Augmented and deflated CMRH method for solving nonsymmetric linear systems

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Abstract. The CMRH (Changing Minimal Residual method based on the Hessenberg process) is an iterative method for solving nonsymmetric linear systems. The method generates a Krylov subspace in which an approximate solution is determined. The CMRH method is generally used with restarting to reduce the storage. Restarting often slows down the convergence. In this paper we present augmentation and deflation techniques for accelerating the convergence of the restarted CMRH method. Augmentation adds a subspace to the Krylov subspace, while deflation removes certain parts from the operator. Numerical experiments show that the new algorithms can be more efficient compared with the CMRH method.

Keywords: Krylov subspace methods, augmentation, deflation, CMRH method, GMRES method, harmonic Ritz values.

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1 Introduction

In this paper we consider the solution of the linear system of equations

\[ Ax = b, \]

where \( A \in \mathbb{R}^{n \times n} \) is a nonsingular matrix and \( b \in \mathbb{R}^n \) is a given vector.

A popular class of iterative methods for solving system (1) is Krylov subspace methods. Krylov subspace methods find an approximate solution

\[ x_m \in x_0 + \mathcal{K}_m(A, r_0), \]

where \( \mathcal{K}_m(A, r_0) \equiv \text{span}\{r_0, Ar_0, \ldots, A^{m-1}r_0\} \) denotes an \( m \)-dimensional Krylov subspace, \( x_0 \) is the initial guess and \( r_0 \) is the initial residual. GMRES is one of the most popular Krylov
subspace methods for solving system (1). Another method is Quasi-Minimal Residual method (QMR) which has low storage rather than GMRES. GMRES often exhibits steady convergence, while QMR convergence curves are characterized by plateaus and sudden drops.

In [36], CMRH method is presented for solving (1) similar to QMR, but uses another basis for the Krylov subspace. This basis is constructed by the Hessenberg process. This technique requires less arithmetic operations and storage than Arnoldi process because, at iteration $k$, it constructs a lower trapezoidal basis $l_1, l_2, \ldots, l_k$, where $l_i$ is a vector having $(i-1)$ leading zero components and one component is equal to one. Hence, for a dense matrix, performing the matrix-vector product $Al_i$ is achieved with a lower cost comparing to Arnoldi process. Similarly to GMRES, this method requires one matrix-vector product per iteration. Heyouni and Sadok [24] proposed an implementation of the CMRH method which minimizes memory requirements. Some recent developments concerning the CMRH method and the Hessenberg process can be found in [1–3, 14, 20, 21, 25, 27, 37, 38, 40]. In [13], Duminil presented an implementation for parallel architectures and an implementation of the left-preconditioned CMRH method.

As GMRES, the CMRH method is often used with restarting strategy to reduce storage, but restarting slows the convergence of the methods and can make them stagnate in some situations.

Deflation and augmentation are two techniques for accelerating the convergence of Krylov subspace methods. In augmentation and deflation approaches, the search space of the method $K_m$ is made larger by an appropriately selected subspace $U$ in every step, or the Krylov subspace method is used for solving a projected problem and then a correction step is applied at the end. The first deflation and augmentation techniques for solving linear systems were presented by Nicolaides [32] and Dostál [9]. For symmetric positive definite matrix $A$, Saad et al. [35] described a deflated version of the CG algorithm. Also, Vuik et al. [39] applied deflated CG with incomplete Cholesky preconditioning for the solution of a class of layered problems with extreme contrasts in the coefficients. For nonsymmetric systems Chapman and Saad [8] and Morgan [29–31] proposed augmentation of Krylov subspaces generated by restarted GMRES method by spaces spanned by certain eigenvectors or Ritz vectors. Convergence properties of Krylov subspace methods augmented by spaces close to invariant subspaces are discussed by Saad [34]. These deflation and augmentation techniques are more suitable for some types of problems than others. They can be very effective when convergence is being hampered by a few eigenvalues [29]. However, they may have little effect on highly non-normal problems [8]. In [12, 18, 19], for some of well-known methods, it was shown that the convergence behavior of Krylov subspaces methods for non-normal matrices does not depend on the eigenvalues of the matrix only. In addition, it may be impossible to obtain useful eigenvalue approximations from either Ritz values [11] or harmonic Ritz values [10] during restart cycles. Baglama and Reichel [4] proposed to augment the Krylov subspace determined by GMRES by an arbitrary linear space of low dimension. Baker et al. [5] used the error approximation for augmenting the next approximation space. For an excellent overview of deflated Krylov subspace methods in the Hermitian and non-Hermitian cases, we refer the reader to [15–17, 22, 23], where extensive bibliographical references and historical comments can be found.

