Journal of Mathematical Modeling
Vol. 9, No. 3, 2021, pp. 331-345

JMM

Fractal Kronig-Penney model involving fractal comb potential

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Abstract. In this article, we suggest a fractal Kronig-Penny model which includes a fractal lattice, a fractal potential energy comb, and a fractal Bloch's theorem on thin Cantor sets. We solve the fractal Schrödinger equation for a given potential, using an exact analytical method. We observe that the allowed band energies and forbidden bands in the fractal lattice are bigger than in the standard lattice. These results show the effect of fractal space-time or their fractal geometry on energy levels.

Keywords: Fractal calculus, fractal Schrödinger equation, local fractal derivative, fractal lattice.
AMS Subject Classification 2010: 28A80, 35J10.

1 Introduction

Fractals are geometric structures whose fractal dimension exceeds their topological dimension. Some examples of fractals are the Sierpinski gasket, Koch curve, and Cantor set [25]. Such sets were actually pathological (or exceptional) counterexamples; for instance, the Koch curve is a compact curve with infinite length, and the Cantor set is an uncountable perfect set with zero Lebesgue measure. As a result, fractals were considered purely mathematical objects. In addition, fractals have attracted great interest in harmonic analysis in the context of Fourier transforms and geometric measure theory that are studied in the early twentieth century by Wiener, Winter, Erdos, Hausdorff, Besicovich, etc. [1,32], but such sets were still not correlated with objects of nature [33,34].

This status had not improved until Mandelbrot suggested the concept of fractals in the 1970s. Mandelbrot argued that many items of nature are not sets of smooth components. He developed this innovative concept and proposed the notion of fractals as a new class of mathematical structures describing nature. The significance of his idea was well-known shortly in several fields of science such as physics, chemistry, and biology. In mathematics, a new field

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Received: 2 September 2020 / Revised: 7 November 2020 / Accepted: 11 November 2020
DOI: 10.22124/jmm.2020.17537.1510

of research called fractal geometry has evolved rapidly on the basis of geometric measure theory, harmonic analysis, dynamic systems and ergodic theory [2–7, 10–13, 24, 29–31, 36, 43, 49].

Parvate and Gangal proposed a new method based on Riemann-like sums of functions which are supported on fractal sets such as fractal Cantor sets, fractal Cantor cubes, fractal Koch curves, and Cantor tartan spaces. The proposed method is simple, and it is more algorithmic, as well as, a more direct approach is feasible. The new calculus has defined integrals and derivatives of order $\alpha \in (0, 1]$ on fractal sets, where α is the dimension of the fractal sets [39–41].

The non-local derivatives and integrals of triadic Cantor sets were identified and implemented in fractal mediums [14]. The Schrödinger equation on fractal sets and curves are given in [15–17]. Sub-division and super-diffusion on thin Cantor sets is studied using mean square displacement [18]. The analogues of the Euler method and logistic equation in fractal calculus were investigated in [19], and other related research studies in [20–22].

Quantum mechanics is a convenient alternative to the classical one to explain physical phenomena [44]. Modeling an electron in a solid is considered very difficult due to the fact that the electron has the collective electrostatic potential of all lattice ions and other electrons. However, the combined potential of the electrons in a solid component represents the symmetry of lattice; hence, the periodicity of the lattice in the situation of a crystal [44]. Every solid includes electrons; therefore, the key question for electrical conductivity is how the electrons react to the applied electrical field. The electrons in the crystals are organized in energy bands isolated by energy regions in which there are no wavelike electron orbitals [26]. These prohibited areas are called energy gaps or band gaps, arising from the contact of the electron conduction waves with the ion cores of the crystal [26]. The crystal acts as an insulator if the permitted energy bands are either filled or empty, so that no electrons may move in an electrical field. The crystal acts as metal if one or more bands are partially filled. The crystal is a semiconductor or semimetal whether one or two bands are slightly filled or slightly blank [26]. In order to understand the distinction among insulators and conductors, we need to expand the free electron model to take into account the periodic lattice of the solid. The new most significant property that arises is the probability of a bandgap [26].

The Kronig-Penney model [27] is a simplified model for an electron with a one-dimensional periodic potential in which the crystal is considered to be infinite. Although the model is one-dimensional, the periodicity of the potential is an important property of electronic band structure. Several studies have been reported dealing with this concept in various approaches. For more details, we refer the readers to [8, 23, 28, 35, 37, 38, 47, 48]. The spectral properties of polariton gas in a quasi-periodic potential have been studied both theoretically and experimentally. To provide the results of this research work, the authors have sculpted the lateral profile of a quasi-1D cavity in the form of a Fibonacci sequence. Imaging the polariton modes in both real and reciprocal spaces, and the characteristic attributes of their fractal energy spectrum have been observed such as the opening of mini-gaps observing the gap labeling theorem and the log-periodic oscillations of the integrated density of states [42, 45, 46].

