis may be extended for all functions of $U(\Omega_+)$, $W(\Omega_+)$ and $H^1_{\Gamma_+}(\Omega_+)$ by density). We apply the variational problem (7) with the test functions:

$$\hat{u} = \begin{cases} \hat{u}_+ & \text{in } \Omega_+, \\ \hat{u}_- = \left(\hat{u}_{+1|\Sigma}, \delta \hat{u}_{+2|\Sigma}\right) & \text{in } \Omega_-, \end{cases} \qquad \hat{w} = \begin{cases} \hat{w}_+ & \text{in } \Omega_+, \\ \hat{w}_- = \hat{w}_{+|\Sigma} + \delta z \partial_n \hat{w}_{+|\Sigma} & \text{in } \Omega_-, \end{cases}$$

$$\hat{\phi} = \begin{cases} \hat{\phi}_+ & \text{in } \Omega_+, \\ \hat{\phi}_- = \hat{\phi}_{+|\Sigma} & \text{in } \Omega_-, \end{cases} \qquad \hat{\theta} = \begin{cases} \hat{\theta}_+ & \text{in } \Omega_+, \\ \hat{\theta}_- = \hat{\theta}_{+|\Sigma} & \text{in } \Omega_-. \end{cases}$$

In order to identify the limit variational problem, we pass through the limit in all the forms involved in the formulation of (7). We can easily identify the limit of the forms indexed by "+" which act on the domain Ω_+. Nevertheless, the nonlinear form $N_+(\cdot\,,\,\cdot\,,\,\cdot\,,\cdot\,)$ requires the use of compactness arguments and Sobolev embedding theorems. We will focus here on the forms acting on the domain of the stiffener. Let us begin by investigating the behavior of the forms involving the thermal components of the solution. To this end, let us first note that $k^\delta_- \delta b^\delta_-(\theta^\delta_-, \hat{\theta}_-)$ may be written in the following form (recall that $\partial_z \hat{\theta}_- = 0$)

$$k^\delta_- \delta b^\delta_-(\theta^\delta_-, \hat{\theta}_-) = \sqrt{k^\delta_- \delta} \int_{\Omega_-} \sqrt{k^\delta_- \delta} \partial_x \theta^\delta_- \partial_x \hat{\theta}_- dx dz,$$

which, using Holder inequality yields

$$\left| k^\delta_- \delta b^\delta_-(\theta^\delta_-, \hat{\theta}_-) \right| \leq \sqrt{k^\delta_- \delta} \left\| \sqrt{k^\delta_- \delta} \partial_x \theta^\delta_- \right\|_{L^2(\Omega_-)} \left\| \partial_x \hat{\theta}_- \right\|_{L^2(\Omega_-)}.$$

Multiplying the previous form by $\zeta(t) \in D(]0,T[)$ and integrating from 0 to T, we get

$$\left| k^\delta_- \delta \int_0^T b^\delta_-(\theta^\delta_-, \hat{\theta}_-) \zeta(t) dt \right| \leq \sqrt{k^\delta_- \delta} \int_0^T \left\| \sqrt{k^\delta_- \delta} \partial_x \theta^\delta_- \right\|_{L^2(\Omega_-)} \left\| \partial_x \hat{\theta}_- \right\|_{L^2(\Omega_-)} |\zeta(t)| \, dt,$$

which, using Cauchy- Schwarz inequality gives

$$\left| k_-^\delta \delta \int_0^T b_-^\delta(\theta_-^\delta, \hat{\theta}_-)\zeta(t)dt \right| \le C\sqrt{k_-^\delta \delta} \left(\int_0^T \left\| \sqrt{k_-^\delta \delta}\partial_x \theta_-^\delta \right\|^2_{L^2(\Omega_-)} \right)^{\frac{1}{2}} \left(\int_0^T \left\| \partial_x \hat{\theta}_- \right\|^2_{L^2(\Omega_-)} \right)^{\frac{1}{2}}$$
$$\le C'\sqrt{k_-^\delta \delta} \left\| \sqrt{k_-^\delta \delta}\partial_x \theta_-^\delta \right\|_{L^2(0,T,L^2(\Omega_-))} \left\| \partial_x \hat{\theta}_+ \right\|_{L^2(\Sigma)},$$

where C, C' are constants independent of δ. The a priori estimates of Proposition 1 state that $\left\| \sqrt{k_-^\delta \delta}\partial_x \theta_-^\delta \right\|^2_{L^2(0,T,L^2(\Omega_-))}$ is bounded independently of δ. Besides, $\left\| \partial_x \hat{\theta}_+ \right\|^2_{L^2(\Sigma)}$ is also independent of δ. Thus, since $\sqrt{k_-^\delta \delta} \to 0$, we deduce that $k_-^\delta \delta b_-^\delta(\theta_-^\delta, \hat{\theta}_-)$ goes to zero, as δ goes to zero in $D'(]0,T[)$.

