Journal of Mathematical Modeling
Vol. 9, No. 3, 2021, pp. 501-516

JMM

A computational method based on Legendre wavelets for solving distributed order fractional diffrential equations

Parisa Rahimkhani, Yadollah Ordokhani*

Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran

Email(s): P.rahimkhani@alzahra.ac.ir, ordokhani@alzahra.ac.ir

Abstract. In the current investigation, the distributed order fractional derivative operational matrix based on the Legendre wavelets (LWs) as the basis functions is derived. This operational matrix is applied together with collocation method for solving distributed order fractional differential equations. Also, convergence analysis of the proposed scheme is given. Finally, numerical examples are presented to show the efficiency and superiority of the mentioned scheme.

Keywords: Legendre wavelets, distributed order fractional diffrential equations, Numerical method, operational matrix.

AMS Subject Classification 2010: 34K28, 68Q85, 65M70.

1 Introduction

The distributed order fractional differential equations (DOFDEs) are used as a source of various mathematical physics, chemistry, biology and engineering problems such as viscoelastic model [1, 12], control systems [34], distributed order oscillator [2], complex system [4, 15], diffusion [5] etc. (The reader can find the concept of the distributed order in the works of Caputo [8, 9]). The existence and the uniqueness of solution of DOFDEs have been studied in [3, 7, 11, 25]. While different approximate approaches have been introduced for solving of fractional differential equations (FDEs) (see e.g., [28,29,31]), there has been less researches into the study of DOFDEs. Most of DOFDEs do not have analytic solutions, so numerical techniques are used for solving DOFDEs. For example, we can mention Petrov-Galerkin scheme [19], meshless scheme [6], Chebyshev collocation scheme [24], Legendre spectral element scheme [10], classical numerical quadrature scheme [21], Bernoulli hybrid functions [22], Laplace and Fourier transforms [16], block pulse function [13], and Legendre operational matrix [14].

*Corresponding author.
Received: 13 January 2021 / Revised: 3 February 2021 / Accepted: 5 February 2021
DOI: 10.22124/jmm.2021.18634.1596

In this work, we adopt the following DOFDEs as

$$\int_{\beta_1}^{\beta_2} \rho(\alpha) D_t^{\alpha} z(t) d\alpha = g(t), \tag{1}$$

with the initial conditions

$$z^{(n)}(0) = \delta_n, \qquad n = 0, \dots, \lceil \beta_2 \rceil - 1, \tag{2}$$

where $\lceil . \rceil$ is the ceiling function.

Wavelets are a special type of functions with different properties such as: compact support, orthogonality, exact representation of polynomials to a certain degree and ability to represent functions at different levels of resolution [14]. In recent years, different wavelets operational matrices for both fractional differentiation and fractional integration have been calculated for solving various kinds of FDEs, for instance operational matrices of fractional-order Legendre wavelets [30], Müntz-Legendre wavelets [27], Bernoulli wavelets [26], Chebyshev wavelets [20], Haar wavelets [32], etc. The operational matrix scheme is usually combined with schemes such as spectral tau scheme, collocation scheme and tau scheme for finding numerical solution of FDEs.

The main goal of this work is to calculate the distributed order fractional derivative operational matrix for the Legendre wavelets. By using this matrix and collocation scheme, we reduce the solution of the Eqs. (1) and (2) to the solution of algebraic equations.

The remainder of this article is organized as follows. In Section 2, we remind some basic definitions of the fractional calculus and Legendre wavelets. In Section 3, distributed order fractional derivative operational matrix of Legendre wavelets is derived. In Section 4, we apply this operational matrix and collocation method for finding numerical solution of DOFDEs. In Section 5, we investigate the error analysis for our method. Section 6 demonstrates results of the proposed technique. In Section 7, we propose a conclusion.

2 Preliminaries and fundamentals

2.1 Fractional calculus

Definition 1. *[29] The Riemann-Liouville fractional integral with order ν is introduced as*

$$I_t^{\nu} z(t) = \tfrac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} z(\tau) d\tau, \qquad t > 0.$$

Definition 2. *[29] The Caputo fractional derivative of order ν is introduced as*

$$D_t^{*\nu} z(t) = \tfrac{1}{\Gamma(n-\nu)} \int_0^t (t-\tau)^{n-\nu-1} z^{(n)}(\tau) d\tau, \qquad n-1 < \nu < n.$$

Proposition 1. *[29] The Caputo fractional derivatives and the Riemann-Liouville fractional integrals satisfy:*
1. $D_t^{*\nu} I_t^{\nu} z(t) = z(t)$,
2. $I_t^{\nu} D_t^{*\nu} z(t) = z(t) - \sum_{i=0}^{n-1} z^{(i)}(0) \frac{t^i}{i!}$,
3. $D_t^{*\nu} z(t) = I_t^{n-\nu} D_t^{*n} z(t)$,

4. $D_t^{*\nu}(\lambda z_1(t) + \theta z_2(t)) = \lambda D_t^{*\nu} z_1(t) + \theta D_t^{*\nu} z_2(t)$,

5. $D_t^{*\nu} t^{\beta} = \begin{cases} 0, & \nu \in N_0,\ \beta < \nu, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} t^{\beta-\nu}, & \textit{otherwise}, \end{cases}$

6. $D_t^{*\nu}\lambda = 0$,

where λ, θ are real constants and $n-1 < \nu \leq n$.