In this paper, we generalize the Kronig-Penney model, which includes the lattice with fractional dimension. The fractal Kronig-Penney model is presented, and its solution and energy bands are also derived. The plan of this paper is as follows: We review fractal calculus in Section 2. We suggest the fractal Kronig-Penney model in Section 3, and we conclude our work in Section 4.

2 Some fundamental tools

2.1 Fractal geometry

In this subsection, the fractal figures are shown in a way that they are between the standard figures with the fractional dimension [33, 34]. Fractal calculus adopts the standard calculus to include functions with fractal support.

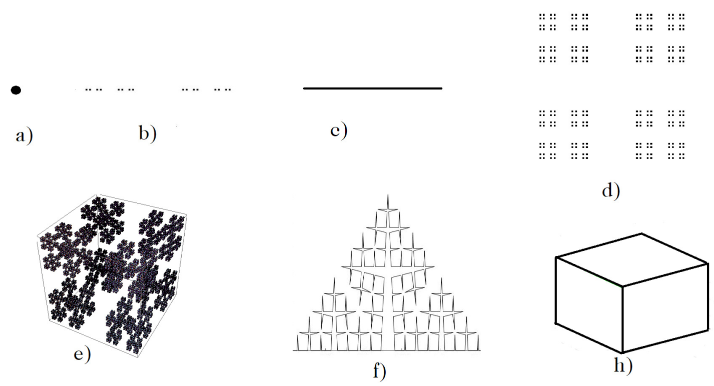


Figure 1: Graph of different shapes with different dimensions.

In Fig. 1, we present in, a) a pint with dimension zero, b) a thin Cantor set with dimension 0.63, c) a line with dimension 1, d) and e) the thin Cantor cubes with dimension 1.2 and 1.8 respectively, f) a fractal curves with dimension 1.82, and h) a cube with dimension 3.

2.2 Staircase functions

In this subsection, we present some basic tools of fractal calculus on the thin Cantor set which is shown in Fig. 5a [9, 21, 39–41].

Definition 1. *[9, 21, 39–41] Let $p[a_1, a_2]$ be a subdivision of an interval $I = [a_1, a_2]$ which is a collection of points $\{a_1 = t_0, t_1, \ldots, t_n = a_2\}$, such that $t_i < t_{i+1}$.*

Definition 2. *[9, 21, 39–41] Assume that $F^\mu \subset \mathbf{R}$(real-line) is a thin Cantor set, and $p[a_1, a_2]$ is a subdivision. The mass function is defined by*

$$\Psi^\alpha (F^\mu, a_1, a_2) = \lim_{\varsigma \to 0} \Psi^\alpha_\varsigma, \tag{1}$$

where

$$\Psi^\alpha_\varsigma = \inf_{\{p[a_1,a_2]:|p|\leq\varsigma\}} \sum_{j=0}^{m-1} \Gamma(\alpha+1)(t_{j+1}-t_j)^\alpha \phi(F^\mu, [t_{j+1}-t_j]), \tag{2}$$

and

$$\phi(F^\kappa, [t_{j+1}-t_j]) = \begin{cases} 1, & F^\mu \cap [t_{j+1}-t_j] \neq \emptyset, \\ 0, & \textit{otherwise}, \end{cases} \tag{3}$$

$$|p| = \max_{0\leq j\leq m} \left(t_{j+1} - t_j\right).$$

Definition 3. *[9, 21, 39–41] Assume that $c_0 \in \mathbf{R}$. The staircase function of order α is defined by*

$$S^{\alpha}_{F^{\mu}}(t) = \begin{cases} \Psi^{\alpha}\left(F^{\mu}, c_0, t\right), & \text{if } t \geq c_0, \\ -\Psi^{\alpha}\left(F^{\mu}, c_0, t\right), & \text{otherwise.} \end{cases} \tag{4}$$

The graph of the integral staircase function is presented in Fig. 2b.

Definition 4. *[9, 21, 39–41] The $\Psi -$ dimension is defined using the mass function as follows:*

$$\begin{aligned} \dim_{\Psi}\left(F^{\mu} \cap [a_1, a_2]\right) &= \inf\left\{\alpha : \Psi^{\alpha}\left(F^{\mu}, a_1, a_2\right) = 0\right\}, \\ &= \sup\left\{\alpha : \Psi^{\alpha}\left(F^{\mu}, a_1, a_2\right) = \infty\right\}. \end{aligned} \tag{5}$$

Fig. 2c presents the $\Psi -$ dimension which is the intersection point of the red line with the blue line.