In the same manner, we can show that the limit of $k_-^\delta \delta b_-^\delta(\phi_-^\delta, \hat{\phi}_-)$ vanish as $\delta \to 0$. For $\lambda_-^\delta \delta c_-^\delta(\phi_-^\delta, \hat{u}_-)$, substituting the test functions introduced above, we get

$$\left| \lambda_-^\delta \delta c_-^\delta(\phi_-^\delta, \hat{u}_-) \right| = \left| \frac{\delta^2 E_- \alpha_-}{2(1-\nu_-)} \int_{\Omega_-} \left(\partial_x \phi_-^\delta \hat{u}_{+1} + \delta^{-1}\partial_z \phi_-^\delta \hat{u}_{+2} \right) dxdz \right|$$
$$= \left| \frac{\delta E_- \alpha_-}{2(1-\nu_-)\sqrt{k_-}} \int_{\Omega_-} \left(\sqrt{k_-^\delta \delta}\partial_x \phi_-^\delta \hat{u}_{+1} + \sqrt{k_-^\delta \delta}\delta^{-1}\partial_z \phi_-^\delta \hat{u}_{+2} \right) dxdz \right|$$
$$\le \frac{\delta E_- \alpha_-}{2(1-\nu_-)\sqrt{k_-}} \left[\left\| \sqrt{k_-^\delta \delta}\partial_x \phi_-^\delta \right\|_{L^2(\Omega_-)} \| \hat{u}_{+1} \|_{L^2(\Sigma)} + \left\| \sqrt{k_-^\delta \delta}\delta^{-1}\partial_z \phi_-^\delta \right\|_{L^2(\Omega_-)} \| \hat{u}_{+2} \|_{L^2(\Sigma)} \right].$$

Thus, using once again Cauchy Schwarz inequality, we obtain, for $\zeta(t) \in D(]0,T[)$,

$$\left| \int_0^T \lambda_-^\delta \delta c_-^\delta(\phi_-^\delta, \hat{u}_-)\zeta(t)dt \right| \le C\frac{\delta E_- \alpha_-}{(1-\nu_-)\sqrt{k_-}} \left[\left\| \sqrt{k_-^\delta \delta}\partial_x \phi_-^\delta \right\|_{L^2(0,T,L^2(\Omega_-))} \| \hat{u}_{+1} \|_{L^2(\Sigma)} + \left\| \sqrt{k_-^\delta \delta}\delta^{-1}\partial_z \phi_-^\delta \right\|_{L^2(0,T,L^2(\Omega_-))} \| \hat{u}_{+2} \|_{L^2(\Sigma)} \right],$$

where C is a constant independent of δ. Owing to the boundedness of $\left\| \sqrt{k_-^\delta \delta}\partial_x \phi_-^\delta \right\|_{L^2(0,T,L^2(\Omega_-))}$ and $\left\| \sqrt{k_-^\delta \delta}\delta^{-1}\partial_z \phi_-^\delta \right\|_{L^2(0,T,L^2(\Omega_-))}$ independently of δ, and since $\dfrac{\delta E_- \alpha_-}{(1-\nu_-)\sqrt{k_-}} \to 0$, we deduce that $\lambda_-^\delta \delta c_-^\delta(\phi_-^\delta, \hat{u}_-)$ goes to zero in $D'(]0,T[)$.

As far as concern $\lambda_-^\delta \delta d_-^\delta((u_-^\delta)', \hat{\phi}_-)$, we can show that

$$\left| \int_0^T \lambda_-^\delta \delta d_-^\delta((u_-^\delta)', \hat{\phi}_-)\zeta(t)dt \right| \le C\frac{\delta E_- \alpha_-}{2(1-\nu_-)\sqrt{\rho_-}} \left\| \sqrt{\rho_-^\delta \delta}(u_{-1}^\delta)' \right\|_{L^\infty(0,T,L^2(\Omega_-))} \left\| \partial_x \hat{\phi}_+ \right\|_{L^2(\Sigma)},$$

where C is independent of δ. The right-hand side of above goes to zero, as δ goes to zero, thanks to the boundedness of $\left\| \sqrt{\rho_-^\delta \delta} (u_{-1}^\delta)' \right\|_{L^\infty(0,T,L^2(\Omega_-))}$ and the fact that $\dfrac{\delta E_- \alpha_-}{2(1-\nu_-)\sqrt{\rho_-}} \to 0$.

Now, we will investigate the behavior of the forms lying on the mechanical behavior of the structure. The limits (10) and (11) permit to conclude that $D_-^\delta \delta a_-^\delta(w_-^\delta, \hat{w}_-)$ converges towards $a_\Sigma(\tilde{w}_+, \hat{w}_+)$. Besides, arguing as for $k_-^\delta \delta b_-^\delta(\theta_-^\delta, \hat{\theta}_-)$, we can show that the limit of $\rho_-^\delta \delta b_-^\delta((w_-^\delta)', \hat{w}_-)'$ vanish, when $\delta \to 0$. As far as concern the nonlinear form $D_-^\delta \delta N_-^\delta(u_-^\delta, w_-^\delta, \hat{u}_-, \hat{w}_-)$, its limit as δ goes to zero may be achieved thanks to Sobolev embeddings, Holder inequality and some nonlinear techniques. To this end, we start by investigating the limit of the first part involved in $D_-^\delta \delta N_-^\delta(u_-^\delta, w_-^\delta, \hat{u}_-, \hat{w}_-)$. Recalling the expressions of $D_-^\delta \delta N_-^\delta(u_-^\delta, w_-^\delta, \hat{u}_-, \hat{w}_-)$ and its limit $N_\Sigma(\tilde{u}_+, \tilde{w}_+, \hat{u}_+, \hat{w}_+)$, we denote by $\mathcal{A}^\delta(u^\delta, w^\delta)$ and $\mathcal{A}(\tilde{u}, \tilde{w})$ the quantities:

$$
\begin{aligned}
\mathcal{A}^\delta(u^\delta, w^\delta) &= \left[\left(\partial_x u_{-1}^\delta + \frac{1}{2} (\partial_x w_-^\delta)^2 \right) + \frac{\nu_-}{\delta^2} \left(\partial_z u_{-2}^\delta + \frac{1}{2} (\partial_z w_-^\delta)^2 \right) \right], \\
\mathcal{A}(\tilde{u}, \tilde{w}) &= (1 - \nu_-^2) \left(\partial_x \tilde{u}_{+1|\Sigma} + \frac{1}{2} (\partial_x \tilde{w}_{+|\Sigma})^2 \right).
\end{aligned}
$$

