Journal of Mathematical Modeling
Vol. 9, No. 2, 2021, pp. 323-330

JMM

Note to the convergence of minimum residual HSS method

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Abstract. The minimum residual HSS (MRHSS) method is proposed in [BIT Numerical Mathematics, 59 (2019) 299–319] and its convergence analysis is proved under a certain condition. More recently in [Appl. Math. Lett. 94 (2019) 210–216], an alternative version of MRHSS is presented which converges unconditionally. In general, as the second approach works with a weighted inner product, it consumes more CPU time than MRHSS to converge. In the current work, we revisit the convergence analysis of the MRHSS method using a different strategy and state the convergence result for general two-step iterative schemes. It turns out that a special choice of parameters in the MRHSS results in an unconditionally convergent method without using a weighted inner product. Numerical experiments confirm the validity of established results.

Keywords: Minimum residual technique, Hermitian and skew-Hermitian splitting, two-step iterative method, Convergence.
AMS Subject Classification 2010: 65F10.

1 Introduction

We first summarize some notations exploited in the paper. For a given square matrix W with real eigenvalues, the minimum and maximum eigenvalues of W are denoted by $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$, respectively. The symmetric and skew-symmetric parts of W are respectively defined by

$$\mathcal{H}(W) = \frac{1}{2}(W + W^T) \quad \text{and} \quad \mathcal{S}(W) = \frac{1}{2}(W - W^T).$$

The notation $\langle x, y\rangle$ refers to the Euclidean inner product of x and y, i.e., $\langle x, y\rangle = x^T y$ and the induced norm is denoted by $\|.\|$. The filed of values of the given matrix W is given by

$$\mathcal{F}(W) := \{\langle Wy, y\rangle / \langle y, y\rangle \mid y \neq 0\}.$$

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Received: 6 November 2020 / Revised: 25 November 2020 / Accepted: 25 November 2020
DOI: 10.22124/jmm.2020.18109.1559

Consider the following linear system of equations,

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a given non-symmetric positive definite matrix, the right-hand side $b \in \mathbb{R}^n$ is given and $x \in \mathbb{R}^n$ is the unknown vector to be determined.

More recently, Yang et al. [4] proposed using the minimum residual technique in conjunction with the well-known Hermitian and skew-Hermitian splitting (HSS) iterative scheme [1]. The corresponding method is called MRHSS and it produces the sequence of approximate solutions $\{x^{(k)}\}_{k=0}^{\infty}$ by the following two-step iterative scheme

$$\begin{aligned} x^{(k+\frac{1}{2})} &= x^{(k)} + \beta_k(\alpha I + \mathcal{H}(A))^{-1} r^{(k)}, \\ x^{(k+1)} &= x^{(k+\frac{1}{2})} + \gamma_k(\alpha I + \mathcal{S}(A))^{-1} r^{(k+\frac{1}{2})}, \end{aligned} \tag{2}$$

where

$$\beta_k = \frac{\left\langle r^{(k)}, A\delta^{(k)}\right\rangle}{\left\|A\delta^{(k)}\right\|^2} \quad \text{and} \quad \gamma_k = \frac{\left\langle r^{(k+\frac{1}{2})}, A\delta^{(k+\frac{1}{2})}\right\rangle}{\left\|A\delta^{(k+\frac{1}{2})}\right\|^2}, \tag{3}$$

in which $r^{(k)} = b - Ax^{(k)}$, $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$, $\delta^{(k)} = (\alpha I + \mathcal{H}(A))^{-1} r^{(k)}$ and $\delta^{(k+\frac{1}{2})} = (\alpha I + \mathcal{S}(A))^{-1} r^{(k+\frac{1}{2})}$. The parameters β_k and γ_k are determined by minimizing the residual norms $\|r^{(k+\frac{1}{2})}\|$ and $\|r^{(k+1)}\|$, respectively. The reported results in [4] illustrate the effectiveness of MHRSS method. However, it is proved that the method is convergent for any initial guess $x^{(0)}$ iff

$$0 \notin \mathcal{F}(A(\alpha I + \mathcal{H}(A))^{-1}) \ \cap \ \mathcal{F}(A(\alpha I + \mathcal{S}(A))^{-1}).$$