In this paper we present deflation and augmentation techniques that have been designed to accelerate the convergence of the CMRH(m) (restarting CMRH) method for the solution of linear systems of equations. We propose two strategy that allow augmentation of the Krylov subspaces generated by CMRH method by a space close to an invariant subspace of $A$. In addition, we
introduce a deflated CMRH method that is analogous to deflated GMRES method\cite{22}, but replaces the Arnoldi process by the Hessenberg process. Approximate spectral information which is required to define the augmentation space is provided by the Hessenberg process. Also deflated CMRH can break down in the same way as deflated GMRES. We show that breakdowns cannot occur if the augmentation space is an exact A-invariant subspace.

The paper is organized as follows. In Section 2, we shortly review the Hessenberg process and CMRH method. We present two augmented CMRH methods in Section 3. In Section 4, we describe a combined deflated and augmented CMRH method and show that under certain conditions, the method determines a solution without breakdown. Section 5 demonstrates the effectiveness of the proposed methods. Conclusions are summarized in Section 6.

Throughout the paper, all vectors and matrices are assumed to be real. We denote the range (or, the image) of a matrix $M$ by $\mathcal{R}(M)$. For the null space (or kernel) of $M$ we write $\mathcal{N}(M)$. For a vector $v$, $\|v\|$ always denotes the Euclidean norm $\|v\| = \sqrt{(v^Tv)}$ and $\|v\|_\infty$ denotes the maximum norm $\|v\|_\infty = \max_{i=1,...,n} |v_i|$, where $v_i$ is the $i$th component of the vector $v$. $Z^\dagger$ denotes the pseudo-inverse of the matrix $Z$; see, for example, \cite{36}. Some MATLAB notation is used; for instance, $H_k(i + 1 : m + 1, 1 : m)$ denotes the portion of $H_k$ with rows from $i + 1$ to $m + 1$ and columns from 1 to $m$. Finally, $I_k$ is the $k \times k$ identity matrix. $e_j$ and $e_j^{(s)}$ denote the $j$th column of identity matrices $I_n$ and $I_s$, respectively. The symbol $\leftrightarrow$ means swapping contents: $x \leftrightarrow y \iff t = x; x = y; y = t$.

2 CMRH algorithm

The CMRH method is an algorithm for solving nonsymmetric linear systems in which the Arnoldi component of GMRES is replaced by the Hessenberg process (see \cite{36}). Given an initial guess $x_0$ for the exact solution $x^* = A^{-1}b$, the Hessenberg process with pivoting constructs a basis of the Krylov subspace $\mathcal{K}_m(A, r_0) = \text{span}\{r_0, Ar_0, \ldots, A^{m-1}r_0\}$, where $r_0 = b - Ax_0$ is the initial residual.

Let $l_1 = r_0/(r_0)_{p_1}$, where $p_1 \in \{1, \ldots, n\}$ is such that $|(r_0)_{p_1}| = \|r_0\|_\infty$. The Hessenberg process with pivoting computes a matrix $L_m = [l_1, \ldots, l_m]$ whose columns form a basis of the Krylov subspace $\mathcal{K}_m(A, r_0)$ by using the formulas

$$w = Al_j - \sum_{i=1}^j h_{i,j}l_i, \quad \text{for } j = 1, \ldots, m,$$

and

$$h_{j+1,j}l_{j+1} = w.$$

The parameters $h_{i,j}$ are determined such that

$$l_{j+1} \perp e_{p_1}, \ldots, e_{p_j} \quad \text{and} \quad (l_{j+1})_{p_{j+1}} = 1,$$

where $p_i \in \{1, 2, \ldots, n\}$ and $p_{j+1} = i_0$, and $i_0$ satisfies $\|w\|_\infty = |(w)_{i_0}|$. Let $\bar{H}_j$ be the $(j + 1) \times j$ upper Hessenberg matrix whose nonzero entries are the $h_{i,j}$, and let $H_j$ be the matrix obtained
from $\bar{H}_j$ by deleting its last row, we have

$$AL_j = L_{j+1}\bar{H}_j = L_jH_j + h_{j+1,j}l_{j+1}(e_j^{(j)})^T.$$  \hspace{1cm} (3)