Note that, in this case, $D_-^\delta \delta$ doesn't depend of δ, so it is omitted in proving the desired limit. Clearly, we can show that $\int_{\Omega_-} \mathcal{A}^\delta(u^\delta, w^\delta) \partial_x \hat{u}_{-1} dxdz$ converges towards $\int_\Sigma \mathcal{A}(\tilde{u}, \tilde{w}) \partial_x \hat{u}_{+1} dx$ in $D'(]0,T[)$ by using the convergences (12) and (13) and the definition of the weak* convergence. Let us now prove that $\int_{\Omega_-} \mathcal{A}^\delta(u^\delta, w^\delta) \partial_x w_-^\delta \partial_x \hat{w}_- dxdz$ converges to $\int_\Sigma \mathcal{A}(\tilde{u}, \tilde{w}) \partial_x \tilde{w}_+ \partial_x \hat{w}_+ dx$.

Note that $\int_\Sigma \mathcal{A}(\tilde{u}, \tilde{w}) \partial_x \tilde{w}_+ \partial_x \hat{w}_+ dx = \int_{-1}^{0} \int_\Sigma \mathcal{A}(\tilde{u}, \tilde{w}) \partial_x \tilde{w}_+ \partial_x \hat{w}_+ dxdz$ and that we can write

$$
\begin{aligned}
&\int_{\Omega_-} \left\{ \mathcal{A}^\delta(u^\delta, w^\delta) \partial_x w_-^\delta \partial_x \hat{w}_- - \mathcal{A}(\tilde{u}, \tilde{w}) \partial_x \tilde{w}_+ \partial_x \hat{w}_+ \right\} dxdz \\
&\quad = \int_{\Omega_-} \left\{ \left[\mathcal{A}^\delta(u^\delta, w^\delta) - \mathcal{A}(\tilde{u}, \tilde{w}) \right] \partial_x \tilde{w}_+ \partial_x \hat{w}_+ + \mathcal{A}^\delta(u^\delta, w^\delta) \partial_x w_-^\delta \left[\partial_x \hat{w}_- - \partial_x \hat{w}_+ \right] \right. \\
&\qquad \left. + \mathcal{A}^\delta(u^\delta, w^\delta) \left[\partial_x w_-^\delta - \partial_x \tilde{w}_+ \right] \partial_x \hat{w}_+ \right\} dxdz.
\end{aligned}
$$

Thus, for $\zeta(t) \in D(]0,T[)$, we obtain

$$
\begin{aligned}
&\left| \int_0^T \left(\int_{\Omega_-} \left\{ \mathcal{A}^\delta(u^\delta, w^\delta) \partial_x w_-^\delta \partial_x \hat{w}_- - \mathcal{A}(\tilde{u}, \tilde{w}) \partial_x \tilde{w}_+ \partial_x \hat{w}_+ \right\} dxdz \right) \zeta(t) dt \right| \\
&\qquad \le \left| \int_0^T \int_{\Omega_-} \left[\mathcal{A}^\delta(u^\delta, w^\delta) - \mathcal{A}(\tilde{u}, \tilde{w}) \right] \partial_x \tilde{w}_+ \partial_x \hat{w}_+ \zeta(t) dxdzdt \right| \\
&\qquad\quad + \int_0^T \left\| \mathcal{A}^\delta(u^\delta, w^\delta) \right\|_{L^2(\Omega_-)} \left\| \partial_x w_-^\delta \right\|_{L^6(\Omega_-)} \left\| \partial_x \hat{w}_- - \partial_x \hat{w}_+ \right\|_{L^3(\Omega_-)} |\zeta(t)| \, dt
\end{aligned}
$$

$$+\int_0^T \left\|\mathcal{A}^\delta(u^\delta,w^\delta)\right\|_{L^2(\Omega_-)} \left\|\partial_x w_-^\delta - \partial_x\tilde{w}_+\right\|_{L^6(\Omega_-)} \|\partial_x\hat{w}_+\|_{L^3(\Omega_-)} |\zeta(t)|\,dt$$
$$\leq \left|\int_0^T \int_{\Omega_-} \left[\mathcal{A}^\delta(u^\delta,w^\delta) - \mathcal{A}(\tilde{u},\tilde{w})\right] \partial_x\tilde{w}_+\partial_x\hat{w}_+\zeta(t)dxdzdt\right|$$
$$+ C\left\{\left\|\mathcal{A}^\delta(u^\delta,w^\delta)\right\|_{L^\infty(0,T,L^2(\Omega_-))} \left\|\partial_x w_-^\delta\right\|_{L^\infty(0,T,L^6(\Omega_-))} \|\partial_x\hat{w}_- - \partial_x\hat{w}_+\|_{L^3(\Omega_-)}\right.$$
$$\left. + \left\|\mathcal{A}^\delta(u^\delta,w^\delta)\right\|_{L^\infty(0,T,L^2(\Omega_-))} \left\|\partial_x w_-^\delta - \partial_x\tilde{w}_+\right\|_{L^\infty(0,T,L^6(\Omega_-))} \|\partial_x\hat{w}_+\|_{L^3(\Omega_-)}\right\},$$