Definition 5. *[9, 21, 39–41] The characteristic function $\chi F^{\mu}(\alpha, t)$ for a given thin Cantor set is defined by*

$$\chi_{F^{\mu}}(\alpha, t) = \begin{cases} \dfrac{1}{\Gamma\left(\alpha+1\right)}, & t \in F^{\mu}, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

2.3 Local fractal calculus

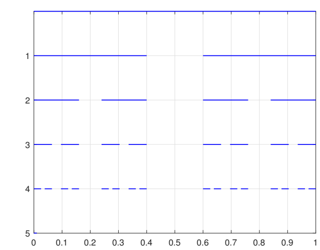
Definition 6. *[9, 39–41] If F^{μ} is α-perfect set, then the F^{α}-derivative of $f(t)$ at t is defined by*

$$D^{\alpha}_{F^{\mu}}\left(f\left(t\right)\right) = \begin{cases} F^{\mu} - \lim\limits_{y\to t} \frac{f(y)-f(t)}{S^{\alpha}_{F^{\mu}}(y)-S^{\alpha}_{F^{\mu}}(t)}, & \text{if } t \in F^{\mu}; \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

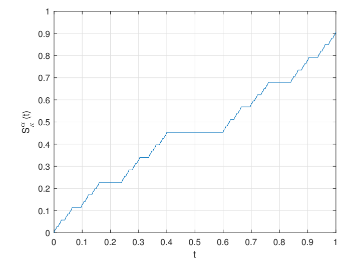
if the limit exists.

Definition 7. *[9, 21, 39–41] The F^{α}-integral of $f(t)$ on $[a_1, a_2]$ is defined by*

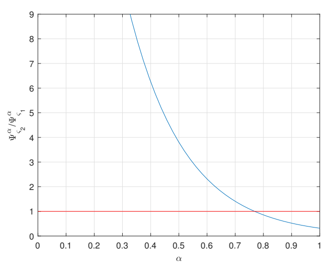
$$\int_{a_1}^{a_2} f\left(t\right) d^{\alpha}_{F^{\mu}} t \approx \sum_{j=1}^{n} f_j\left(t\right)\left(S^{\alpha}_{F^{\mu}}\left(t_j\right) - S^{\alpha}_{F^{\mu}}\left(t_{j-1}\right)\right). \tag{8}$$



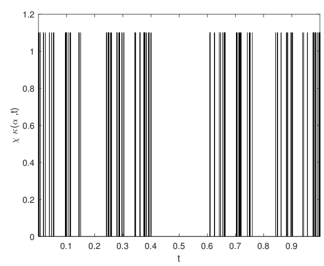
(a) The thin Cantor-like set ($\mu = 1/5$) by iteration.



(b) The integral staircase function for the thin Cantor set F^{μ} for the case of ($\mu = 1/5$).



(c) Ψ − dimension of the thin Cantor set ($\mu = 1/5$).



(d) Characteristic function thin Cantor set with ($\mu = 1/5$).

Figure 2: Graphs corresponding to the thin Cantor set with $\mu = 1/5$.

The following are some important formulas of local fractal calculus:

$$D^{\alpha}_{F^{\mu}} c\chi_{F^{\mu}} = 0, \quad c \quad \text{is constant}, \tag{9}$$

$$D^{\alpha}_{F^{\mu}} S^{\alpha}_{F^{\mu}}(t) = \chi_{F^{\mu}}(\alpha, t), \tag{10}$$

$$D^{\alpha}_{F^{\mu}} S^{\alpha}_{F^{\mu}}(t)^{m} = m\chi_{F^{\mu}} S^{\alpha}_{F^{\mu}}(t)^{m-1}, \tag{11}$$

$$D^{\alpha}_{F^{\mu}} \sin(S^{\alpha}_{F^{\mu}}(t)) = \chi_{F^{\mu}}(\alpha, t)\cos(S^{\alpha}_{\kappa}(t)), \tag{12}$$

$$D^{\alpha}_{F^{\mu}} \cosh(aS^{\alpha}_{F^{\mu}}(t)) = a\chi_{F^{\mu}}(\alpha, t)\sinh(aS^{\alpha}_{\kappa}(t)), \tag{13}$$

$$D^{\alpha}_{F^{\mu}}(f(t)g(t)) = D^{\alpha}_{F^{\mu}}(f(t))g(t) + f(t)D^{\alpha}_{F^{\mu}}(g(t)), \quad \text{(Local fractal Leibniz rule)}, \tag{14}$$

$$D^{\alpha}_{F^{\mu}} \exp(S^{\alpha}_{F^{\mu}}(t)) = \chi_{F^{\mu}} \exp(S^{\alpha}_{F^{\mu}}(t)), \tag{15}$$

$$\sin(S^{\alpha}_{F^{\mu}}(t)) = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{S^{\alpha}_{F^{\mu}}(t)^{2i-1}}{(2i-1)!}, \quad \text{(Fractal Maclaurin series)}, \tag{16}$$

$$\cos\left(aS^{\alpha}_{F^{\mu}}(t)\right) = \left(\frac{\mathrm{e}^{\mathrm{i}aS^{\alpha}_{F^{\mu}}(t)} + \mathrm{e}^{-\mathrm{i}aS^{\alpha}_{F^{\mu}}(t)}}{2}\right), \tag{17}$$

$$\sin\left(aS^{\alpha}_{F^{\mu}}(t)\right) = \left(\frac{\mathrm{e}^{\mathrm{i}aS^{\alpha}_{F^{\mu}}(k)} - \mathrm{e}^{-\mathrm{i}aS^{\alpha}_{F^{\mu}}(t)}}{2i}\right), \tag{18}$$

$$\sinh\left(aS^{\alpha}_{F^{\mu}}(t)\right) = \left(\frac{\mathrm{e}^{aS^{\alpha}_{F^{\mu}}(t)} - \mathrm{e}^{-aS^{\alpha}_{F^{\mu}}(t)}}{2}\right), \tag{19}$$

$$\cosh\left(aS^{\alpha}_{F^{\mu}}(t)\right) = \left(\frac{\mathrm{e}^{aS^{\alpha}_{F^{\mu}}(t)} + \mathrm{e}^{aS^{\alpha}_{F^{\mu}}(t)}}{2}\right). \tag{20}$$