2.2 The distributed order differential

Definition 3. *[17] The distributed order fractional differential is defined by means of*

$$D_t^{*\rho(\alpha)} z(t) = \int_{\beta_1}^{\beta_2} \rho(\alpha) D_t^{*\alpha} z(t) d\alpha, \tag{3}$$

where $\rho(\alpha)$ is the weight function of distribution with order α that is a non-negative and a generalized function. Also, β_1 and β_2 are non-negative real numbers.

Proposition 2. *The operator $D_t^{*\rho(\alpha)}$ satisfies the following properties*

- $D_t^{*\rho(\alpha)} \left(\sum_{k=1}^{\ell} \theta_\ell z_\ell(t) \right) = \sum_{k=1}^{\ell} \theta_\ell D_t^{*\rho(\alpha)} z_\ell(t)$,
- $D_t^{*\rho(\alpha)} \lambda = 0$,

where $\{\theta_\ell\}_{\ell=1}^{k}$ and λ are constants.

Remark 1. *If $\rho(\alpha) = \delta(\alpha - \mu), \beta_1 < \mu < \beta_2$, then we have*

$$D_t^{*\rho(\alpha)} z(t) = \int_{\beta_1}^{\beta_2} \delta(\alpha - \mu) D_t^{*\alpha} z(t) d\alpha = D_t^{*\mu} z(t),$$

where δ is the Dirac delta function.

2.3 Legendre wavelets

The Legendre wavelets $\psi_{nm}(t)$ is introduced over interval $[0, 1)$ as [23]

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2m+1} 2^{\frac{k-1}{2}} L_m(2^{k-1}t - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $m = 0, 1, 2, \ldots, M-1$; $n = 1, 2, \ldots, 2^{k-1}$ and $L_m(t)$ are the Legendre polynomials of order m which satisfy the recurrence relation

$$\begin{aligned} L_{m+1}(t) &= \frac{(2m+1)(2t-1)}{m+1} L_m(t) - \frac{m}{m+1} L_{m-1}(t), \qquad m = 1, 2, \ldots, \\ L_0(t) &= 1, \qquad L_1(t) = 2t - 1. \end{aligned}$$

The Legendre polynomials are orthogonal with

$$\int_0^1 L_m(t)L_n(t)dt = \begin{cases} \frac{1}{2m+1}, & m = n, \\ 0, & m \neq n. \end{cases}$$

Also, the Legendre polynomials have the following explicit analytic form

$$L_m(t) = \sum_{r=0}^{m} (-1)^{m+r} \frac{(m+r)!}{(m-r)!(r!)^2} t^r.$$

Any function $z(t)$ over $[0,1)$ can be approximated by means of the Legendre wavelets as follows

$$z(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{5}$$

where the coefficients c_{nm} are given by

$$c_{nm} = < z, \psi_{nm} > = \int_0^1 f(t)\psi_{nm}(t)dt, \tag{6}$$

where $<,>$ denotes the inner product in $L^2[0,1]$. If the infinite series in Eq. (5) is truncated, then we get

$$z(t) \simeq z_{\hat{m}}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T \Psi(t),$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$, vectors given by

$$\begin{aligned} C &= [c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}]^T \\ &= [c_1, c_2, c_3, \ldots, c_{\hat{m}}]^T, \qquad \hat{m} = 2^{k-1}M, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \Psi(t) &= [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1(M-1)}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}(M-1)}(t)]^T \\ &= [\psi_1(t), \psi_2(t), \psi_3(t), \ldots, \psi_{\hat{m}}(t)]^T. \end{aligned} \tag{8}$$

2.4 Transformation matrix of Legendre wavelets to piecewise Taylor functions

The piecewise Taylor functions (PTFs) is defined on $[0,1)$ as [23]

$$\phi_{n,m}(t) = \begin{cases} t^m, & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where $m = 0, 1, 2, \ldots, M-1$ and $n = 1, 2, \ldots, 2^{k-1}$. The LWs can be expanded into $\hat{m} = 2^{k-1}M$ set of the PTFs as

$$\Psi_{\hat{m}\times 1}(t) = P^{-1}_{\hat{m}\times\hat{m}} \Phi_{\hat{m}\times 1}(t), \tag{10}$$

where P^{-1} is the transformation matrix of the LWs to the PTFs. This matrix is obtained in [23] as

$$\phi_i = \sum_{j=1}^{\hat{m}} = p_{ij}\psi_j(t), \qquad i = 1, 2, \ldots, \hat{m},$$

where

$$\begin{aligned} \Phi(t) &= [\phi_{10}(t), \phi_{11}(t), \ldots, \phi_{1(M-1)}(t), \ldots, \phi_{2^{k-1}0}(t), \ldots, \phi_{2^{k-1}(M-1)}(t)]^T \\ &= [\phi_1(t), \phi_2(t), \phi_3(t), \ldots, \phi_{\hat{m}}(t)]^T. \end{aligned} \tag{11}$$

and $p_{ij} =< \phi_i, \psi_j >$, $i, j = 1, 2, \ldots, \hat{m}$.

3 Derivative operators

3.1 Distributed order fractional derivative operator

In this section, we construct distributed order fractional derivative operator of LWs, for this aim, first we present the following Lemmas.