where the above inequalities come after many uses of the Hölder inequality: $\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$, for any p, q, such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$; $0 < p < q < +\infty$. Hence, since $\partial_x\tilde{w}_+$ and $\partial_x\hat{w}_+$ belong to $H^1(\Sigma)$, their product is in $L^2(\Sigma)$. This yields $\partial_x\tilde{w}_+\partial_x\hat{w}_+\zeta(t) \in L^1(0,T,L^2(\Sigma))$. Then, we deduce from the convergence (12) and (13) and the definition of the weak * convergence that the first part of the right hand side of the above inequality goes to zero as $\delta \to 0$. Besides, since $\partial_x w_-^\delta \longrightarrow \partial_x\tilde{w}_-$ weakly* in $L^\infty(0,T,H^1(\Omega_-))$, we deduce, using a compactness argument, that $\partial_x w_-^\delta \longrightarrow \partial_x\tilde{w}_-$ strongly in $L^\infty(0,T,H^{1-\varepsilon}(\Omega_-))$ for all $\varepsilon > 0$. Owing to the Sobolev embedding $H^{1-\varepsilon}(\Omega_-) \hookrightarrow L^{\frac{2}{\varepsilon}}(\Omega_-)$, we obtain that $\partial_x w_-^\delta \longrightarrow \partial_x\tilde{w}_- = \partial_x\tilde{w}_{+|\Sigma}$ strongly in $L^\infty(0,T,L^6(\Omega_-))$ as δ goes to zero. This embedding implies also that $\left\|\partial_x w_-^\delta\right\|_{L^\infty(0,T,L^6(\Omega_-))}$ is bounded independently of δ. Combining these results with the fact that $\left\|\mathcal{A}^\delta(u^\delta,w^\delta)\right\|_{L^\infty(0,T,L^2(\Omega_-))}$ and $\|\partial_x\hat{w}_+\|_{L^3(\Omega_-)}$ are bounded independently of δ, we deduce that the remaining parts of the above inequality vanish at the limit.

Using the same arguments, we prove that the other parts involved in $D_-^\delta\delta N_-^\delta(u_-^\delta, w_-^\delta, \hat{u}_-, \hat{w}_-)$ go to zero in $D'(]0,T[)$. Finally, taking into account the above assertions, the passage through the limit in both the variational formulation and the initial conditions leads to the limit approximate model. □

Remark 1. *The limit variational problem obtained above is, formally, equivalent to the following boundary value problem:*

- *Equations of motion in* $\Omega_+ \times (0,T)$*:*

$$\rho_+\tilde{u}_+'' - div\Big\{\mathcal{C}\left[\epsilon(\tilde{u}_+) + f(\nabla\tilde{w}_+)\right]\Big\} + \lambda\nabla\tilde{\phi}_+ = 0,$$
$$\rho_+\left[I - \Delta\right]\tilde{w}_+'' + D_+\Delta^2\tilde{w}_+ - div\Big\{\mathcal{C}\left[\epsilon(\tilde{u}_+) + f(\nabla\tilde{w}_+)\right]\nabla\tilde{w}_+\Big\} + \lambda_+\Delta\tilde{\theta}_+ = 0,$$
$$\rho_+\tilde{\phi}_+' - k_+\Delta\tilde{\phi}_+ + \lambda_+ div\tilde{u}_+' = 0,$$
$$\rho_+\tilde{\theta}_+' - k_+\Delta\tilde{\theta}_+ - \lambda_+\Delta\tilde{w}_+' = 0.$$

- *Clamped boundary conditions on* $\Gamma_+ \times (0,T)$*:*

$$\tilde{u}_+ = 0,\ \tilde{w}_+ = \partial_n\tilde{w}_+ = 0,\ \tilde{\theta}_+ = 0,\ \tilde{\phi}_+ = 0,$$

- *New approximate boundary conditions on* $\Sigma \times (0,T)$*:*

$$
\begin{aligned}
&{}^t\tau\mathcal{C}\left[\epsilon(\tilde{u}_+) + f(\nabla\tilde{w}_+)\right] n = E_-\partial_x\Big(\partial_x\tilde{u}_{+1} + \frac{1}{2}(\partial_x\tilde{w}_+)^2\Big),\\
&{}^t n\mathcal{C}\left[\epsilon(\tilde{u}_+) + f(\nabla\tilde{w}_+)\right] n = 0,\\
&D_+\left[\Delta\tilde{w}_+ + (1-\nu_+)\,B_1\tilde{w}_+\right] = -Q(\tilde{w}_+),\\
&D_+\left[\partial_n\Delta\tilde{w}_+ + (1-\nu_+)\partial_\tau B_2\tilde{w}_+\right] - \rho_+\partial_n\tilde{w}_+'' - C\left[\epsilon(\tilde{u}_+) + f(\nabla\tilde{w}_+)\right] n.\nabla\tilde{w}_+ + \lambda_+\partial_n\tilde{\theta}_+\\
&\qquad = P(\tilde{w}_+) - E_-\partial_x\left[(\partial_x\tilde{u}_{+1} + \frac{1}{2}(\partial_x\tilde{w}_+)^2)\partial_x\tilde{w}_+\right],\\
&k_+\partial_n\tilde{\theta}_+ + \lambda_+\partial_n\tilde{w}_+' = 0, \qquad k_+\partial_n\tilde{\phi}_+ - \lambda_+\tilde{u}_+'n = 0,
\end{aligned}
$$

where:

$$
P(\tilde{w}_+) = E_-\partial_x^4\tilde{w}_+ \quad \textit{and} \quad Q(\tilde{w}_+) = -\frac{2E_-}{1+\nu_-}\partial_x^2\partial_n\tilde{w}_+.
$$