3 Fractal Kronig-Penney model

The Schrödinger equation plays an important role in quantum mechanics like the Newton's second law in classical mechanics [26, 27, 44]. It is a wave equation where the probability of occurrence of phenomena in quantum mechanics can be obtained. We define α-dimensional Schrödinger equation to generalize the Kronig-Penney model.

A fractal crystal lattice with α-dimension, and period a is built by lattice translation vector $T = ma\hat{i},\ m = \pm 1, \pm 2, \ldots$ that operates on the thin Cantor sets.

A fractal potential energy function on thin Cantor set is defined by

$$U(x) = \begin{cases} U_0, & -b < x < 0 \quad x \in F^{\mu}; \\ 0, & 0 < x < a; \\ U_0, & a < x < b, \quad x \in F^{\mu}, \end{cases} \tag{21}$$

where U_0 is constant, and $U(x) = U(x+a)$ is called the potential energy function of the **fractal Kronig-Penney** model or the **fractal potential energy comb** [8,23,28,35,37,38,42,45,47,48].

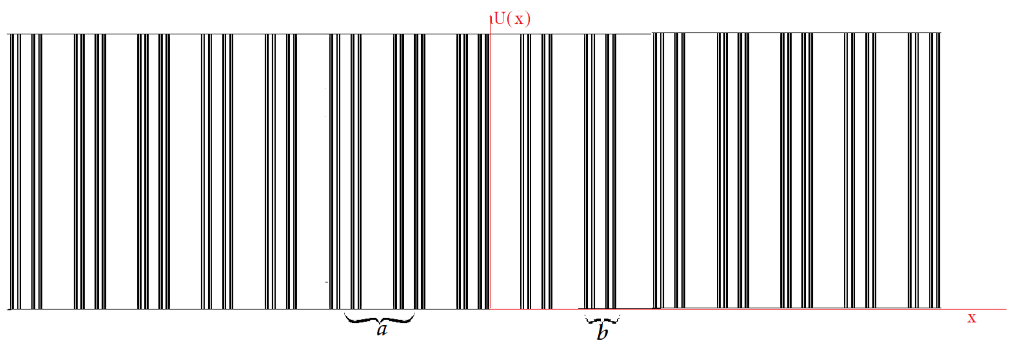


Figure 3: Graph of potential of the fractal Kronig-Penney model.

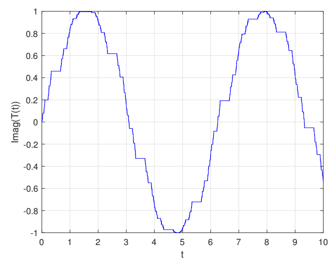
In Fig. 3 we have plotted the potential of the fractal Kronig- Penney model on thin Cantor set with $\mu = 1/3$.

Consider **the α-dimensional Schrödinger equation** given by

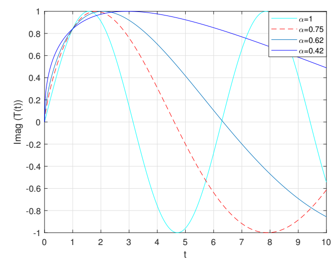
$$-\frac{\hbar^2}{2m} D^{2\alpha}_{F^{\mu},x} \Psi(x,t) + U(x)\Psi(x,t) = i\hbar D^{\alpha}_{F^{\mu},t} \Psi(x,t), \quad x, t \in F^{\mu}, \tag{22}$$

where $i = \sqrt{-1}$, $\hbar$ is the reduced Planck's constant, t is fractal time [6], x is fractal position, $\Psi(x,t)$ is called **fractal wave function**, and the left hand side of Eq. (22) is equivalent to the fractal Hamiltonian energy operator acting on $\Psi(x,t)$. Eq. (22) is called the general form of the Schrödinger equation on fractal time-space. We will investigate the fractal Schrödinger time-dependent equation and time-independent for the given fractal potential energy.

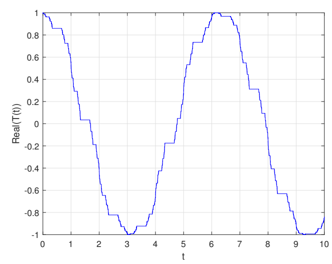
Let the fractal wave function $\Psi(x,t)$ is separable. In addition, the fractal wave function of two variables are represented as the product of two fractal separate functions of a fractal single



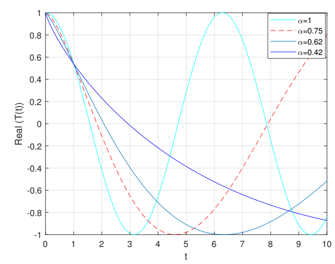
(a) The graph of imaginary part of Eq. (26) $\mu = 1/3$.