Lemma 1. *Let η_j and ω_j be the nodes and the weights of the Legendre-Gauss quadrature rule, respectively. Then the following relation for $i \in \mathbb{N}, i \geq \lceil \beta_2 \rceil$ holds:*

$$D_t^{*\rho(\alpha)} t^i = \frac{\beta_2 - \beta_1}{2} \sum_{j=1}^{s} \omega_j \rho(\frac{\beta_2 - \beta_1}{2}\eta_j + \frac{\beta_2 + \beta_1}{2}) \frac{i! t^{i - \frac{\beta_2 - \beta_1}{2}\eta_j - \frac{\beta_2 + \beta_1}{2}}}{\Gamma(i + 1 - \frac{\beta_2 - \beta_1}{2}\eta_j - \frac{\beta_2 + \beta_1}{2})}. \tag{12}$$

Proof. By applying Eqs. (3) and the Legendre-Gauss quadrature formula, yield

$$\begin{aligned} D_t^{*\rho(\alpha)} t^i &= \int_{\beta_1}^{\beta_2} \rho(\alpha) D_t^{*\alpha} t^i d\alpha = \int_{\beta_1}^{\beta_2} \rho(\alpha) \frac{i! t^{i-\alpha}}{\Gamma(i + 1 - \alpha)} d\alpha \\ &= \frac{\beta_2 - \beta_1}{2} \int_{-1}^{1} \rho(\frac{\beta_2 - \beta_1}{2}\alpha' + \frac{\beta_2 + \beta_1}{2}) \frac{i! t^{i - \frac{\beta_2 - \beta_1}{2}\alpha' - \frac{\beta_2 + \beta_1}{2}}}{\Gamma(i + 1 - \frac{\beta_2 - \beta_1}{2}\alpha' - \frac{\beta_2 + \beta_1}{2})} d\alpha' \\ &= \frac{\beta_2 - \beta_1}{2} \sum_{j=1}^{s} \omega_j \rho(\frac{\beta_2 - \beta_1}{2}\eta_j + \frac{\beta_2 + \beta_1}{2}) \frac{i! t^{i - \frac{\beta_2 - \beta_1}{2}\eta_j - \frac{\beta_2 + \beta_1}{2}}}{\Gamma(i + 1 - \frac{\beta_2 - \beta_1}{2}\eta_j - \frac{\beta_2 + \beta_1}{2})}, \end{aligned}$$

which completes the proof. □

Lemma 2. *For $i = 0, 1, \ldots, \lceil \beta_2 \rceil - 1$, we get $D_t^{*\rho(\alpha)} t^i = 0$.*

Proof. The proof is established by Eq. (3) and Lemma 1. □

Now, by considering Lemmas 1, 2 and Eq. (10), we get

$$D_t^{*\rho(\alpha)} \Psi(t) = R(\beta_1, \beta_2, \rho(\alpha), t) = P^{-1} \Lambda(\beta_1, \beta_2, \rho(\alpha), t), \tag{13}$$

where $R(\beta_1, \beta_2, \rho(\alpha), t)$ and $\Lambda(\beta_1, \beta_2, \rho(\alpha), t)$ are distributed order fractional derivative operators of the LWs and the PTFs, respectively.

The Distributed order fractional derivative operator of the PTFs is given as:

$$\begin{aligned}\Lambda(\beta_1, \beta_2, \rho(\alpha), t) = & [0, 0, \dots, \theta(\lceil\beta_2\rceil), \dots, \theta(M-1), 0, 0, \dots, \theta(\lceil\beta_2\rceil), \\ & \dots, \theta(M-1), \dots, 0, 0, \dots, \theta(\lceil\beta_2\rceil), \dots, \theta(M-1)],\end{aligned} \tag{14}$$

where

$$\theta(i) = \frac{\beta_2 - \beta_1}{2} \sum_{j=1}^{s} \omega_j \rho(\frac{\beta_2 - \beta_1}{2}\eta_j + \frac{\beta_2 + \beta_1}{2}) \frac{i! t^{i - \frac{\beta_2 - \beta_1}{2}\eta_j - \frac{\beta_2 + \beta_1}{2}}}{\Gamma(i + 1 - \frac{\beta_2 - \beta_1}{2}\eta_j - \frac{\beta_2 + \beta_1}{2})}, \tag{15}$$

for $i = \lceil\beta_2\rceil, \dots, M-1$.

3.2 Integer-order derivative operator

Now, we obtain integer-order derivative operator of Legendre wavelets $D^*(n, t)$ as

$$D_t^{(n)}\Psi(t) = D^*(n, t) = P^{-1}D(n, t), \tag{16}$$

where $D(n, t)$ is integer-order derivative operator of the piecewise Taylor functions. This operator is obtained as

$$\begin{aligned}D(n, t) = & [0, 0, \dots, \tau(n), \dots, \tau(M-1), 0, 0, \dots, \tau(n), \dots, \\ & \tau(M-1), \dots, 0, 0, \dots, \tau(n), \dots, \tau(M-1)],\end{aligned} \tag{17}$$

where $\tau(i) = \prod_{j=0}^{n-1}(i-j)t^{i-n}$, $i = n, n+1, \dots, M-1$.

4 Implementation of the scheme to the DOFDEs

Consider distributed order fractional differential equations of the form

$$\int_{\beta_1}^{\beta_2} \rho(\alpha) D_t^{*\alpha} z(t) d\alpha = g(t), \tag{18}$$

with the initial conditions

$$z^{(n)}(0) = \delta_n, \qquad n = 0, \dots, \lceil\beta_2\rceil - 1. \tag{19}$$

For finding numerical solution of problem (18) - (19), we approximate $z(t)$ by LWs as

$$z(t) \simeq z_{\hat{m}}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t). \tag{20}$$

By combing Eq. (13) with Eq. (20), the following relation is obtained:

$$D_t^{*\rho(\alpha)} z(t) \simeq D_t^{*\rho(\alpha)} z_{\hat{m}}(t) = C^T D_t^{*\rho(\alpha)}\Psi(t) = C^T R(\beta_1, \beta_2, \rho(\alpha), t). \tag{21}$$