If $p$ denotes the permutation vector defined by $p_1, p_2, \ldots, p_n$ (as in Algorithm 1) and

$$P_n = [e_{p_1}, e_{p_2}, \ldots, e_{p_n}]^T,$$

denotes the $n \times n$ permutation matrix defined by the vector $p$, then we can easily check that $P_nL_j$ is a unit lower trapezoidal matrix. Using (3), the $j$th iterates of CMRH is defined by

$$x_j = x_0 + L_jy_j,$$

where $y_j$ minimizes the following problem:

$$\min_{y \in \mathbb{R}^j} \| \bar{H}_jy - (r_0)p_1e_1^{(j+1)} \|.$$

So, the iterate $x_j$ can be written as

$$x_j = x_0 + L_j\bar{H}_j^\dagger(r_0)p_1e_1^{(j+1)}.$$  \hspace{1cm} (4)

Notice that if $\|w\|_\infty = 0$ at step $j$, then, in exact arithmetic, the degree of the minimal polynomial of $A$ with respect to the vector $r_0$ is $j$ [26] and we have constructed an invariant subspace and the process must be stopped. In this case, $x_j$ is the exact solution of (1), (see Theorem 3 of [36]). CMRH(m) algorithm can be summarized as shown in Algorithm 1. More details about the CMRH algorithm can be found in [36].

Finally, we recall the definition of a harmonic Ritz pair [33] which is required to define the augmentation space.

**Definition 1.** Consider a subspace $\mathcal{W}$ of $\mathbb{C}^n$. Given a matrix $A \in \mathbb{C}^{n \times n}$, $\bar{\theta} \in \mathbb{C}$, and $\bar{y} \in \mathcal{W}$, $(\bar{\theta}, \bar{y})$ is a harmonic Ritz pair of $A$ with respect to $\mathcal{W}$ if and only if

$$A\bar{y} - \bar{\theta}\bar{y} \perp \mathcal{W},$$

or equivalently, for the canonical scalar product,

$$\forall \bar{w} \in \mathcal{R}(AW), \quad \bar{w}^H(A\bar{y} - \bar{\theta}\bar{y}) = 0.$$  

We call $\bar{y}$ a harmonic Ritz vector associated with the harmonic Ritz value $\bar{\theta}$.

Assume that the columns of $W_s = [w_1, w_2, \ldots, w_s] \in \mathbb{R}^{n \times s}$ constitute a basis of $\mathcal{W}$, the harmonic Ritz pairs $(\bar{\theta}_i, \bar{y}_i), i = 1, 2, \ldots, s$, can be obtained by solving the small generalized eigenvalue problem

$$W_s^TA^TAW_sg_i = \bar{\theta}_iW_s^TA^TW_sg_i, \quad \bar{y}_i = W_s g_i, \quad i = 1, 2, \ldots, s.$$  \hspace{1cm} (5)
Algorithm 1. CMRH(m) method

Choose \( m \).

Start: for given \( x_0 \), compute \( r_0 = b - Ax_0 \) and set \( p = [1, 2, \ldots, n]^T \).

Determine \( i_0 \) such that \( \| (r_0)_i \|_\infty = \| r_0 \|_\infty \), \( l_1 = r_0 / (r_0)_i \), \( p_1 \leftrightarrow p_i \).

Iterate: For \( j = 1, \ldots, m \)

\[ u = A l_j, \]

For \( i = 1, \ldots, j \)

\[ h_{i,j} = (u)_{p_i}, \]

\[ u = u - h_{i,j} l_i, \]

end

If \( (j < n \text{ and } u \neq 0) \) then

Determine \( i_0 \in \{j + 1, \ldots, n\} \) such that \( \| (u)_{p_{i_0}} \| = \| (u)_{p_{j+1:n}} \|_\infty \),

\[ h_{j+1,j} = (u)_{p_{i_0}}, \]

\[ l_{j+1} = u / h_{j+1,j}, \]

