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$d-$Fibonacci and $d-$Lucas polynomials

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Abstract. Riordan arrays give us an intuitive method of solving combinatorial problems. They also help to apprehend number patterns and to prove many theorems. In this paper, we consider the Pascal matrix, define a new generalization of Fibonacci and Lucas polynomials called $d-$Fibonacci and $d-$Lucas polynomials (respectively) and provide their properties. Combinatorial identities are obtained for the defined polynomials and by using Riordan method we get factorizations of Pascal matrix involving $d-$Fibonacci polynomials.

Keywords: $d-$Fibonacci polynomials, $d-$Lucas polynomials, Riordan arrays, Pascal matrix, Q_d-Fibonacci matrix.

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1 Introduction

The Fibonacci numbers F_n are defined by the recurrence sequence $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 1$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. This famous sequence appears in many areas of mathematics and it has been generalized in many research fields. We recall here the generalization of Falcon and Plaza [4], where a general Fibonacci sequence was introduced. It generalizes, among others, both the classical Fibonacci sequence and Pell sequence. These general $k-$Fibonacci numbers $F_{k,n}$ are defined by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}, n \geq 2$, with the initial values $F_0 = 0$ and $F_1 = 1$. We note that the Pell numbers are $2-$Fibonacci numbers. It should be reminded that the $k-$Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the well-known 4-triangle longest-edge partition. Besides that, the $k-$Fibonacci numbers were given in an explicit way and many properties were proven in [12]. They were related with the so-called Pascal $2-$triangle. In fact, the Fibonacci polynomials are defined by using the Fibonacci-like recursion relations. They were studied in 1883 by the Belgian mathematician Catalan and Jacobsthal. In what follows we

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present polynomials the $F_n(x)$ studied by Catalan, defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), n \geq 3, \text{ where } F_1(x) = 1 \text{ and } F_2(x) = x.$$

And the Fibonacci polynomials studied by Jacobsthal are defined by

$$J_n(x) = xJ_{n-1}(x) + J_{n-2}(x), n \geq 3, \text{ where } J_1(x) = J_2(x) = 1.$$

Finally, the Lucas polynomials studied in 1970 by Bicknell are given by the following recurrence relation

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 2, \text{ where } L_0(x) = 2 \text{ and } L_1(x) = x.$$

In [13], Nalli and Haukkanen introduced $h(x)-$Fibonacci polynomials that generalize both Catalans Fibonacci polynomials $F(x)$ and Byrds Fibonacci polynomials $\phi_n(x)$ and also the $k-$Fibonacci numbers $F_{k;n}$. The $h(x)-$Fibonacci polynomial sequence, denoted by $\{F_{h,n}(x)\}_{n\geq 0}$, is defined by the following recurrence relation

$$\begin{aligned} &F_{h,0}(x) = 0, \quad F_{h,1}(x) = 1, \\ &F_{h,n+1}(x) = h(x) F_{h,n}(x) + F_{h,n-1}(x), \ n \geq 1. \end{aligned} \tag{1}$$

Lee and Asci [6] generalized the $h(x)-$Fibonacci polynomial to the $(p,q)-$Fibonacci polynomial $F_{p,q,n}(x)$, which is defined by

$$F_{p,q,n+1}(x) = p(x) F_{p,q,n}(x) + q(x) F_{p,q,n-1}(x), \ n \geq 1. \tag{2}$$

They obtained combinatorial identities and by using Riordan method they gave tow factorizations of Pascal matrix involving $(p,q)-$Fibonacci polynomials.

Brawer and Pirovino [1] defined the $n\times n$ lower triangular Pascal as matrix $P_n = [P_{i,j}]_{i,j=1,2,\ldots,n}$

$$P_{i,j} = \begin{cases} \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The Pascal matrix is particularly important in studying combinatorial identities and it has many applications in numerical analysis, probability, surface reconstruction and combinatorics. In [9], Lee et al. factorized the Pascal matrix involving the Fibonacci matrix defined by

$$\mathcal{F}_n = (F_{i,j}) = \begin{cases} F_{i-j+1}, & \text{if } \ i-j+1 \geq 0, \\ 0, & \text{if } \ i-j+1 < 0. \end{cases}$$

The inverse of this matrix is given as follows

$$\mathcal{F}_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & \\ 0 & -1 & -1 & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & -1 & 1 \end{bmatrix}.$$

The Riordan group was introduced by Shapiro et al. [15] as a set of infinite lower-triangular integer matrices where each matrix is defined by a pair of formal power series $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $f(z) = \sum_{n=1}^{\infty} f_n z^n$ with $g_0 \neq 0$ and $f_1 \neq 0$.