Also, Eq. (18) can be written in the following relation:

$$C^T R(\beta_1, \beta_2, \rho(\alpha), t) - g(t) = 0. \tag{22}$$

Now, we collocate Eq. (22) at the $\hat{m} - \lceil \beta_2 \rceil$ zeros of shifted Legendre polynomials as

$$C^T R(\beta_1, \beta_2, \rho(\alpha), t_i) - g(t_i) = 0, \qquad i = 1, 2, \ldots, \hat{m} - \lceil \beta_2 \rceil. \tag{23}$$

Also, using Eqs. (16), (19) and (20), we obtain

$$Z^{(n)}(0) \simeq C^T D^*(n, 0) = \delta_n, \qquad n = 0, 1, \ldots, \lceil \beta_2 \rceil - 1. \tag{24}$$

Hence, Eqs. (23) and (24) create a system of $\hat{m}$ algebraic equations. This system can be solved using Newton's iterative scheme for finding the unknown vector C. Thus function $z(t)$ calculated by Eq. (20).

5 Error analysis

In this part, we derive the error estimates of numerical approximation using Legendre wavelets. Let $z_{\hat{m}}(t)$ is the approximation of $z(t)$ as:

$$z_{\hat{m}}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t). \tag{25}$$

Therefore,

$$z(t) - z_{\hat{m}}(t) = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm} \psi_{nm}(t). \tag{26}$$

Theorem 1. *Suppose that function $z(t)$ is approximated by truncated Legendre wavelets series $z_{\hat{m}}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t)$ and $|z''(t)| < Q$ (Q is a finite constant), then the upper bound of error can be obtained as follows:*

$$\|z(t) - z_{\hat{m}}(t)\|_2 < \sqrt{\frac{3Q^2}{8} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5 (2m-3)^4}}. \tag{27}$$

Proof. By using the orthonormality of the Legendre wavelets, we ge

$$\begin{aligned} \|z(t) - z_{\hat{m}}(t)\|_2^2 &= \int_0^1 |z(t) - z_{\hat{m}}(t)|^2 dt = \int_0^1 \left| \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}\psi_{nm}(t) \right|^2 dt \\ &\leq \int_0^1 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}\psi_{nm}(t)|^2 dt = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \int_0^1 |c_{nm}|^2 |\psi_{nm}(t)|^2 dt \\ &= \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2 \int_0^1 |\psi_{nm}(t)|^2 dt = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2. \end{aligned} \tag{28}$$

From Eqs. (4) and (6), we obtain

$$c_{nm} = \int_0^1 z(t)\psi_{nm}(t)dt = \int_{\frac{\hat{n}}{2^{k-1}}}^{\frac{\hat{n}+1}{2^{k-1}}} z(t)\sqrt{2m+1}2^{\frac{k-1}{2}}L_m(2^{k-1}t-\hat{n})dt. \tag{29}$$

Now, we let $x = 2^{k-1}t - \hat{n}$, then

$$\begin{aligned}
c_{nm} &= \int_0^1 z(\frac{x+\hat{n}}{2^{k-1}})(\frac{2m+1}{2})^{\frac{1}{2}}2^{\frac{k}{2}}L_m(x)\frac{1}{2^{k-1}}dx = \left(\frac{2m+1}{2^{k-1}}\right)^{\frac{1}{2}}\int_0^1 z(\frac{x+\hat{n}}{2^{k-1}})L_m(x)dx \\
&= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}}\int_0^1 z(\frac{x+\hat{n}}{2^{k-1}})d[L_{m+1}(x)-L_{m-1}(x)]dx \\
&= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}}\left(z(\frac{x+\hat{n}}{2^{k-1}})(L_{m+1}(x)-L_{m-1}(x))\right)\Big|_0^1 \\
&\quad -\left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}}\int_0^1 z'(\frac{x+\hat{n}}{2^{k-1}})\frac{1}{2^{k-1}}(L_{m+1}(x)-L_{m-1}(x))dx \\
&= -\left(\frac{1}{2^{3k-1}(2m+1)}\right)^{\frac{1}{2}}\int_0^1 z'(\frac{x+\hat{n}}{2^{k-1}})(L_{m+1}(x)-L_{m-1}(x))dx \\
&= -\left(\frac{1}{2^{3k-1}(2m+1)}\right)^{\frac{1}{2}}\int_0^1 z'(\frac{x+\hat{n}}{2^{k-1}})d\left[\frac{L_{m+2}(x)-L_m(x)}{2(2m+3)} - \frac{L_m(x)-L_{m-2}(x)}{2(2m-1)}\right]dx \\
&= -\left(\frac{1}{2^{3k-1}(2m+1)}\right)^{\frac{1}{2}} z'(\frac{x+\hat{n}}{2^{k-1}})\left[\frac{L_{m+2}(x)-L_m(x)}{2(2m+3)} - \frac{L_m(x)-L_{m-2}(x)}{2(2m-1)}\right]_0^1 \\
&= \left(\frac{1}{2^{5k-3}(2m+1)}\right)^{\frac{1}{2}}\int_0^1 z''(\frac{x+\hat{n}}{2^{k-1}})\left[\frac{L_{m+2}(x)-L_m(x)}{2(2m+3)} - \frac{L_m(x)-L_{m-2}(x)}{2(2m-1)}\right]dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|c_{nm}|^2 &= \left|\left(\frac{1}{2^{5k-3}(2m+1)}\right)^{\frac{1}{2}}\int_0^1 z''(\frac{x+\hat{n}}{2^{k-1}})\left[\frac{L_{m+2}(x)-L_m(x)}{2(2m+3)} - \frac{L_m(x)-L_{m-2}(x)}{2(2m-1)}\right]dx\right|^2 \\
&= \frac{1}{2^{5k-1}(2m+1)}\left|\int_0^1 z''(\frac{x+\hat{n}}{2^{k-1}})\right. \\
&\quad \left.\left[\frac{(2m-1)L_{m+2}(x)-(4m+2)L_m(x)+(2m+3)L_{m-2}(x)}{(2m+3)(2m-1)}\right]dx\right|^2 \\
&\leq \frac{1}{2^{5k-1}(2m+1)}\int_0^1 \left|z''(\frac{x+\hat{n}}{2^{k-1}})\right|^2 dx \\
&\quad \int_0^1 \left|\frac{(2m-1)L_{m+2}(x)-(4m+2)L_m(x)+(2m+3)L_{m-2}(x)}{(2m+3)(2m-1)}\right|^2 dx \\
&< \frac{Q^2}{2^{5k-1}(2m+1)}\int_0^1 \frac{(2m-1)^2L_{m+2}^2(x)+(4m+2)^2L_m^2(x)+(2m+3)^2L_{m-2}^2(x)}{(2m+3)^2(2m-1)^2}dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Q^2}{2^{5k-1}(2m+1)(2m+3)^2(2m-1)^2} \\
&\quad \left[(2m-1)^2 \int_0^1 L_{m+2}^2(x)dx + (4m+2)^2 \int_0^1 L_m^2(x)dx + (2m+3)^2 \int_0^1 L_{m-2}^2(x)dx\right] \\
&= \frac{Q^2}{2^{5k-1}(2m+1)(2m+3)^2(2m-1)^2}\Bigg[(2m-1)^2 \frac{1}{2m+5} \\
&\quad + (4m+2)^2 \frac{1}{2m+1} + (2m+3)^2 \frac{1}{2m-3}\Bigg] \\
&= \frac{Q^2}{2^{5k-1}(2m+1)(2m+3)^2(2m-1)^2} \frac{6(2m+3)^2}{2m-3} \\
&= \frac{6Q^2}{2^{5k-1}(2m+1)(2m-1)^2(2m-3)} = \frac{6Q^2}{2^{5k-1}(2m-3)^4} \\
&< \frac{12Q^2}{(2n)^5(2m-3)^4} = \frac{3}{8}\frac{Q^2}{n^5(2m-3)^4}. \qquad (30)
\end{aligned}
$$