(b) The graph of imaginary part of Eq. (27).



(c) The graph of real part of Eq. (26) $\mu = 1/3$.



(d) The graph of real part Eq. (27) for different α.

Figure 4: The sketch of Eqs. (26)-(27) setting $E^{\alpha}_{F^{\mu}} = 1, \hbar = 1$.

variable:

$$\Psi\left(x,t\right) = \psi\left(x\right)T\left(t\right). \tag{23}$$

Then, in view of conjugacy fractal calculus with ordinary calculus and by using mathematical methods for solving partial differential equations, it is noticeable that the fractal wave equation can be written into two separate differential equations as follows:

$$ihD^{\alpha}_{F^{\mu},t}T\left(t\right) = E^{\alpha}_{F^{\mu}}T\left(t\right), \tag{24}$$

and

$$-\frac{\hbar^2}{2m}D^{2\alpha}_{F^{\mu},x}\psi\left(x\right) + U\left(x\right)\psi\left(x\right) = E^{\alpha}_{F^{\mu}}\psi\left(x\right), \tag{25}$$

which is called the fractal Schrödinger time-independent equation. Note that Eq. (24) depends on the fractal time $T\left(t\right)$ [6], while Eq. (25) depends only on the fractal position $\psi\left(x\right)$.Also $E^{\alpha}_{F^{\mu}}$ is the fractal energy eigenvalue. Eq. (24) can be solved immediately to give the following:

$$T\left(t\right) = \exp\left(\frac{-iE^{\alpha}_{F^{\mu}}S^{\alpha}_{F^{\mu}}\left(t\right)}{\hbar}\right) \tag{26}$$

$$\approx \exp\left(\frac{-iE^{\alpha}_{F^{\mu}}t^{\alpha}}{\hbar}\right). \tag{27}$$

In the above solution, we have used $c_1 t^{\alpha} \leq S^{\alpha}_{F^{\mu}}(t) \leq c_2 t^{\alpha}$ to write the approximation. In Fig. 4, we have sketched Eqs. (26)-(27). Solutions of Eq. (24) depend on the fractal potential energy function $U(x)$. In the following part, we present an approach to solve Eq. (24) for the fractal potential energy comb.

The **Fractal Bloch theorem** states that the solutions of the fractal Schrödinger equation Eq. (22) for a fractal periodic potential Eq. (21) can be expressed in the following form:

$$\psi(x) = u_k(x)\, e^{\left(iS_{F^\mu}^{\alpha}(k) S_{F^\mu}^{\alpha}(x)\right)}, \tag{28}$$

where $u_k(x)$ has a period of the **fractal crystal lattice** with $u_k(x) = u_k(x+a)$, and k is a fractal crystal momentum of a particle.

Proof: To prove this, we first note that for the fractal lattice with the period a, we have

$$|\psi(x)|^2 = |\psi(x+a)|^2, \tag{29}$$

where $p(x,t) = |\psi(x)|^2$ is called the fractal probability density function of the particle moving under the applied **fractal force** corresponding to the fractal potential function $U(x)$.
Eq. (29) refers to the following:

$$\psi(x+a) = \mathcal{C}\psi(x), \tag{30}$$

where $|\mathcal{C}|^2 = 1$. Then, one may write $\mathcal{C} = \exp(iS_{F^\mu}^{\alpha}(a) S_{F^\mu}^{\alpha}(k))$, where k is an arbitrary parameter. Then, we can write

$$\psi(x) = \exp(-iS_{F^\mu}^{\alpha}(a) S_{F^\mu}^{\alpha}(k))\psi(x+a). \tag{31}$$

Multiplying both sides of Eq. (31) by $\exp(-iS_{F^\mu}^{\alpha}(x) S_{F^\mu}^{\alpha}(k))$, we get

$$\exp(-iS_{F^\mu}^{\alpha}(x) S_{F^\mu}^{\alpha}(k))\psi(x) = \exp(-iS_{F^\mu}^{\alpha}(k)(S_{F^\mu}^{\alpha}(x) + S_{F^\mu}^{\alpha}(a))\psi(x+a). \tag{32}$$

Eq. (32) shows that

$$u_k(x) = \exp(-iS_{F^\mu}^{\alpha}(x) S_{F^\mu}^{\alpha}(k))\psi(x), \tag{33}$$

which is a fractal periodic function with period a. By rewriting Eq. (33), we obtain

$$\psi(x) = u_k(x) \exp(iS_{F^\mu}^{\alpha}(k) S_{F^\mu}^{\alpha}(x)), \tag{34}$$

which completes the proof.