Recall that ${}^t n$ *(resp.* ${}^t\tau$*) is the transposed vector of* n *(resp.* τ*). The above system is associated with the initial conditions (15). Note that the boundary value problem described above follows from the variational problem* $(\mathcal{P}^0)$*. The clamped boundary conditions are incorporated in the spaces* $W(\Omega_+)$ *and* $U(\Omega_+)$*. As far as concern the new approximate boundary conditions, they come naturally by making use of the Green's Formulas in* $(\mathcal{P}^0)$*.*

Remark 2. *As a perspective, the question of existence and uniqueness of the solution of the model obtained here represents a matter of further investigation. One may try to establish the well-posedness of the limit problem by adapting the analysis carried out in [8] for a single plate. The unicity of the solution allows the convergence of the whole sequence.*

Remark 3. *The asymptotic modeling we carried out in this section leads to a new model posed only over the domain* Ω_+*, which is nothing but the reference configuration of the plate. The domain of the layer disappears in the formulation of this new model. Only the mechanical effect of this later appears in the limit problem, which is expressed by the additive terms involved in the right-hand sides of the boundary conditions imposed on the interface* Σ*. Moreover, this effect is also expressed by means of the new initial conditions imposed on* Σ*. These new boundary and initial conditions are in some sense the "memory" of the vanishing layer. Besides, they are not standard: they involve derivatives of order equal to that of the interior differential operator. This type of Boundary conditions is called "Ventcel conditions" in the literature.*

Remark 4. *Note that the choice of the behavior of* α_-^δ *is not fortuitous. When* k_-^δ *varies as* δ *and* E_-^δ *varies as* δ^{-1}*, the coefficient of thermal expansion must behave as* $\delta^{1+\varepsilon}$ *;* $\varepsilon > 0$ *in order to obtain a vanishing thermal effect at the limit.*

4.2 Case of an insulating moderately rigid layer

The goal of this section is to show that an insulating layer may be rigid but not sufficiently enough to induce mechanical effects on the plate. It's the case when the rigidity E_-^δ vary as δ^{-a}, where a is any real number satisfying $0 < a < 1$. To this end, we make the assumption that

$E_-^\delta = \delta^{-a} E_-$ and let the other physical coefficients vary as for the insulating high rigid stiffener case.

Owing to the a priori estimates stated in Proposition 1, we can assert that, up to a subsequence, there exist $\tilde{w}_+$, $\tilde{u}_+$, $\tilde{\phi}_+$ and $\tilde{\theta}_+$ such that

- $w_+^\delta \longrightarrow \tilde{w}_+$, $u_+^\delta \longrightarrow \tilde{u}_+$, weakly* in $L^\infty(0,T;H^2(\Omega_+))$ and $L^\infty(0,T;(H^1(\Omega_+))^2)$ respectively.
- $\phi_+^\delta \longrightarrow \tilde{\phi}_+$, $\theta_+^\delta \longrightarrow \tilde{\theta}_+$ weakly* in $L^\infty(0,T,L^2(\Omega_+))$ and weakly in $L^2(0,T,H^1(\Omega_+))$.

Remark 5. *Unlike the high stiff layer case, the a priori estimates of Proposition 1 do not lead to the boundedness of w_-^δ and u_-^δ in $L^\infty(0,T;H^2(\Omega_-))$ and $L^\infty(0,T;(H^1(\Omega_-))^2)$ respectively. In this situation, it's $\sqrt{\delta^{1-a}E_-}w_-^\delta$ and $\sqrt{\delta^{1-a}E_-}u_-^\delta$ that benefit from this property. Nevertheless, the forthcoming analysis requires some estimates about the displacement components of the solution. Thus, we next establish some auxiliary results and statements which play a key role in the identification of the limit problem.*

Lemma 1. *Let $v \in H^1(\Omega_-)$. The following estimate holds true:*

$$\|v\|^2_{L^2(\Omega_-)} \leq C\left[\|\partial_z v\|^2_{L^2(\Omega_-)} + \left\|v_{|\Sigma}\right\|^2_{L^2(\Sigma)}\right],$$

where C is a constant independent of δ.

Proof. For all z, $-1 < z < 0$, we can write

$$v^2(x,z) = v^2(x,0) - 2\int_z^0 v(x,\xi)\partial_\xi v(x,\xi)d\xi.$$

Thus, using Holder inequality, it follows

$$\begin{aligned}\int_0^1 \left|v^2(x,z)\right| dx &\leq \int_0^1 \left|v^2(x,0)\right| dx + 2\int_0^1\int_{-1}^0 |v\partial_z v|\, dxdz \\ &\leq \left\|v_{|\Sigma}\right\|^2_{L^2(\Sigma)} + 2\|v\|_{L^2(\Omega_-)}\|\partial_z v\|_{L^2(\Omega_-)}.\end{aligned}$$

Integrating from -1 to 0, with respect to the variable z, and owing to the generalized Young's inequality: $ab < \dfrac{a^2}{2\varepsilon} + \dfrac{\varepsilon b^2}{2}$, for $\varepsilon = 2$, we get the desired estimate. □

Proposition 4. *Let $v \in W^\delta(\Omega)$. We have:*

$$\begin{aligned}&\|\partial_z v_-\|^2_{L^2(\Omega_-)} \leq C\left[\left\|\partial_z^2 v_-\right\|^2_{L^2(\Omega_-)} + \delta^2\left\|\partial_n v_{+|\Sigma}\right\|^2_{L^2(\Sigma)}\right], \\ &\|v_-\|^2_{L^2(\Omega_-)} \leq C\left[\left\|\partial_z^2 v_-\right\|^2_{L^2(\Omega_-)} + \|v_+\|^2_{H^2(\Omega_+)}\right],\end{aligned}$$

where C is a constant independent of δ.