Therefore

$$
\|z(t) - z_{\hat{m}}(t)\|_2^2 \leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2 < \frac{3Q^2}{8} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5(2m-3)^4}. \qquad (31)
$$

The theorem is established by taking the square roots. From above theorem, we conclude that

$$
k, M \to \infty, \qquad \text{then} \qquad \|z(t) - z_{\hat{m}}(t)\|_2 \to 0,
$$

which completes the proof. □

6 Numerical results and comparisons

In this part, we consider some numerical examples are conducted to validate the mentioned scheme. We performed our computations using Mathematica 10.

Example 1. Consider the equation

$$
\int_0^2 \frac{\Gamma(6-\alpha)}{120} D_t^{*\alpha} z(t) d\alpha = \frac{t^5 - t^3}{\ln t}, \qquad (32)
$$

with the initial conditions $z(0) = z'(0) = 0$. The aforementioned problem has the following exact solution $z(t) = t^5$. Table 1 presents details of absolute error in $z(0.5)$ along with those given in Refs. [11] and [18] for comparison. Also, Figure 1 demonstrates absolute error of the proposed scheme with $k = 1, M = 6$ and Ref. [18]. From Table 1 and Figure 1, we conclude that obtained numerical results have a good accuracy. Moreover, numerical solution and the exact solution for $k = 1$ and $M = 5, 6$ are depicted in Figure 2.

Example 2. Consider the equation

$$
\int_0^2 \Gamma(4-\alpha)\sinh(\alpha) D_t^{*\alpha} z(t) d\alpha = \frac{6t(t^2 - \cosh(2) - \sinh(2)\ln t)}{(\ln t)^2 - 1}, \qquad (33)
$$

Table 1: Comparison of absolute error for $k = 1$ and $t = 0.5$ with Refs. [11,18] (Example 1).

Ref. [11]	Absolute error	CPU
$k = 4$	6.39×10^{-4}	–
$k = 8$	1.59×10^{-4}	–
$k = 16$	3.73×10^{-5}	–
Ref. [18]		
$k = 4$	6.36×10^{-4}	–
$k = 8$	1.47×10^{-4}	–
$k = 16$	1.71×10^{-5}	–
Present method		
$M = 5$	1.66×10^{-3}	0.031
$M = 6$	2.84×10^{-7}	0.047

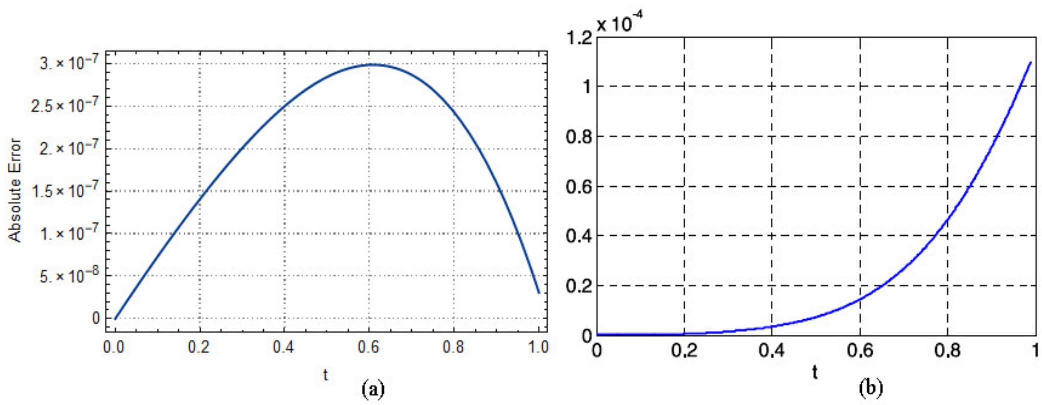

Figure 1: Comparison of absolute errors of (a) : present method for $k = 1, M = 6$ (b) : Ref. [18] for Example 1.