The fractal Kronig-Penney model is suggested by considering the fractal Schrödinger time-independent equation Eq. (25), involving the fractal potential energy comb Eq. (21). We present the following solution of the model:
In the region $0 < x < a$ in which $U(x) = 0$, the eigenfunction is a linear combination of plane waves traveling to the right and to the left, namely

$$\psi(x) = A\mathrm{e}^{iS_{F^\mu}^{\alpha}(K) S_{F^\mu}^{\alpha}(x)} + B\mathrm{e}^{-iS_{F^\mu}^{\alpha}(K) S_{F^\mu}^{\alpha}(x)}, \tag{35}$$

where $A,\ B$, are constant and

$$E_{F^\mu}^{\alpha} = \frac{\hbar^2 S_{F^\mu}^{\alpha}(K)^2}{2m}, \tag{36}$$

which is called the **fractal energy**. In the region $-b < x < 0$ with the **fractal barrier**, the solution is of the following form:

$$\psi(x) = C\mathrm{e}^{S_{F^\mu}^{\alpha}(Q) S_{F^\mu}^{\alpha}(x)} + D\mathrm{e}^{S_{F^\mu}^{\alpha}(Q) S_{F^\mu}^{\alpha}(x)}, \tag{37}$$

where C, D are constants, and

$$E^{\alpha}_{F^{\mu}} = U_0 - \frac{h^2 S^{\alpha}_{F^{\mu}}(Q)^2}{2m}. \tag{38}$$

Consequently, using the fractal Bloch theorem Eq. (28), the solution in the region $a < x < a+b$ is related to the solution of Eq. (37) in the region $-b < x < 0$ as follows:

$$\psi\left(a < x < a+b\right) = \psi\left(-b < x < 0\right) e^{iS^{\alpha}_{F^{\mu}}(k)(a+b)}, \tag{39}$$

where k is index to label the solution. By boundary conditions, ψ and $D^{\alpha}_{F^{\mu},x}\psi$ are continuous at $x = 0$ and $x = a$. We can find A, B, C, D as follows:
Using Eqs. (35)-(37) with Eq. (39), we have

$$A\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(x)} + B\mathrm{e}^{-iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(x)} = \left(C\mathrm{e}^{S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(x)} + D\mathrm{e}^{-S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(x)}\right)\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(k)(a+b)}, \tag{40}$$

By differentiating Eq. (40) with respect to x, we get

$$\begin{aligned} &iS^{\alpha}_{F^{\mu}}\left(K\right)\chi_{F^{\mu}}\left(A\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(x)} - B\mathrm{e}^{-iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(x)}\right) \\ &\quad = S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}C\mathrm{e}^{S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(x)}\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(k)(a+b)} - S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}D\mathrm{e}^{-S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(x)}\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(k)(a+b)}. \end{aligned} \tag{41}$$

At $x = 0$, Eqs. (40)-(41) respectively, become

$$A + B - C - D = 0, \tag{42}$$

$$iS^{\alpha}_{F^{\mu}}\left(K\right)\chi_{F^{\mu}}\left(x\right)A - iS^{\alpha}_{F^{\mu}}\left(K\right)\chi_{F^{\mu}}\left(x\right)B - S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}\left(x\right)C + S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}\left(x\right)D = 0. \tag{43}$$

At $x = a$, by using Eqs. (40)-(41), respectively, we obtain the following:

$$\begin{aligned} A\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(a)} + B\mathrm{e}^{-iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(a)} - C\mathrm{e}^{-S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(b)+iS^{\alpha}_{F^{\mu}}(k)(a+b)} \\ - D\mathrm{e}^{S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(b)+iS^{\alpha}_{F^{\mu}}(k)(a+b)} = 0 \end{aligned} \tag{44}$$

and

$$\begin{aligned} &iS^{\alpha}_{F^{\mu}}\left(K\right)\chi_{F^{\mu}}\left(x\right)\left(A\mathrm{e}^{iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(b)} - Bi\mathrm{e}^{-iS^{\alpha}_{F^{\mu}}(K)S^{\alpha}_{F^{\mu}}(b)}\right) \\ &- S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}\left(x\right)\left(C\mathrm{e}^{-S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(b)+iS^{\alpha}_{F^{\mu}}(k)(a+b)} + D\mathrm{e}^{S^{\alpha}_{F^{\mu}}(Q)S^{\alpha}_{F^{\mu}}(x)+iS^{\alpha}_{F^{\mu}}(k)(a+b)}\right) = 0. \end{aligned} \tag{45}$$