An infinite lower triangular matrix $D = [d_{n,k}]_{n,k\geq 0}$ is called a Riordan array, if its $i^{-\text{th}}$ column generating function is $g(x)[f(x)]^i$ for $i \geq 0$ (the first column being indexed by 0). With little loss of generality, we also assume $d_{0,0} = g_0 = 1$. The matrix corresponding to the pairs g and f is denoted by (g, f). The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The identity of this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = \left(1/\left(g \circ \bar{f}\right), \bar{f}\right)$ where $\bar{f}$ is the reversion or compositional inverse of f. The reversion of f is the power series $\bar{f}$ such that $\left(f \circ \bar{f}\right)(x) = x$. It may be written sometimes as $\bar{f} = \mathrm{Rev} f$. This group will be denoted by $\mathcal{R}$.

In [3], Cheon et al. presented many results, including a generalized Lucas polynomial sequence from Riordan arrays and combinatorial interpretation for a pair of generalized Lucas polynomial sequences.

In this paper, let $\{p_i(x)\}_{i=1}^{d+1}$ be a polynomial sequence with real coefficients. In the first section, we introduce $d-$Fibonacci polynomials that generalize both Catalan's Fibonacci polynomials $F_n(x)$ and Byrd's Fibonacci polynomials $\varphi_n(x)$ and also the $k-$Fibonacci numbers $F_{k,n}$. Then, we provide properties for the $d-$Fibonacci polynomials. Next, we introduce $d-$Lucas polynomials that generalize the Lucas polynomials and present properties of these polynomials. In the second section, we introduce the matrix $Q_d(x)$ that generalizes the $Q-$matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

whose powers generate the Fibonacci numbers. We show some properties of the classical type for the $d-$Fibonacci and $d-$Lucas polynomials and the matrix $Q_d(x)$. Finally, we give a new factorization of Pascal matrix involving $d-$ Fibonacci polynomials by using Riordan method.

2 Generalization of Fibonacci and Lucas polynomials

In this section, we introduce a new generalization of Fibonacci polynomials. For a real polynomials $\{p_i(x)\}_{i=1}^{d+1}$, let us define the $d-$Fibonacci polynomials $\{F_n(x)\}$ by their recurrence relation as follows

$$F_{n+1}(x) = p_1(x) F_n(x) + p_2(x) F_{n-1}(x) + \cdots + p_{d+1}(x) F_{n-d}(x), \ n \geq 1, \tag{3}$$

with the initial conditions $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$. We define the $d-$Lucas polynomials $\{L_n\}$ in terms of $d-$Fibonacci as follows

$$L_n(x) = F_{n+1}(x) + p_2(x) F_{n-1}(x) + \cdots + p_{d+1}(x) F_{n-d}(x), \ n \geq 1, \tag{4}$$

with the initial conditions $L_n(x) = 0$ for $n < 0$, $L_0(x) = 2$ and $L_1(x) = p_1(x)$.

Now, the corresponding generating functions of $\{F_n(x)\}$ and $\{L_n(x)\}$ are, respectively,

$$G(x,t)=\sum_{n=0}^{\infty}F_n(x)\,t^n \quad \text{and} \quad H(x,t)=\sum_{n=0}^{\infty}L_n(x)\,t^n. \tag{5}$$

Using the recurrence relation (3), we can easily obtain the explicit form of the generating functions [2,14]

$$G(x,t)=t\left[1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}\right]^{-1}, \tag{6}$$

and

$$H(x,t)=\left[2-p_1(x)\,t\right]\left[1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}\right]^{-1}. \tag{7}$$

Hence, the polynomials L_n may be expressed in terms of F_n as follows

$$L_n=2F_{n+1}-p_1F_n, \quad n\geq 0. \tag{8}$$

Furthermore, using the recurrence formula of $\{F_n\}$ we can prove by induction on n that the $d-$Lucas polynomials satisfy in addition the following recurrence relation.