with the initial conditions $z(0) = z'(0) = 0$. The aforementioned problem has the following exact solution $z(t) = t^3$. Figure 3 compares absolute errors of the mentioned scheme for $k = 1, M = 6$ with Ref. [33]. Table 2 lists absolute error for $k = 1$ and various cases of M. Also, numerical solution and the exact solution for $k = 1$ and $M = 4$ are depicted in Figure 4.

Example 3. Consider the equation

$$\int_{0.2}^{1.5} \Gamma(3-\alpha) D_t^{*\alpha} z(t) d\alpha = 2\frac{t^{1.8} - t^{0.5}}{\ln t}, \tag{34}$$

with the initial conditions $z(0) = z'(0) = 0$. The aforementioned problem has the following exact solution $z(t) = t^2$. Table 3 compares scheme by [18] and the present scheme with $k = 1$ in $z(0.9)$. In Table 4, we have compared obtained absolute errors for various numbers of M. Based on

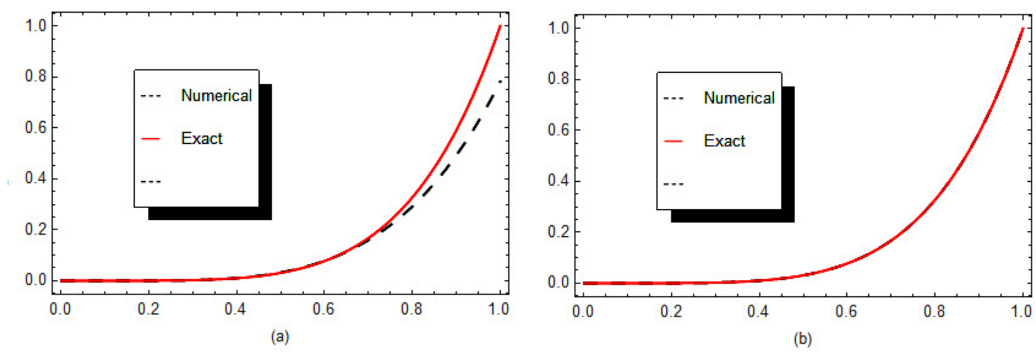

Figure 2: Comparison of numerical solution and the exact solution with $(a) : M = 5$ $(b) : M = 6$ for Example 1.

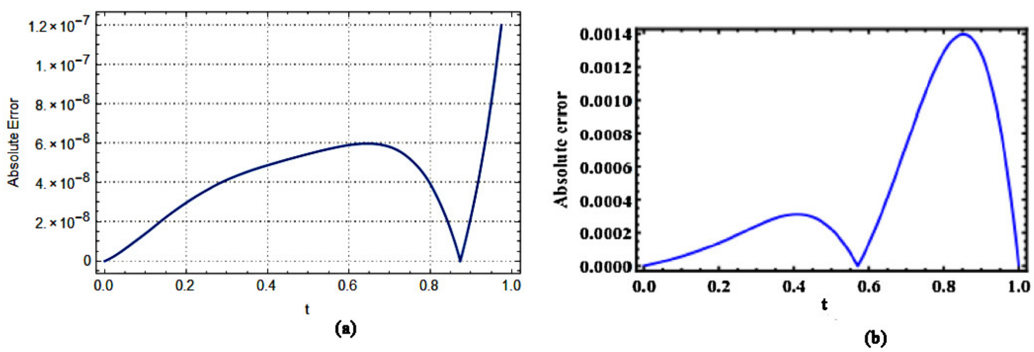

Figure 3: The comparison of absolute errors of (a) : present method for $k = 1, M = 6$ (b) : Ref. [33] for Example 2.

this table, we observe that accuracy of the mentioned scheme increases by increasing number of LWs. Also, Figure 5 demonstrates absolute error of the proposed scheme with $k = 1, M = 6$ and Ref. [18].

7 Conclusion

In this work, a class of DOFDEs has been solved numerically. Firstly, the Legendre wavelets derivative operators of distributed order fractional and integer-order has been constructed. Secondly, these operators and colocation technique has been utilized to transform the under study problem into a system of algebraic equations. Finally, the proposed technique for four problems have been tested to show the efficiency of the new technique.

Table 2: The absolute error for $k = 1$ and different values of M (Example 2).

t	$M = 4$	$M = 5$	$M = 6$
0.1	2.33×10^{-6}	1.19×10^{-7}	1.41×10^{-8}
0.2	7.93×10^{-6}	3.47×10^{-7}	2.97×10^{-8}
0.3	1.48×10^{-5}	5.45×10^{-7}	4.12×10^{-8}
0.4	2.07×10^{-5}	6.55×10^{-7}	4.88×10^{-8}
0.5	2.38×10^{-5}	6.94×10^{-7}	5.45×10^{-8}
0.6	2.19×10^{-5}	7.61×10^{-7}	5.89×10^{-8}
0.7	1.29×10^{-5}	1.03×10^{-6}	5.80×10^{-8}
0.8	5.18×10^{-6}	1.75×10^{-6}	3.91×10^{-8}
0.9	3.44×10^{-5}	3.26×10^{-6}	2.24×10^{-8}
CPU	0.001	0.016	0.032

Table 3: Comparison of relative error for $k = 1, t = 0.9$ with Refs. [18] (Example 3).