Proof. v_- being in $H^2(\Omega_-)$, we apply the previous lemma for $\partial_z v_-$ and deduce

$$\|\partial_z v_-\|^2_{L^2(\Omega_-)} \leq C \left[\left\|\partial_z^2 v_-\right\|^2_{L^2(\Omega_-)} + \left\|\partial_z v_{-|\Sigma}\right\|^2_{L^2(\Sigma)} \right].$$

Making use of the transmission condition $\partial_z v_{-|\Sigma} = \delta \partial_n v_{+|\Sigma}$, we get the first estimate stated in this proposition. Finally, since $\delta^2 << 1$, the second estimate follows by applying once again the Lemma and taking advantage of the transmission conditions and the continuity of the trace operator. □

Proposition 5. *The component w_-^δ verifies*

$$\left\|w_-^\delta\right\|_{L^\infty(0,T,L^2(\Omega_-))} \leq C,$$

where C is a constant independent of δ.

Proof. The a priori estimates stated in Proposition 1 assert that $\sqrt{\delta^{1-a}E_-}\delta^{-2}\partial_z^2 w_-^\delta$ is bounded independently of δ in $L^\infty(0,T,L^2(\Omega_-))$, which implies that $\dfrac{1}{\sqrt{\delta^{a+3}}}\left\|\partial_z^2 w_-^\delta\right\|_{L^\infty(0,T,L^2(\Omega_-))} \leq C$ and consequently, $\left\|\partial_z^2 w_-^\delta\right\|_{L^\infty(0,T,L^2(\Omega_-))} \leq C\sqrt{\delta^{a+3}} \leq C'$, since $\delta << 1$ (C and C' are constants independent of δ). Besides, w_+^δ is bounded in $L^\infty(0,T,H^2(\Omega_+))$. The conclusion follows from Proposition 4. □

In order to pass through the limit in (7), we need the following lemma (which can easily be proved):

Lemma 2. *Let $v^\delta \in L^\infty(0,T,H^1(\Omega_-))$. If $v^\delta \longrightarrow \tilde{v}$ weakly* in $L^\infty(0,T,L^2(\Omega_-))$ and for $0 < a < 1$, $\sqrt{\delta^{1-a}E_-}\partial_x v^\delta \longrightarrow \tilde{h}$ weakly* in $L^\infty(0,T,L^2(\Omega_-))$, then $\tilde{h} = 0$.*

Theorem 2. *The weak limit $(\tilde{u}_+, \tilde{w}_+, \tilde{\phi}_+, \tilde{\theta}_+)$ verifies:*

- $\tilde{w}_+ \in L^\infty(0,T,H^2_{|\Gamma_+}(\Omega_+))$, $(\tilde{w}_+)' \in L^\infty(0,T,H^1_{|\Gamma_+}(\Omega_+))$,
- $\tilde{u}_+ \in L^\infty(0,T,(H^1_{|\Gamma_+}(\Omega_+))^2)$, $(\tilde{u}_+)' \in L^\infty(0,T,[L^2(\Omega_+)]^2)$,
- $\tilde{\theta}_+, \tilde{\phi}_+ \in L^\infty(0,T,L^2(\Omega_+)) \cap L^2(0,T,H^1_{|\Gamma_+}(\Omega_+))$, *and solve the variational problem*

$$(\tilde{\mathcal{P}}_0) \begin{cases} \rho_+\langle \tilde{u}'_+, \hat{u}_+\rangle'_{\Omega_+} + \rho_+\langle \tilde{w}'_+, \hat{w}_+\rangle'_{\Omega_+} + \rho_+ b_+(\tilde{w}'_+, \hat{w}_+)' + a_+(\tilde{w}_+, \hat{w}_+) + \lambda_+ c_+(\tilde{\phi}_+, \hat{u}_+) \\ -\lambda_+ b_+(\tilde{\theta}_+, \hat{w}_+) + \rho_+\langle \tilde{\phi}_+, \hat{\phi}_+\rangle'_{\Omega_+} + k_+ b_+(\tilde{\phi}_+, \hat{\phi}_+) - \lambda_+ d_+(\tilde{u}'_+, \hat{\phi}_+) \\ +\rho_+\langle \tilde{\theta}_+, \hat{\theta}_+\rangle'_{\Omega_+} + \lambda_+ b_+(\tilde{w}'_+, \hat{\theta}_+) + k_+ b_+(\tilde{\theta}_+, \hat{\theta}_+) + N_+(\tilde{u}_+, \tilde{w}_+, \hat{u}_+, \hat{w}_+) = 0, \end{cases}$$

for all $(\hat{u}_+, \hat{w}_+, \hat{\phi}_+, \hat{\theta}_+) \in (H^1_{|\Gamma_+}(\Omega_+))^2 \times H^2_{|\Gamma_+}(\Omega_+) \times H^1_{|\Gamma_+}(\Omega_+) \times H^1_{|\Gamma_+}(\Omega_+)$, with the initial conditions:

$$\tilde{u}_+(0) = u_+^*,\ \tilde{u}'_+(0) = u_+^{**},\ \tilde{w}_+(0) = w_+^*, \quad in\ \ \Omega_+ \tag{16}$$

$$\tilde{w}'_+(0) = w_+^{**}, \tilde{\phi}_+(0) = \phi_+^*,\ \tilde{\theta}_+(0) = \theta_+^* \qquad in\ \ \Omega_+. \tag{17}$$