The Eqs. (42)-(45) are written in the matrix form as follows:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ iS^{\alpha}_{F^{\mu}}\left(K\right)\chi_{F^{\mu}}\left(a\right) & -iS^{\alpha}_{F^{\mu}}\left(K\right)\chi_{F^{\mu}}\left(a\right) & -S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}\left(a\right) & S^{\alpha}_{F^{\mu}}\left(Q\right)\chi_{F^{\mu}}\left(a\right) \\ H & I & J & L \\ M & N & O & P \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{46}$$

where

$$\begin{aligned}
H &= \mathrm{e}^{iS_{F^\mu}^{\alpha}(K)S_{F^\mu}^{\alpha}(a)},\\
I &= \mathrm{e}^{-iS_{F^\mu}^{\alpha}(K)S_{F^\mu}^{\alpha}(a)},\\
J &= -\mathrm{e}^{-S_{F^\mu}^{\alpha}(Q)S_{F^\mu}^{\alpha}(a)+iS_{F^\mu}^{\alpha}(k)(a+b)},\\
L &= -\mathrm{e}^{S_{F^\mu}^{\alpha}(Q)S_{F^\mu}^{\alpha}(a)+iS_{F^\mu}^{\alpha}(k)(a+b)},\\
M &= iS_{F^\mu}^{\alpha}(K)\,\chi_{F^\mu}(a)\,\mathrm{e}^{iS_{F^\mu}^{\alpha}(K)S_{F^\mu}^{\alpha}(a)},\\
N &= -iS_{F^\mu}^{\alpha}(K)\,\chi_{F^\mu}(a)\,\mathrm{e}^{-iS_{F^\mu}^{\alpha}(k)S_{F^\mu}^{\alpha}(a)},\\
O &= -S_{F^\mu}^{\alpha}(Q)\,\chi_{F^\mu}(a)\,\mathrm{e}^{-S_{F^\mu}^{\alpha}(Q)S_{F^\mu}^{\alpha}(b)+iS_{F^\mu}^{\alpha}(k)(a+b)},\\
P &= S_{F^\mu}^{\alpha}(Q)\,\chi_{F^\mu}(a)\,\mathrm{e}^{S_{F^\mu}^{\alpha}(Q)S_{F^\mu}^{\alpha}(b)+iS_{F^\mu}^{\alpha}(k)(a+b)}.
\end{aligned}$$

Eq. (46) has a solution that exists only if the determination of the coefficients of A, B, C, D are zero. By expanding determinant of the matrix and using trigonometric identities, we have

$$\begin{aligned}
&S_{F^\mu}^{\alpha}(K)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)-\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}-S_{F^\mu}^{\alpha}(K)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)+\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}\\
&-S_{F^\mu}^{\alpha}(K)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)-\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}+(S_{F^\mu}^{\alpha}(K))^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)+\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}\\
&+4\mathrm{i}S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,(\chi(a))^2+4\mathrm{i}S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,(\chi(a))^2e^{2\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)}-2\mathrm{i}S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,(\chi(a))^2\\
&\quad e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)-\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}-2\mathrm{i}S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)+\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}\\
&-2\mathrm{i}S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)-\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}\\
&-2\mathrm{i}S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)+\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}\\
&-S_{F^\mu}^{\alpha}(Q)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)-\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}+S_{F^\mu}^{\alpha}(Q)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)+\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}\\
&+S_{F^\mu}^{\alpha}(Q)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)-\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}-S_{F^\mu}^{\alpha}(Q)^2(\chi(a))^2e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)+\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}=0.
\end{aligned} \tag{47}$$

Dividing Eq. (47) by $\left(-i8(\chi(a))^2S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)\,e^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)}\right)$ and after some simplifications, we have

$$\begin{aligned}
&\left(\frac{S_{F^\mu}^{\alpha}(Q)^2-S_{F^\mu}^{\alpha}(K)^2}{2S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)}\right)\left(\frac{\mathrm{e}^{\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}-\mathrm{e}^{-\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}-\mathrm{e}^{\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}+\mathrm{e}^{-\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}}{4i}\right)\\
&+\left(\frac{\mathrm{e}^{\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}+\mathrm{e}^{-\mathrm{i}aS_{F^\mu}^{\alpha}(K)+bS_{F^\mu}^{\alpha}(Q)}+\mathrm{e}^{\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}+\mathrm{e}^{-\mathrm{i}aS_{F^\mu}^{\alpha}(K)-bS_{F^\mu}^{\alpha}(Q)}}{4}\right)\\
&-\left(\frac{\mathrm{e}^{\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)}+\mathrm{e}^{-\mathrm{i}(a+b)S_{F^\mu}^{\alpha}(k)}}{2}\right)=0.
\end{aligned} \tag{48}$$

Using Eqs. (17)-(20) in Eq. (48), we have

$$\begin{aligned}
&\left(\frac{S_{F^\mu}^{\alpha}(Q)^2-S_{F^\mu}^{\alpha}(K)^2}{2S_{F^\mu}^{\alpha}(K)\,S_{F^\mu}^{\alpha}(Q)}\right)\sinh\left(bS_{F^\mu}^{\alpha}(Q)\right)\sin\left(aS_{F^\mu}^{\alpha}(K)\right)+\cosh\left(bS_{F^\mu}^{\alpha}(Q)\right)\cos\left(aS_{F^\mu}^{\alpha}(K)\right)\\
&=\cos\left((a+b)\,S_{F^\mu}^{\alpha}(k)\right).
\end{aligned} \tag{49}$$

The result is simplified if the potential is presented via the **fractal periodic Dirac delta function** which is obtained when we pass to the limit $b=0$ and $U_0=\infty$ in such a way that

$baS_{F^\mu}^\alpha(Q)^2/2 = P$ is a finite quantity. In this limit, $S_{F^\mu}^\alpha(Q) \gg S_{F^\mu}^\alpha(K)$ and $bS_{F^\mu}^\alpha(Q) \ll 1$. Then, Eq. (49) reduces to

$$f(K) = \left(\frac{P}{aS_{F^\mu}^\alpha(K)}\right)\sin\left(aS_{F^\mu}^\alpha(K)\right) + \cos\left(aS_{F^\mu}^\alpha(K)\right) = \cos\left(aS_{F^\mu}^\alpha(k)\right), \tag{50}$$

which might be called the dispersion relation for Kronig-Penney model with the **fractal Dirac comb potential energy**.