Ref. [18]	Absolute error	CPU
$k = 4$	5.10×10^{-3}	–
$k = 8$	1.20×10^{-3}	–
$k = 16$	1.96×10^{-4}	–
$k = 32$	6.73×10^{-5}	–
$k = 64$	1.40×10^{-4}	–
Present method		
$M = 3$	2.01×10^{-4}	0.001
$M = 4$	7.88×10^{-6}	0.016
$M = 5$	2.01×10^{-7}	0.031
$M = 6$	7.53×10^{-9}	0.031

Table 4: The absolute error for $k = 1$ and different values of M (Example 3).

t	$M = 3$	$M = 4$	$M = 5$	$M = 6$
0.1	2.01×10^{-6}	8.33×10^{-8}	2.14×10^{-9}	9.37×10^{-9}
0.2	8.04×10^{-6}	2.52×10^{-7}	4.76×10^{-9}	1.66×10^{-8}
0.3	1.81×10^{-5}	3.85×10^{-7}	4.56×10^{-9}	2.17×10^{-8}
0.4	3.22×10^{-5}	3.60×10^{-7}	1.46×10^{-9}	2.48×10^{-8}
0.5	5.02×10^{-5}	5.61×10^{-8}	1.46×10^{-9}	2.58×10^{-8}
0.6	7.24×10^{-5}	6.49×10^{-7}	2.14×10^{-9}	2.47×10^{-8}
0.7	9.85×10^{-5}	1.88×10^{-6}	2.17×10^{-8}	2.15×10^{-8}
0.8	1.29×10^{-4}	3.75×10^{-6}	7.00×10^{-8}	1.55×10^{-8}
0.9	1.63×10^{-4}	6.38×10^{-6}	1.63×10^{-7}	6.10×10^{-9}

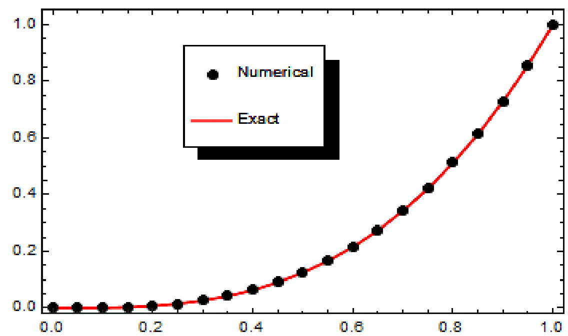

Figure 4: Comparison of numerical solution and the exact solution with $k = 1, M = 4$ for Example 2.

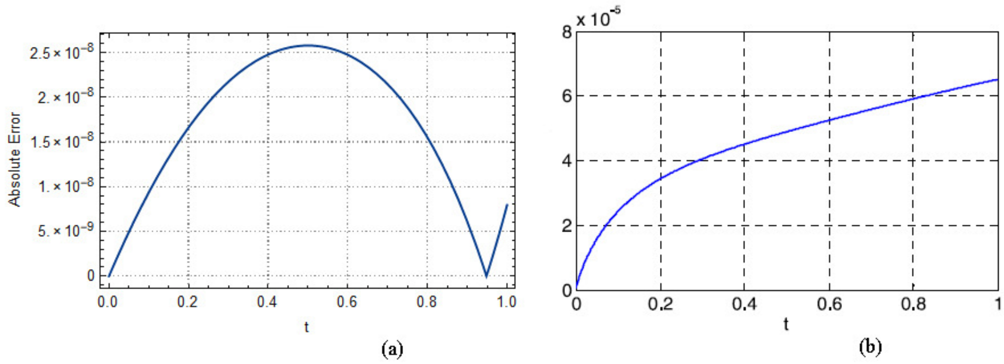

Figure 5: The comparison of absolute errors of (a) : present method for $k = 1, M = 6$ (b) : Ref. [18] for Example 3.

Acknowledgements

The second author is supported by the Alzahra university within project 99/1/159. Also, we express our sincere thanks to the anonymous referees for valuable suggestions that improved the paper.

References

[1] T.M. Atanackovic, *A generalized model for the uniaxial isothermal deformation of a viscoelastic body*, Acta Mech. **159** (2002) 77−86.

[2] T.M. Atanackovic , M. Budincevic , S. Pilipovic , *On a fractional distributed-order oscillator*, J. Phys. A Math. General **38** (2005) 6703−6713.

[3] T.M. Atanackovic, L. Oparnica, S. Pilipovic, *On a nonlinear distributed order fractional differential equation*, J. Math. Anal. Appl. **328** (2007) 590−608.

[4] T.M. Atanackovic, L. Oparnica , S. Pilipovic , *Semilinear ordinary differential equation coupled with distributed order fractional differential equation*, Nonlinear Anal. Theory Methods Appl. **72** (2010) 4101−4114.

[5] T.M. Atanackovic, S. Pilipovic, D. Zorica, *Distributed-order fractional wave equation on a finite domain: stress relaxation in a rod*, Int. J. Eng. Sci. **49** (2011) 175−190.

[6] M. Abbaszadeh, M. Dehghan, *An improved meshless method for solving two-dimensional distributed order time-fractional diffusion-wave equation with error estimate*, Numer. Algorithms **75** (2017) 173−211.