Obviously, it is not easy to check the above condition in the general case. Therefore, in another work, Yang [5] shows that if the second parameter is determined by minimizing a weighted norm of residual then the resulting iterative scheme is unconditionally convergent. More precisely, in the second step of (2), the parameter γ_k is replaced by

$$\gamma_k = \frac{\left\langle Mr^{(k+\frac{1}{2})}, MA\delta^{(k+\frac{1}{2})}\right\rangle}{\left\|MA\delta^{(k+\frac{1}{2})}\right\|^2}, \tag{4}$$

which is the minimizer of

$$\min_{\gamma} \left\|r^{(k+\frac{1}{2})} - \gamma A\delta^{(k+\frac{1}{2})}\right\|_M.$$

Here $M = (\alpha I + \mathcal{H}(A))^{-1}$ and $\|x\|_M := \|Mx\|$. The corresponding method works as good as (2) in terms of the required number of iterations for the convergence. However, it consumes more CPU-time than MRHSS due to using the weighted inner product. In this paper, we establish the convergence of two-step iterative schemes in conjunction with minimum residual technique giving a simple proof. Basically, we use the fact that every single step of MHRSS is a one-dimensional oblique projection technique [3, Chapter 5]. This point of view sheds light on constructing convergent two-step iterative schemes such as the following one which is particularly mentioned in the current work,

$$\begin{aligned} x^{(k+\frac{1}{2})} &= x^{(k)} + \beta_k(\alpha I + \mathcal{H}(A))^{-1} r^{(k)}, \\ x^{(k+1)} &= x^{(k+\frac{1}{2})} + \gamma_k(\eta I + \mathcal{S}(A))^{-1} r^{(k+\frac{1}{2})}, \end{aligned} \tag{5}$$

where β_k and γ_k are given by (3).

The remainder of the paper is organized as follows: in the second section, we discuss the convergence of two-step iterative schemes in conjunction with minimum residual technique. In section 3, we disclose numerical comparison results between the proposed approach and the ones given in [4,5] and brief conclusive remarks are given in section 4.

2 Main results

In this section, we study the convergence of two-step iterative schemes obtained after applying minimum residual technique. To this end, we assume that two splittings $A = \tilde{M} - \tilde{N}$ and $A = \hat{M} - \hat{N}$ are given. Consider the following two-step iteration method as follows:

$$\begin{aligned} x^{(k+\frac{1}{2})} &= x^{(k)} + \beta_k \tilde{M}^{-1} r^{(k)}, \\ x^{(k+1)} &= x^{(k+\frac{1}{2})} + \gamma_k \hat{M}^{-1} r^{(k+\frac{1}{2})}, \end{aligned} \tag{6}$$

where β_k and γ_k are given by (3). Evidently, the above method reduces to (5) for $\tilde{M} = \alpha I + \mathcal{H}(A)$ and $\hat{M} = \eta I + \mathcal{S}(A)$. Setting,

$$\delta^{(k)} = \tilde{M}^{-1} r^{(k)} \quad \text{and} \quad \delta^{(k+\frac{1}{2})} = \hat{M}^{-1} r^{(k+\frac{1}{2})},$$

we may rewrite (6) in the following form

$$x^{(k+\frac{1}{2})} = x^{(k)} + \beta_k \delta^{(k)} \quad \text{and} \quad x^{(k+1)} = x^{(k+\frac{1}{2})} + \gamma_k \delta^{(k+\frac{1}{2})} \ .$$

Considering formulas (3) for the parameters, it can be verified that

$$r^{(k+\frac{1}{2})} \perp A\delta^{(k)} \quad \text{and} \quad r^{(k+1)} \perp A\delta^{(k+\frac{1}{2})}. \tag{7}$$

Note that

$$r^{(k+\frac{1}{2})} = r^{(k)} - \beta_k A\delta^{(k)} \quad \text{and} \quad r^{(k+1)} = r^{(k+\frac{1}{2})} - \gamma_k A\delta^{(k+\frac{1}{2})}, \tag{8}$$

The decomposition $A = M - N$ is called splitting, if M is nonsingular [3, Chapter 4].