Subsequently, we approximate Eqs. (49)-(50) for the continuous case and the experimental data. In view of $c_1 t^\alpha \le S_{F^\mu}^\alpha(t) \le c_2 t^\alpha$, we have

$$\left(\frac{Q^{2\alpha} - K^{2\alpha}}{2K^\alpha Q^\alpha}\right)\sinh(bQ^\alpha)\sin(aK^\alpha) + \cosh(bQ^\alpha)\cos(aK^\alpha) \approx \cos((a+b)k^\alpha). \tag{51}$$

Here, we introduce reduced variables as follows:

$$b_a = b/a, \quad \varepsilon = \frac{2ma^{2/\alpha}E_{F^\mu}^\alpha}{\hbar^2}, \quad q = \frac{2ma^{2/\alpha}U_0}{\hbar^2}, \quad aK^\alpha = \varepsilon^{\alpha/2}, \quad aQ^\alpha = (q-\varepsilon)^{\alpha/2}, \tag{52}$$

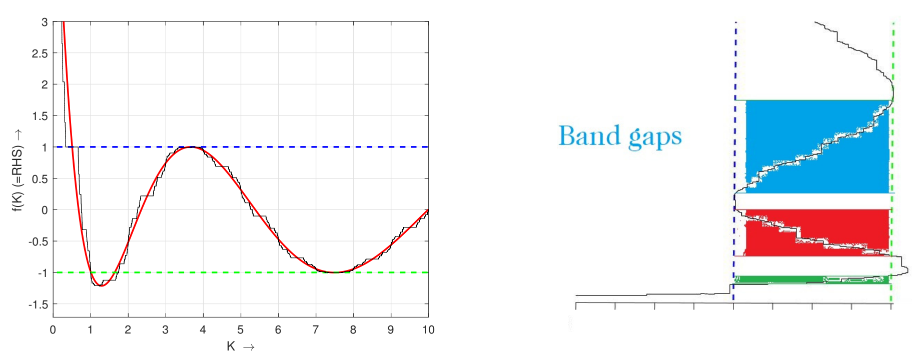
where ε is called **fractal reduced energy**. Then, we obtain

$$\left(\frac{(q-\varepsilon)^\alpha - \varepsilon^\alpha}{2(\varepsilon(q-\varepsilon))^{\alpha/2}}\right)\sinh\left(b_a(q-\varepsilon)^{\alpha/2}\right)\sin\left(\varepsilon^{\alpha/2}\right) + \cosh\left(b_a(q-\varepsilon)^{\alpha/2}\right)\cos\left(\varepsilon^{\alpha/2}\right)$$
$$\approx \cos\left((a+b)\,k^\alpha\right). \tag{53}$$

Eq. (50) can be written in the following form:

$$\left(\frac{b_a(q-\varepsilon)^\alpha}{2\varepsilon^{\alpha/2}}\right)\sin\left(\varepsilon^{\alpha/2}\right) + \cos\left(\varepsilon^{\alpha/2}\right) \approx \cos\left((a)\,k^\alpha\right), \tag{54}$$

which might be called **fractal dispersion relation.**



(a) Graph of $f(K)$ versus to K.

(b) Band gaps for the case of $\mu = 1/3$.

Figure 5: Graphs of Eq. (50) setting thin Cantor set with $\mu = 1/3$.

In Fig. 5, we have shown that the fractal lattice has wider band energies.

Remark 1. *We note that all results leads to standard one when we choose $\alpha = 1$.*

4 Conclusions

In this research, we have solved the fractal Schrödinger equation for the fractal Kronig-Penny model. The main goal of this research study has been successfully achieved by applying the fractal potential energy comb and fractal Bloch's theorem. Furthermore, simulation analysis has been performed in order to shed light on the important physical characteristics of the guiding equation. In physics, people have usually studied the laws of physics in spaces with higher dimensions. Instead, we have studied the laws of physics in the spaces of non-integer dimensions which can be seen in nature and laboratories.The obtained results can be applied in nanophysics, known as the fractal structures of solid crystals, which can give us new properties which are changeable with their fractal dimensions. The mathematical computations were conducted using the symbolic calculus software Matlab-R2013b. In addition, we demonstrated that when we set $\alpha = 1$ which is the fractal dimension of the support of the solutions, one can obtain the same results in the standard version.

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