[7] R.L. Bagley, P.J. Torvik, *On the existence of the order domain and the solution of distributed order equations* -part II, Int. J. Appl. Mech. **8** (2000) 965−987.

[8] M. Caputo, *Mean fractional-order-derivatives differential equations and filters*, Ann. Univers. Ferr. **41** (1995) 73–84.

[9] M. Caputo, *Distributed order differential equations modelling dielectric induction and diffusion*, Fract. Calc. Appl. Anal. **4** (2001) 421–442.

[10] M. Dehghan, M. Abbaszadeh, *A Legendre spectral element method (SEM) based on the modified bases for solving neutral delay distributed-order fractional damped diffusion-wave equation*, Math. Methods Appl. Sci. **41** (2018) 3476–3494.

[11] K. Diethelma, N.J. Ford, *Numerical analysis for distributed-order differential equations*, J. Comput. Appl. Math. **225** (2009) 96−104.

[12] W. Ding, S. Patnaik, S. Sidhardh, F. Semperlotti, *Applications of distributed-order fractional operators*, Entropy, **23** (2021) 110.

[13] P.L.T. Duong, E. Kwok, M. Lee, *Deterministic analysis of distributed order systems using operational matrix*, Appl. Math. Model. **40** (2016) 1929–1940 .

[14] M. Pourbabaee, A. Saadatmandi, *A novel Legendre operational matrix for distributed order fractional differential equations*, Appl. Math. Comput. **361** (2019) 215–231.

[15] M. Fukunaga, N. Shimizu, *Nonlinear fractional derivative models of viscoelastic impact dynamics based on entropy elasticity and generalized Maxwell law*, Comput. Nonlinear Dyn. **6** (2011) 021005.

[16] R. Gorenflo, Y. Luchko, M. Stojanovic, *Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density*, Fract. Calc. Appl. Anal. **16** (2013) 297–316.

[17] Z. Jiao, Y. Chen, I. Podlubny, *Distributed-order Dynamic Systems: Stability, Simulation, Applications and Perspectives*, Springer, New York, 2012.

[18] J.T. Katsikadelis, *Numerical solution of distributed order fractional differential equations*, J. Comput. Phys. **259** (2014) 11–22.

[19] E. Kharazmi, M. Zayernouri, G.E. Karniadakis, *Petrov-Galerkin and spectral collocation methods for distributed order differential equations*, SIAM J. Sci. Comput. **39** (2017) A1003–A1037.

[20] Y. Li, *Solving a nonlinear fractional differential equation using Chebyshev wavelets*, Commun. Nonlinear Sci. Numer. Simul. **15** (2010) 2284-2292.

[21] X.Y. Li, B.Y. Wu, *A numerical method for solving distributed order diffusion equations*, Appl. Math. Lett. **53** (2016) 92–99.

[22] S. Mashayekhi, M. Razzaghi, *Numerical solution of distributed order fractional differential equations by hybrid functions*, J. Comput. Phys. **315** (2016) 169–181.

[23] Yi-Ming Chen, Yan-Qiao Wei, Da-Yan Liu, Hao Yu, *Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre wavelets*, Appl. Math. Lett. **46** (2015) 83–88.

[24] M. Morgado, M. Rebelo, L. Ferras, N. Ford, *Numerical solution for diffusion equations with distributed order in time using a Chebyshev collocation method*, Appl. Numer. Math. **114** (2017) 108–123.

[25] H. Noroozi, A. Ansari, *Basic results on distributed order fractional hybrid differential equations with linear perturbations*, J. Math. Model. **2** (2014) 55–73.

[26] P. Rahimkhani, R. Moeti, *Numerical Solution of the Fractional Order Duffing-van der Pol Oscillator equation by using Bernoulli wavelets collocation method*, Int. J. Appl. Comput. Math. **4** (2018) 59.

[27] P. Rahimkhani, Y. Ordokhani, *Numerical solution a class of 2D fractional optimal control problems by using 2D Muntz-Legendre wavelets*, Optim. Contr. Appl. Met. **39** (2018) 1916–1934.

[28] P. Rahimkhani, Y. Ordokhani, E. Babolian, *A numerical scheme for solving nonlinear fractional Volterra integro-differential equations*, Iran. J. Math. Sci. Inform. **13** (2018) 111–132.

[29] P. Rahimkhani, Y. Ordokhani, E. Babolian, *Fractional-order Bernoulli wavelets and their applications*, Appl. Math. Model. **40** (2016) 8087–8107.

[30] P. Rahimkhani, Y. Ordokhani, E. Babolian, *Fractional-order Legendre wavelets and their applications for solving fractional-order differential equations with initial/boundary conditions*, Comput. methods differ. Equ. **5** (2017) 117–140.

[31] P. Rahimkhani, Y. Ordokhani, E. Babolian, *Müntz-Legendre wavelet operational matrix of fractional-order integration and its applications for solving the fractional pantograph differential equations*, Numer. Algorithms **77** (2081) 1283–1305.

[32] M.U. Rehman , R.A. Khan , *A numerical method for solving boundary value problems for fractional differential equations*, Appl. Math. Model. **36** (2012) 894–907.

[33] M.S. Semary, H.N. Hassan, A.G. Radwan, *Modified methods for solving two classes of distributed order linear fractional differential equations*, Appl. Math. Comput. **323** (2018) 106–119.

[34] F. Zhou, Y. Zhao, Y. Li, Y.Q. Chen, Design, *Implementation and application of distributed order PI control*, ISA Trans. **52** (2013) 429–437.