Proposition 1. *The $d-$Lucas polynomials satisfy the following recurrence relation*

$$L_{n+1}(x)=p_1(x)\,L_n(x)+p_2(x)\,L_{n-1}(x)+\cdots+p_{d+1}(x)\,L_{n-d}(x)\,,\ n\geq 1. \tag{9}$$

The Binet formula is well known in the theory of Fibonacci numbers. These formula can also be carried out for the $d-$Fibonacci polynomials from the characteristic equation

$$t^{d+1}-p_1(x)\,t^d-\cdots-p_{d+1}(x)=0. \tag{10}$$

That is,

$$\frac{t}{1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}}=\sum_{i=1}^{d+1}\frac{A_i(x)}{1-\sigma_i(x)\,t}. \tag{11}$$

Now, identifying the coefficient of t^n in the both sides of (3), we easily obtain the following Binet's formula

$$F_n(x)=\sum_{i=1}^{d+1}A_i(x)\left[\sigma_i(x)\right]^n. \tag{12}$$

More precisely, the multinomial coefficients allow us to give the explicit form of the $d-$Fibonacci polynomials. Indeed,

Theorem 1. *For $n\geq 0$, we have*

$$F_n(x)=\sum_{\substack{n_1,n_2,\ldots,n_{d+1}\\ 1+n_1+2n_2+\cdots+(d+1)n_{d+1}=n}}\binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}}p_1^{n_1}(x)p_2^{n_2}(x)\cdots p_{d+1}^{n_{d+1}}(x). \tag{13}$$

Proof. By developing the right hand side of (11), we get

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n+1}(x)\,t^n &= \frac{1}{1-p_1(x)t-p_2(x)t^2-\cdots-p_{d+1}(x)t^{d+1}} \\ &= \sum_{n=0}^{\infty}\left[p_1(x)t+p_2(x)t^2+\cdots+p_{d+1}(x)t^{d+1}\right]^n \\ &= \sum_{n=0}^{\infty}\left[\sum_{n_1+n_2+\cdots+n_{d+1}=n}\binom{n}{n_1,n_2,\cdots,n_{d+1}}p_1^{n_1}(x)p_2^{n_2}(x)\cdots p_{d+1}^{n_{d+1}}(x)\right]t^{n_1+2n_2+\cdots+(d+1)n_{d+1}} \\ &= \sum_{n=0}^{\infty}\left[\sum_{\substack{n_1,n_2,\cdots,n_{d+1}\\ n_1+2n_2+\cdots+(d+1)n_{d+1}=n}}\binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\cdots,n_{d+1}}p_1^{n_1}(x)p_2^{n_2}(x)\cdots p_{d+1}^{n_{d+1}}(x)\right]t^n.\end{aligned}$$

Whence the desired result. □

In [6], Lee and Asci introduced the matrix

$$Q_{p,q}=\begin{bmatrix} p(x) & q(x)\\ 1 & 0\end{bmatrix}$$

that plays the $Q-$Fibonacci matrix with

$$Q=\begin{bmatrix}1&1\\1&0\end{bmatrix}.$$

They proved some results which generalized the classical representation of the Fibonacci polynomials as well as the Fibonacci sequence in terms of the matrix Q. Indeed, in our case, we introduce a further generalization of this kind of matrices. To this end, let us define the Q_d matrix as follows

$$Q_d=\begin{bmatrix} p_1(x) & p_2(x) & \cdots & & p_{d+1}(x)\\ 1 & 0 & & & 0\\ 0 & \ddots & & & \\ & \ddots & & & \\ 0 & & 0 & 1 & 0\end{bmatrix}. \tag{14}$$

It follows immediately that, the determinant of the matrix Q_d is the polynomial $(-1)^d\,p_{d+1}(x)$.

Theorem 2. *We have the following representation*

$$Q^n=\begin{bmatrix} F_{n+1} & p_2F_n+\cdots+p_{d+1}F_{n-d+1} & p_3F_n+\cdots+p_{d+1}F_{n-d+2} & \cdots & p_{d+1}F_n\\ F_n & p_2F_{n-1}+\cdots+p_{d+1}F_{n-d} & p_3F_{n-1}+\cdots+p_{d+1}F_{n-d+1} & \cdots & p_{d+1}F_{n-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ F_{n-d+1} & p_2F_{n-d}+\cdots+p_{d+1}F_{n-2d+1} & p_3F_{n-d}+\cdots+p_{d+1}F_{n-2d+2} & \cdots & p_{d+1}F_{n-d}\end{bmatrix}. \tag{15}$$

Proof. The formula is easy to prove by induction on n. □

We have the following consequence which is also a generalization of the result presented in [6].