therefore, orthogonality conditions (7) imply that

$$\begin{aligned}
\left\langle r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})} \right\rangle &= \left\langle r^{(k)}, r^{(k+\frac{1}{2})} \right\rangle \\
&= \left\langle r^{(k)}, r^{(k)} - \beta_k A\delta^{(k)} \right\rangle \\
&= \left\langle r^{(k)}, r^{(k)} \right\rangle - \beta_k \left\langle r^{(k)}, A\delta^{(k)} \right\rangle \\
&= \left\langle r^{(k)}, r^{(k)} \right\rangle \left(1 - \beta_k \frac{\langle r^{(k)}, A\delta^{(k)} \rangle}{\langle r^{(k)}, r^{(k)} \rangle} \right) \\
&= \left\langle r^{(k)}, r^{(k)} \right\rangle \left(1 - \frac{\langle r^{(k)}, A\delta^{(k)} \rangle^2}{\langle r^{(k)}, r^{(k)} \rangle \langle A\delta^{(k)}, A\delta^{(k)} \rangle} \right) \\
&= \left\langle r^{(k)}, r^{(k)} \right\rangle (1 - \cos^2 \angle_k) \\
&= \left\langle r^{(k)}, r^{(k)} \right\rangle \sin^2 \angle_k.
\end{aligned}$$

Consequently, we have

$$\left\| r^{(k+\frac{1}{2})} \right\|^2 = \sin^2 \angle_k \left\| r^{(k)} \right\|^2,$$

where $\angle_k$ is the angle between $r^{(k)}$ and $A\delta^{(k)}$. With the same strategy, we see that

$$\left\| r^{(k+1)} \right\|^2 = \sin^2 \angle_{k+\frac{1}{2}} \left\| r^{(k+\frac{1}{2})} \right\|^2,$$

where $\angle_{k+\frac{1}{2}}$ is the angle between $r^{(k+\frac{1}{2})}$ and $A\delta^{(k+\frac{1}{2})}$. Hence, we have

$$\left\| r^{(k+1)} \right\|^2 = \sin^2 \angle_k \times \sin^2 \angle_{k+\frac{1}{2}} \left\| r^{(k)} \right\|^2,$$

which is equivalent to say that

$$\left\| r^{(k+1)} \right\| = |\sin \angle_k| \left| \sin \angle_{k+\frac{1}{2}} \right| \left\| r^{(k)} \right\|.$$

Notice that the iterative scheme (6) is convergent, if $|\sin \angle_k|$ and $|\sin \angle_{k+\frac{1}{2}}|$ are not both equal to "one" at the same time while $r^{(k+1)} \neq 0$. Otherwise, we have

$$\left\langle r^{(k)}, A\delta^{(k)} \right\rangle = 0, \tag{9}$$

and

$$\left\langle r^{(k+\frac{1}{2})}, A\delta^{(k+\frac{1}{2})} \right\rangle = 0. \tag{10}$$

Note that $A\tilde{M}^{-1} = (\tilde{M} - \tilde{N})\tilde{M}^{-1} = I - \tilde{N}\tilde{M}^{-1}$ and $A\hat{M}^{-1} = (\hat{M} - \hat{N})\hat{M}^{-1} = I - \hat{N}\hat{M}^{-1}$. Consequently, we have

$$\frac{\left\langle r^{(k)}, \tilde{N}\tilde{M}^{-1} r^{(k)} \right\rangle}{\langle r^{(k)}, r^{(k)} \rangle} = 1,$$

and

$$\frac{\left\langle r^{(k+\frac{1}{2})}, \hat{N}\hat{M}^{-1}r^{(k+\frac{1}{2})}\right\rangle}{\left\langle r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}\right\rangle} = 1,$$

in the case that (9) and (10) hold simultaneously.

We comment that for the above computations, the positive definiteness of A is not used. We summarize the above discussions in the following theorem which covers the result established in [4, Theorem 2].