Proof. Let $\hat{u}_+ \in D(\Omega_+) \cap (H^1_{|\Gamma_+}(\Omega_+))^2$, $\hat{w}_+ \in D(\Omega_+) \cap H^2_{|\Gamma_+}(\Omega_+)$, $\hat{\phi}_+ \in D(\Omega_+) \cap H^1_{|\Gamma_+}(\Omega_+)$ and $\hat{\theta}_+ \in D(\Omega_+) \cap H^1_{|\Gamma_+}(\Omega_+)$. We apply the variational problem (7) with the test functions:

$$\hat{u} = \begin{cases} \hat{u}_+ & \text{in } \Omega_+, \\ \hat{u}_- = \left(\hat{u}_{+1|\Sigma}, \delta \hat{u}_{+2|\Sigma}\right) & \text{in } \Omega_-, \end{cases} \qquad \hat{w} = \begin{cases} \hat{w}_+ & \text{in } \Omega_+, \\ \hat{w}_- = \hat{w}_{+|\Sigma} + \delta z \partial_n \hat{w}_{+|\Sigma} & \text{in } \Omega_-, \end{cases}$$

$$\hat{\phi} = \begin{cases} \hat{\phi}_+ & \text{in } \Omega_+, \\ \hat{\phi}_- = \hat{\phi}_{+|\Sigma} & \text{in } \Omega_-, \end{cases} \qquad \hat{\theta} = \begin{cases} \hat{\theta}_+ & \text{in } \Omega_+, \\ \hat{\theta}_- = \hat{\theta}_{+|\Sigma} & \text{in } \Omega_-. \end{cases}$$

In the limit study of the problem $(\mathcal{P}^\delta)$, only the behavior of the forms $D^\delta_- \delta a^\delta_-(w^\delta_-, \hat{w}_-)$ and $D^\delta_- \delta N^\delta_-(u^\delta_-, w^\delta_-, \hat{u}_-, \hat{w}_-)$ change from the high stiff layer case. Using same arguments as for the justification of the limit of $k^\delta_- \delta b^\delta_-(\theta^\delta_-, \hat{\theta}_-)$ in the proof of Theorem 1, we can show that $D^\delta_- \delta a^\delta_-(w^\delta_-, \hat{w}_-)$ goes to zero, as δ goes to zero in $D'(0,T)$. Let us focus on the behavior of $D^\delta_- \delta N^\delta_-(u^\delta_-, w^\delta_-, \hat{u}_-, \hat{w}_-)$ and prove that its limit is equal to zero in $D'(0,T)$. We start by investigating the first part of this form. Using the same notations as for the proof of Theorem 1, recalling the fact that $D^\delta_- \delta = \dfrac{\delta^{1-a} E_-}{1-\nu_-^2}$, using Hölder inequalities, we obtain, for $\zeta(t) \in D(]0,T[)$:

$$\begin{aligned} &\left| \int_0^T \left(D^\delta_- \delta \int_{\Omega_-} \mathcal{A}^\delta(u^\delta, w^\delta) \partial_x w^\delta_- \partial_x \hat{w}_- dx dz \right) \zeta(t) dt \right| \\ &= \left| \int_0^T \left(\frac{1}{1-\nu_-^2} \int_{\Omega_-} \left(\sqrt{\delta^{1-a} E_-} \mathcal{A}^\delta(u^\delta, w^\delta) \right) \left(\sqrt{\delta^{1-a} E_-} \partial_x w^\delta_- \right) \partial_x \hat{w}_- dx dz \right) \zeta(t) dt \right| \\ &\leq \frac{C}{1-\nu_-^2} \left\| \sqrt{\delta^{1-a} E_-} \mathcal{A}^\delta(u^\delta, w^\delta) \right\|_{L^\infty(0,T,L^2(\Omega_-))} \left\| \sqrt{\delta^{1-a} E_-} \partial_x w^\delta_- \right\|_{L^\infty(0,T,L^6(\Omega_-))} \left\| \partial_x \hat{w}_{+|\Sigma} \right\|_{L^3(\Omega_-)}, \end{aligned}$$