Corollary 1. *For $m, n \geq 0$, we have*

$$\begin{aligned} F_{n+m+1} &= F_{n+1}F_{m+1} + p_1(x)F_nF_m + p_2(x)\left(F_nF_{m-1} + F_{n-1}F_m\right) \\ &\quad + p_3(x)\left(F_nF_{m-2} + F_{n-1}F_{m-1} + F_{n-2}F_m\right) + \cdots \\ &\quad + p_{d+1}(x)\left(F_nF_{m-d+1} + F_{n-1}F_{m-d+2} + \cdots + F_{n-d+1}F_m\right). \end{aligned} \tag{16}$$

Proof. Writing $Q^{n+m} = Q^nQ^m$ and then by identification we get the result since the first entries of matrices Q^{n+m} is the production of the first row in Q^n by the first column of Q^m. □

We have also the following generalization of the result given in [6], and it is again easily proved by induction on n.

Theorem 3. *For $n \geq d$, we have*

$$\begin{aligned} t^n &= F_{n-d+1}t^d + \left(p_2(x)F_{n-d} + \cdots + p_{d+1}(x)F_{n-2d+1}\right)t^{d-1} \\ &\quad + \left(p_3(x)F_{n-d} + \cdots + p_{d+1}(x)F_{n-2d+2}\right)t^{d-2} + \cdots + p_{d+1}(x)F_{n-d}. \end{aligned} \tag{17}$$

Now, we are able to generalize several properties of the Fibonacci polynomials as well as for the generalized Fibonacci polynomials. First, we begin with the following

Theorem 4. *For $d \geq 2$ and $n \geq 0$, we have the following expression*

$$\sum_{\substack{n_1, n_2, \cdots, n_{d+1} \\ (d+1)n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1, n_2, \cdots, n_{d+1}} p_1^{n_1}(x)p_2^{n_2}(x)\cdots p_{d+1}^{n_{d+1}}(x)F_{n-[n_1+n_2+\cdots+n_{d+1}]} = F_{n(d+1)}. \tag{18}$$

Proof. For $n = 1$, we obtain

$$F_{d+1} = p_1(x)F_d + p_2(x)F_{d-1} + \cdots + p_{d+1}(x)F_0.$$

Let us denote the right hand side of (18) by RH. Then, for $n \geq 2$, we have

$$\begin{aligned} &RH \\ &= \sum_{\substack{n_1, n_2, \ldots, n_{d+1} \\ (d+1)n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1, n_2, \cdots, n_{d+1}} p_1^{n_1}(x)p_2^{n_2}(x)\cdots p_{d+1}^{n_{d+1}}(x)\left[\sum_{i=1}^{d+1} A_i\sigma_i^{n-[n_1+n_2+\cdots+n_{d+1}]}\right] \\ &= \sum_{\substack{n_1, n_2, \ldots, n_{d+1} \\ (d+1)n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1, n_2, \cdots, n_{d+1}} p_1^{n_1}(x)p_2^{n_2}(x)\cdots p_{d+1}^{n_{d+1}}(x)\left[\sum_{i=1}^{d+1} A_i\sigma_i^{dn_1+(d-1)n_2+\cdots+n_{d+1}}\right] \\ &= A_1 \sum_{\substack{n_1, n_2, \ldots, n_{d+1} \\ (d+1)n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1, n_2, \cdots, n_{d+1}} \left[\sigma_1^d p_1(x)\right]^{n_1}\left[\sigma_1^{d-1}p_2(x)\right]^{n_2}\cdots\left[p_{d+1}(x)\right]^{n_{d+1}} + \cdots \\ &\quad + A_{d+1} \sum_{\substack{n_1, n_2, \ldots, n_{d+1} \\ (d+1)n_1 + dn_2 + \cdots + n_{d+1} = n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1, n_2, \cdots, n_{d+1}} \left[\sigma_d^d p_1(x)\right]^{n_1}\left[\sigma_d^{d-1}p_2(x)\right]^{n_2}\cdots\left[p_{d+1}(x)\right]^{n_{d+1}} \end{aligned}$$

that is,

$$\begin{aligned} RH = \quad & A_1\left[\sigma_1^d p_1(x)+\sigma_1^{d-1}p_2(x)+\cdots+p_{d+1}(x)\right]^n+\cdots \\ & +A_{d+1}\left[\sigma_{d+1}^d p_1(x)+\sigma_{d+1}^{d-1}p_2(x)+\cdots+p_{d+1}(x)\right]^n. \end{aligned}$$