Theorem 1. *Assume that $A \in \mathbb{R}^{n\times n}$ is nonsingular. The iterative scheme (6) converges to the exact solution of $Ax = b$ for any initial guess, if*

$$0 \notin \mathcal{F}(A\tilde{M}^{-1}) \ \cap \ \mathcal{F}(A\hat{M}^{-1}),$$

or equivalently,

$$1 \notin \mathcal{F}(\tilde{N}\tilde{M}^{-1}) \ \cap \ \mathcal{F}(\hat{N}\hat{M}^{-1}). \tag{11}$$

We end this part with a remark which shows that the iterative method (3) is convergent under certain condition. To do so, we first need to recall the following proposition which can be immediately concluded from [2, Proposition 2.1].

Proposition 1. *Assume that $A \in \mathbb{R}^{n\times n}$ is nonsingular. If*

$$\lambda_{\max}(\mathcal{H}(A))\lambda_{\min}(\mathcal{H}(A)) > -\lambda_{\max}((\mathcal{S}(A))^T\mathcal{S}(A)), \tag{12}$$

then there exists an η such that $\|\tilde{\mathcal{S}}^{-1}\|\|\tilde{\mathcal{H}}\| < 1$ where $\tilde{\mathcal{H}} = \mathcal{H} - \eta I$ and $\tilde{\mathcal{S}} = \eta I + \mathcal{S}$. In particular, the parameter η can be chosen by

$$\eta^* = \frac{\lambda_{\max}(\mathcal{H}(A)) + \lambda_{\min}(\mathcal{H}(A))}{2}, \tag{13}$$

for which the value of $\|\tilde{\mathcal{S}}^{-1}\|\|\tilde{\mathcal{H}}\|$ is minimized.

Remark 1. Assume that $r^{(k+\frac{1}{2})} \neq 0$. The Cauchy–Schwarz inequality ensures that

$$\frac{\left\langle r^{(k+\frac{1}{2})}, \hat{N}\hat{M}^{-1}r^{(k+\frac{1}{2})}\right\rangle}{\left\langle r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}\right\rangle} \le \frac{\|\hat{N}\hat{M}^{-1}r^{(k+\frac{1}{2})}\|}{\|r^{(k+\frac{1}{2})}\|} \le \|\hat{N}\hat{M}^{-1}\|.$$

It is immediate to conclude that $\|\hat{N}\hat{M}^{-1}\| < 1$ is a sufficient condition for the convergence of (6). In particular, by Proposition 1, we can conclude that if (12) holds then the iterative method (5) is convergent for any choice of α (such that $\alpha \notin \sigma(\mathcal{H}(A))$) after replacing η by (13). We further comment that if A is a positive definite matrix (i.e., $\mathcal{H}(A)$ is symmetric positive definite), then condition (12) is satisfied.

The spectrum of a given square matrix W is denoted by $\sigma(W)$

3 Numerical experiments

In this section, we examine the performance of the iterative scheme (5), MRHSS iterative method [4] and its weighted version [5] for two cases of a test problem mentioned in [4, 5]. The iterative methods are respectively called by MRHSS(α, η^*), MRHSS(α) and WMRHSS(α) where η^* is computed by (13). All of the reported experiments were performed on a 64-bit 2.45 GHz core i7 processor and 8.00GB RAM using some Matlab codes on MATLAB version 8.3.0532. Right-hand sides associated with random solution vectors were used in all of the experiments, performing ten runs and then averaging the CPU-times and rounding iteration numbers to the nearest integer. We report CPU-times and iteration counts under "CPU" and "Iter" in the table below. Furthermore, under "Err" we report the relative error $\|x^{(k)} - x^*\|/\|x^*\|$ averaged over the ten runs where $x^{(k)}$ is the k-th approximate solution and x^* is the exact solution. The initial guess was taken to be the zero vector and the iterations were stopped once $\|b - Ax^{(k)}\| \leq 10^{-7}\|b\|$.

We comment that linear system with the coefficient matrix $\alpha I + \mathcal{H}(A)$ is solved by sparse Cholesky factorization with the symmetric approximate minimum degree (SYMAMD) reordering. The LU factorization in combination with the column approximate minimum degree (COLAMD) reordering is exploited for solving shifted linear systems associated with skew-symmetric part of A.