where C is a positive constant independent of δ. Owing to the Proposition 1, we deduce that $\left\| \sqrt{\delta^{1-a} E_-} \mathcal{A}^\delta(u^\delta, w^\delta) \right\|_{L^\infty(0,T,L^2(\Omega_-))}$ is bounded independently of δ. Besides, $\left\| \partial_x \hat{w}_{+|\Sigma} \right\|_{L^3(\Omega_-)}$ doesn't depend on δ. It remains then to show that the limit of $\left\| \sqrt{\delta^{1-a} E_-} \partial_x w^\delta_- \right\|_{L^\infty(0,T,L^6(\Omega_-))}$ vanish as δ goes to zero, which will essentially be done by making use of Lemma 2. Indeed, the Proposition 5 shows that w^δ_- is bounded independently of δ in $L^\infty(0,T,L^2(\Omega_-))$, thus, up to a subsequence, $w^\delta_- \longrightarrow w_0$ weakly* in $L^\infty(0,T,L^2(\Omega_-))$. Moreover, from the a priori estimates of Proposition 1, we get that $\sqrt{\delta^{1-a} E_-} w^\delta_-$ is bounded in $L^\infty(0,T,H^2(\Omega_-))$, which implies the boundedness of $\sqrt{\delta^{1-a} E_-} \partial_x w^\delta_-$ in $L^\infty(0,T,H^1(\Omega_-))$. So, up to a subsequence, $\sqrt{\delta^{1-a} E_-} \partial_x w^\delta_- \longrightarrow w_1$ weakly* in $L^\infty(0,T,H^1(\Omega_-))$. Using a compactness argument, we obtain $\sqrt{\delta^{1-a} E_-} \partial_x w^\delta_- \longrightarrow w_1$ strongly in $L^\infty(0,T,H^{1-\varepsilon}(\Omega_-))$ for all $\varepsilon > 0$. By the Sobolev embedding $H^{1-\varepsilon}(\Omega_-) \hookrightarrow L^{\frac{2}{\varepsilon}}(\Omega_-)$, we deduce that $\sqrt{\delta^{1-a} E_-} \partial_x w^\delta_- \longrightarrow w_1$ strongly in $L^\infty(0,T,L^p(\Omega_-))$, for all $p \geq 1$. Using Lemma 2, we deduce that $w_1 = 0$ and consequently, the right-hand side of the above inequality goes to zero as $\delta \to 0$.

In the same manner, using same arguments, we show that all the other parts of the nonlinear form $D^\delta_- \delta N^\delta_-(u^\delta_-, w^\delta_-, \hat{u}_-, \hat{w}_-)$ goes to zero as $\delta \to 0$. By adopting a similar formulation as for the inequality above, it suffices to show, in addition, that $\sqrt{\delta^{1-a} E_-} \delta^{-1} \partial_z w^\delta_-$ converges to zero in $L^\infty(0,T,L^6(\Omega_-))$ when $\delta \to 0$, which follows from the Proposition 2, the consequences of Proposition 1 and from the Sobolev embeddings. □

Remark 6. *As far as concern the boundary value problem associated with the above asymptotic model, it is described by the same equations of motion on Ω_+ and clamped boundary conditions on Γ_+ as those identified in the case of a high rigid stiffener. Nevertheless, the difference lies in the limit initial conditions (16) and the new approximate boundary conditions on the interface Σ, which in the case of the moderately rigid stiffener read:*

- ***New approximate boundary conditions** on $\Sigma \times (0,T)$:*

$$
\begin{aligned}
&{}^t\tau\mathcal{C}\left[\epsilon(\tilde{u}_+)+f(\nabla\tilde{w}_+)\right]n=0, \qquad {}^t n\mathcal{C}\left[\epsilon(\tilde{u}_+)+f(\nabla\tilde{w}_+)\right]n=0,\\
&D_+\left[\Delta\tilde{w}_+ + (1-\nu_+)B_1\tilde{w}_+\right]=0,\\
&D_+\left[\partial_n\Delta\tilde{w}_+ + (1-\nu_+)\partial_\tau B_2\tilde{w}_+\right]-\rho_+\partial_n\tilde{w}_+'' - C\left[\epsilon(\tilde{u}_+)+f(\nabla\tilde{w}_+)\right]n.\nabla\tilde{w}_+ + \lambda_+\partial_n\tilde{\theta}_+=0,\\
&k_+\partial_n\tilde{\theta}_+ + \lambda_+\partial_n\tilde{w}_+'=0, \qquad k_+\partial_n\tilde{\phi}_+ - \lambda_+\tilde{u}_+' n=0.
\end{aligned}
$$

Note that in the expression of these new boundary conditions, neither the Thermal effect nor the mechanical effect of the stiffener are taken into account. The influence of the layer is here asymptotically negligible in comparison with the effect obtained for the high stiff layer case. This conclusion is in accordance with the physical intuition: the material constituting the coating is not sufficiently rigid, so its effect on the deformation of the plate vanish.

5 Conclusion

In this paper, we have modelled the behavior of reinforced plate with an insulating thin stiff layer on a portion of its boundary. Using a variational asymptotic approach, we have studied the asymptotic behavior of this structure when the thickness of the stiffener approaches zero. We have identified asymptotic models that reflect the effect of this latter on the plate, according to its rigidity: very stiff or moderately stiff. In both cases, the thermal effect of this inserted body doesn't appear in the asymptotic model, which is in concordance with the physical intuition, because its constitutive material is insulating. Nevertheless, its mechanical effect is taken into account only when it is very stiff. For a moderately rigid material, neither thermal nor mechanical effect of the stiffener appear in the limit model. As a perspective, the question of well-posedness of the asymptotic model obtained in this paper, which is rather delicate, represents a matter of further investigation and elaboration in a forthcoming paper. The idea consists in trying to adapt the analysis carried out in [8], for a single plate. Besides, one of challenging other questions, which motivated the modeling we have carried out here is also the numerical analysis of the obtained model. Finally, let us notice that the asymptotic analysis we have described here is of wide applicability: many extensions may be made to other elastic models and other geometrical configurations.

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