Now, from the characteristic equation, we see that $\sigma_i^d p_1+\sigma_i^{d-1}p_2+\cdots+p_{d+1}=\sigma_i^{d+1}$. Using this last property, we obtain

$$RH=\sum_{i=1}^{d+1}A_i\left[\sigma_i^{d+1}\right]^n=F_{n(d+1)},$$

which completes the proof. □

We have also the following generalization of [6].

Theorem 5. *For $n \geq 0$, we have*

$$\begin{aligned} & \sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ (d+1)n_1+dn_2+\cdots+n_{d+1}=n}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} p_1^{n_1}(x)p_2^{n_2}(x)\cdots(-p_{d+1}(x))^{n_{d+1}}\,F_{n-[n_1+n_2+\cdots+n_{d+1}]} \\ & \qquad = \sum_{k=0}^{n}\binom{k}{n}\left(-2p_{d+1}(x)\right)^k F_{(d+1)(n-k)}. \end{aligned} \tag{19}$$

Proof. Let us denote the right-hand side of (19) by RH. Then, by regarding the proof of the Theorem 4, we can easily get

$$\begin{aligned} RH \quad &= A_1\left[\left(\sigma_1^d p_1(x)+\sigma_1^{d-1}p_2(x)+\cdots+\sigma_1 p_d(x)\right)-p_{d+1}(x)\right]^n+\cdots \\ & \quad + A_{d+1}\left[\left(\sigma_{d+1}^d p_1(x)+\sigma_{d+1}^{d-1}p_2(x)+\cdots+\sigma_{d+1} p_d(x)\right)-p_{d+1}(x)\right]^n. \end{aligned}$$

Using again the characteristic equation, we obtain

$$\begin{aligned} RH \quad &= A_1\left[\sigma_1^{d+1}-2p_{d+1}(x)\right]^n+\cdots+A_{d+1}\left[\sigma_{d+1}^{d+1}-2p_{d+1}(x)\right]^n \\ &= A_1\sum_{k=0}^{n}\binom{k}{n}\left(-2p_{d+1}(x)\right)^k\sigma_1^{(d+1)(n-k)}+\cdots+A_{d+1}\sum_{k=0}^{n}\binom{k}{n}\left(-2p_{d+1}(x)\right)^k\sigma_{d+1}^{(d+1)(n-k)}. \end{aligned}$$

Whence the desired result. □

3 The Infinite $d-$Fibonacci and $d-$Lucas Polynomials Matrix

In this section, we define a new matrix called $d-$Fibonacci polynomials matrix. The infinite $d-$Fibonacci polynomials matrix

$$\mathcal{F}(x)=\left[F_{p_1,p_2,\ldots,p_{d+1},i,j}(x)\right]$$

is defined as follows

$$\digamma(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ p_1(x) & 1 & 0 & \vdots \\ p_1(x)^2+p_2(x) & p_1(x) & 1 & \vdots \\ p_1(x)^3+2p_1(x)p_2(x)+p_3(x) & p_1(x)^2+p_2(x) & p_1(x) & 1 \\ t_1(x) & t_2(x) & p_1(x)^2+p_2(x) & \ddots \\ \vdots & \cdots & \cdots & \ddots \end{bmatrix} = \left(g_{\digamma(x)}(t), f_{\digamma(x)}(t)\right),$$

where

$$\begin{aligned} t_1(x) &= p_1(x)^4+2p_1(x)^2p_2(x)+p_2(x)^2+p_1(x)p_3(x)+p_4(x), \\ t_2(x) &= p_1(x)^3+2p_1(x)p_2(x)+p_3(x). \end{aligned}$$

The matrix $\digamma(x)$ is an element of the set of Riordan matrices. Since, the first column of $\digamma(x)$ is $\left(1, p_1(x), p_1(x)^2+p_2(x), t_2(x), t_1(x), \ldots\right)^T$, then it obvious that

$$g_{\digamma(x)}(t) = \sum_{n=0}^{\infty} F_{p_1,p_2,\cdots,p_{d+1},n}(x)\, t^n = \left[1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}\right]^{-1}.$$