The test problem arises from using five-point central difference discretization of the following two-dimensional convection-diffusion equation,

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = f(x,y), \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega, \tag{14}$$

where $\Omega = [0,1] \times [0,1]$. The coefficient functions $a(x,y)$ and $b(x,y)$ are chosen as

- Case I. $a(x,y) = x\sin(x+y)$ and $b(x,y) = y\cos(xy)$;
- Case II. $a(x,y) = 5y\ \exp(xy)$ and $b(x,y) = 5x\ \exp(x+y)$.

The mesh size $h = 1/\ell$ and the matrix $A \in \mathbb{R}^{(\ell-1)^2\times(\ell-1)^2}$, in the resulting linear system $Ax = b$, is symmetric positive definite. We comment that for deriving matrix A, we first multiply both sides of (14) by $-h^2$.

The obtained numerical results are reported in Table 1. The results demonstrate that the iterative scheme (5) works as efficient as the MRHSS method. As anticipated, the weighted version consumes more time due to extra computations of using a weighted inner product. The value of α_{exp} is the optimal parameter for MRHSS method which is obtained experimentally in [4]. Our numerical observations suggest that α_{exp} is a good approximation for the optimal values of α in the iterative scheme (5) and the weighted version of the MRHSS method [5]. For further details, we plot the required number of iterations with respect to α for each of three iterative schemes in Figure 1.

Method	Case I				Case II			
	$\ell = 80$ (α_{exp} =2e-4)		$\ell = 160$ (α_{exp} =1e-4)		$\ell = 80$ (α_{exp} =9e-3)		$\ell = 160$ (α_{exp} =3e-3)	
	Iter (CPU)	Err	Iter (CPU)	Err	Iter (CPU)	Err	Iter (CPU)	Err
MRHSS(α)	4(0.037)	2.02e-06	4(0.198)	1.86e-05	32(0.341)	5.01e-06	28(1.79)	5.19e-05
WMRHSS(α)	4(0.048)	1.89e-06	5(0.359)	1.57e-06	32(0.407)	6.29e-06	27(2.60)	4.59e-05
MRHSS(α, η^*)	4(0.039)	2.22e-06	4(0.216)	1.86e-05	31(0.338)	5.63e-06	28(1.86)	5.34e-05

Table 1: Average of experimental results over ten runs for Cases I and II.

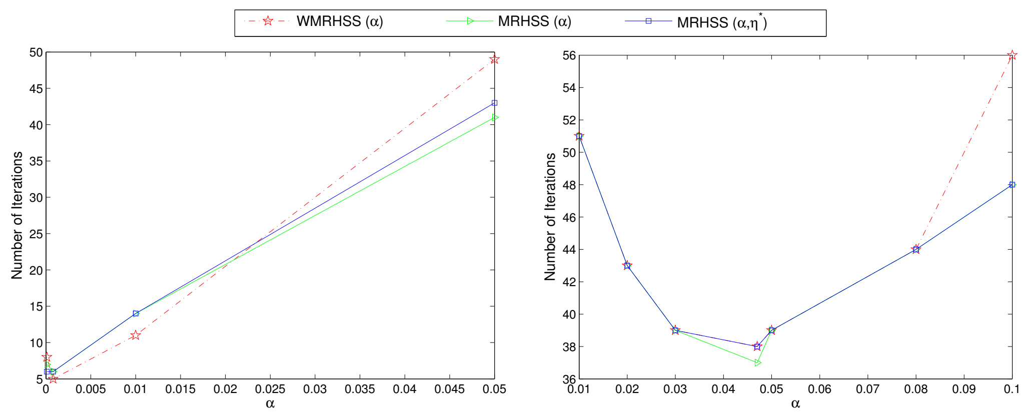

Figure 1: Thee value of α versus the required number of iterations for the convergence in Cases I (left) and II (right) with $\ell = 40$.

4 Conclusion

We established the convergence of two-step iterative schemes in conjunction with the minimum residual technique. The presented results cover the convergence analysis of the recently proposed MRHSS method. In particular, an approach was given based on using the Euclidean inner product which converges for a special choice of parameters. Numerical results illustrated the efficiency of the proposed iterative method. Further research can focus on developing the idea of MRHSS for solving the Saddle point linear system of equations.

Acknowledgements

The authors would like to thank Somayeh Mollahsani for her help in writing the Matlab code for producing the test problem.

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