In the matrix $\digamma(x)$ each entry has a rule with upper two rows, that is,

$$F_{n+1,j}(x) = p_1(x)\,F_{n,j}(x)+p_2(x)\,F_{n-1,j}(x)+\cdots+p_{d+1}(x)\,F_{n-d,j}(x)$$

Then $f_{\digamma(x)}(t)=t$, that is,

$$\begin{aligned} \digamma(x) &= \left(g_{\digamma(x)}(t), f_{\digamma(x)}(t)\right) \\ &= \left(\frac{1}{1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}}, t\right). \end{aligned}$$

Hence, $\digamma(x) \in \mathcal{R}$.

Similarly, we can define the $d-$Lucas polynomial matrix. The infinite $d-$Lucas polynomial matrix is given by

$$\chi(\mathbf{x}) = \left[L_{p_1,p_2,\ldots,p_{d+1},i,j}(x)\right]$$

and it can be written as

$$\begin{aligned} \chi(\mathbf{x}) &= \left(g_{\chi(x)}(t), f_{\chi(x)}(t)\right) \\ &= \left(\frac{2-p_1(x)t}{1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)\,t^{d+1}}, t\right). \end{aligned}$$

In the rest of this section, we give two factorizations of Pascal matrix involving the $d-$Fibonacci polynomial matrix. For these factorizations we need to define two matrices. Firstly, we define an infinite matrix $B(x)=(b_{i,j}(x))$ as follows

$$b_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-2}{j-1} - p_2(x)\binom{i-3}{j-1} - \cdots - p_{d+1}(x)\binom{i-d-2}{j-1}. \tag{20}$$

So, we have

$$B(x)=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-p_1(x) & 1 & 0 & 0 & \\ 1-p_1(x)-p_2(x) & 2-p_1(x) & 1 & 0 & \\ 1-p_1(x)-p_2(x)-p_3(x) & 3-2p_1(x)-p_2(x) & 3-p_1(x) & 1 & \vdots \\ \vdots & \vdots & 6-3p_1(x)-p_2(x) & 4-p_1(x) & \ddots \\ l_1(x) & l_2(x) & \ddots & & \ddots \\ l_3(x) & l_4(x) & & & \ddots \\ \vdots & \cdots & & & \end{bmatrix}, \tag{21}$$

where

$$\begin{aligned} l_1(x) &= 1-p_1(x)-p_2(x)-\cdots-p_d(x), \\ l_2(x) &= d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-2p_{d-2}(x)-p_{d-1}(x), \\ l_3(x) &= 1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x) \\ l_4(x) &= (d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-2p_{d-1}(x)-p_d(x). \end{aligned}$$

Now, we give the first factorization of the infinite Pascal matrix via the infinite $d-$Fibonacci polynomials matrix and the infinite matrix $B(x)$ by the following theorem.

Theorem 6. *Let $B(x)$ be the infinite matrix as in (21) and $\mathcal{F}(x)$ be the infinite $d-$Fibonacci polynomial matrix, then*

$$P(x)=\mathcal{F}(x)*B(x)$$

where P is the Pascal matrix.

Proof. From the definitions of the infinite Pascal matrix and the infinite $d-$Fibonacci polynomials matrix, we have the following Riordan representing

$$P=\left(\frac{1}{1-t},\frac{t}{1-t}\right),\ \mathcal{F}(x)=\left(\frac{1}{1-p_1(x)\,t-p_2(x)\,t^2-\cdots-p_{d+1}(x)t^{d+1}},t\right).$$

Now, we can find the Riordan representation of the infinite matrix

$$B(x)=\left(g_{B(x)}(t),f_{B(x)}(t)\right)$$

as follows

$$B(x)=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-p_1(x) & 1 & 0 & 0 & \\ 1-p_1(x)-p_2(x) & 2-p_1(x) & 1 & 0 & \\ 1-p_1(x)-p_2(x)-p_3(x) & 3-2p_1(x)-p_2(x) & 3-p_1(x) & 1 & \vdots \\ \vdots & \vdots & 6-3p_1(x)-p_2(x) & 4-p_1(x) & \ddots \\ l_1(x) & l_2(x) & \ddots & & \ddots \\ l_3(x) & l_4(x) & & & \ddots \\ \vdots & \cdots & & & \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned}
l_1(x) &= 1 - p_1(x) - p_2(x) - \cdots - p_d(x), \\
l_2(x) &= d - (d-1)p_1(x) - (d-2)p_2(x) - \cdots - 2p_{d-2}(x) - p_{d-1}(x), \\
l_3(x) &= 1 - p_1(x) - p_2(x) - \cdots - p_{d+1}(x), \\
l_4(x) &= (d+1) - dp_1(x) - (d-1)p_2(x) - \cdots - 2p_{d-1}(x) - p_d(x).
\end{aligned}$$

From the first column of the matrix $B(x)$, we obtain

$$g_{B(x)}(t) = \frac{1 - p_1(x)\,t - p_2(x)\,t^2 - \cdots - p_{d+1}(x)\,t^{d+1}}{1-t}.$$

From the rule of the matrix $B(x)$,

$$f_{B(x)}(t) = \frac{t}{1-t}.$$

Thus,

$$B(x) = \left(\frac{1 - p_1(x)\,t - p_2(x)\,t^2 - \cdots - p_{d+1}(x)\,t^{d+1}}{1-t}, \frac{t}{1-t}\right) \tag{23}$$

which completes the proof. $\square$

Now, we define the $n \times n$ matrix $R(x) = (r_{i,j}(x))$ as follows

$$r_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-1}{j} - p_2(x)\binom{i-1}{j+1} - \cdots - p_{d+1}(x)\binom{i-1}{j+d}.$$

We have the infinite matrix $R(x)$ as follows

$$R(x) = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
1 - p_1(x) & 1 & 0 & 0 & \\
1 - 2p_1(x) - p_2(x) & 2 - p_1(x) & 1 & 0 & \\
1 - 3p_1(x) - 3p_2(x) - p_3(x) & 3 - 2p_1(x) - p_2(x) & 3 - p_1(x) & 1 & \vdots \\
\vdots & \vdots & 6 - 3p_1(x) - p_2(x) & 4 - p_1(x) & \\
s_1(x) & s_2(x) & \ddots & & \ddots \\
s_3(x) & s_4(x) & & & \ddots \\
\vdots & \cdots & & &
\end{bmatrix},$$

where

$$\begin{aligned}
s_1(x) &= 1 - dp_1(x) - \frac{d(d-1)}{2!}p_2(x) - \cdots - p_d(x), \\
s_2(x) &= d - (d-1)p_1(x) - (d-2)p_2(x) - \cdots - 2p_{d-2}(x) - p_{d-1}(x), \\
s_3(x) &= 1 - (d+1)p_1(x) - \frac{(d+1)d}{2!}p_2(x) - \cdots - p_{d+1}(x), \\
s_4(x) &= (d+1) - dp_1(x) - (d-1)p_2(x) - \cdots - 2p_{d-1}(x) - p_d(x).
\end{aligned}$$

Now, we give the second factorization of the infinite Pascal matrix via the infinite $d-$Fibonacci polynomials matrix by the following corollary.

Corollary 2. *Let $R(x)$ be the matrix as in (22). Then*

$$P = R(x) * \digamma(x).$$

We can find easily the inverses of the matrices by using the Riordan representations of the given matrices.

Corollary 3. *One has*

$$\digamma^{-1}(x) = (1 - p_1(x)\,t - p_2(x)\,t^2 - \cdots - p_{d+1}(x)\,t^{d+1}, t), \tag{24}$$

$$B^{-1}(x) = \frac{(1+t)^d}{D},$$

where

$$\begin{aligned} D &= 1 + (d+1-p_1(x))\,t + \left(\frac{(d+1)d}{2!} - dp_1(x) - p_2(x)\right)t^2 \\ &\quad + \left(\frac{(d+1)d(d-1)}{3!} - \frac{d(d-1)}{2!}p_1(x) - (d-1)p_2(x) - p_3(x)\right) \\ &\quad + \cdots + (1 - p_1(x) - p_2(x) - \cdots - p_{d+1}(x))\,t^{d+1}, \end{aligned}$$

and

$$\chi^{-1}(x) = \left(\frac{1 - p_1(x)\,t - p_2(x)\,t^2 - \cdots - p_{d+1}(x)\,t^{d+1}}{2 - p_1(x)t}, t\right). \tag{25}$$

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