\( p_{j+1} \leftrightarrow p_{i_0} \)

else

\[ h_{j+1,j} = 0, \text{ Stop.} \]

end

If (an estimate of) \( \| b - Ax_j \|_2 \) is small enough or \( j = n \) then

\[ x_j = x_0 + [l_1, \ldots, l_j] \ast y_j, \text{ where } y_j \text{ minimizes } \| \tilde{H} y - (r_0)_{p_1} e_1^{(j+1)} \|, y \in \mathbb{R}^l. \]

Stop iteration

end

\[ \text{end} \]

\[ x_m = x_0 + [l_1, \ldots, l_m] \ast y_m, \text{ where } y_m \text{ minimizes } \| \tilde{H} y - (r_0)_{p_1} e_1^{(m+1)} \|, y \in \mathbb{R}^m. \]

\[ x_0 := x_m, \text{ go to Start.} \]

3 Adding approximate eigenvector to the subspace

Restarting CMRH may lead to poor convergence and even stagnation. The convergence can be improved in many situations. For accelerating the convergence of restarting CMRH, as restarting GMRES \([28, 29, 31]\), we propose that approximate eigenvectors corresponding to a few of the smallest eigenvalues in magnitude be formed and added to the Krylov subspace for CMRH. We present two implementations of the augmented CMRH method. In the implementations presented here, the subspace of projection is of the form \( K = K_m + U_k \), where \( K_m \) is the standard Krylov subspace whose dimension is \( m \) and \( U_k = \text{span}\{U_k\} \) is a subspace with dimension \( k \), where \( U_k = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k] \) with \( \tilde{y}_i, i = 1, \ldots, k \), being the harmonic Ritz vectors corresponding to the \( k \) smallest harmonic Ritz values (in magnitude).

Let \( s = m + k \). In the first algorithm (called augmented-CMRH algorithm), the solution space \( K \) is defined by

\[ K = \text{span}\{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k, l_{k+1}, l_{k+2}, \ldots, l_s\}, \]

where \( l_{k+1}, l_{k+2}, \ldots, l_s \) are the Hessenberg vectors. Let \( W_s = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k, l_{k+1}, l_{k+2}, \ldots, l_s] \) be the \( n \times s \) matrix whose columns are basis vectors for the augmented subspace \( K \). We first compute the LU factorization with pivoting

\[ AU_k = L_k R_k, \] (6)
where $R_k \in \mathbb{R}^{k \times k}$ is upper triangular and $L_k \in \mathbb{R}^{n \times k}$. Let $p = [p_1, p_2, \ldots, p_n]^T$ be the permutation vector and $P_n = [e_{p_1}, e_{p_2}, \ldots, e_{p_n}]^T$ be the permutation matrix defined by this LU factorization with pivoting, so, $L_k = P_n L_k$ is the lower trapezoidal matrix.

Letting $P_k = [e_{p_1}, e_{p_2}, \ldots, e_{p_k}]^T$ and using the oblique projector $Q = L_k(P_k L_k)^{-1} P_k$ on the subspace generated by the basis $L_k = [l_1, l_2, \ldots, l_k]$, and along the subspace generated by the basis $P_k^T = [e_{p_1}, e_{p_2}, \ldots, e_{p_k}]$, we define the normalized vector

$$l_{k+1} = \frac{(I - L_k C_n)r_0}{((I - L_k C_n)r_0)\tilde{p}_{k+1}},$$

(7)

where $C_n = (P_k L_k)^{-1} P_k$, $r_0$ is the residual vector from previous cycle, $\tilde{p}_{k+1} = i_0$, and $i_0$ satisfies $\| (I - L_k C_n)r_0 \|_\infty = |((I - L_k C_n)r_0)_{i_0}|$. Using $\tilde{p}_{k+1}$, we update the permutation vector $p$. We then append columns to $L_k$, which are determined by a modified Hessenberg process with pivoting. The initial vectors for the modified Hessenberg process is chosen to be $l_{k+1}$. The generated vectors are appended to the matrix $L_k$ as they become available. After $m$ steps of this modified Hessenberg process with pivoting, we have

$$AW_s = L_{s+1} \bar{H}_s,$$

(8)

where $L_{s+1} = [L_k \ l_{k+1}, \ldots, l_{s+1}] \in \mathbb{R}^{n \times (s+1)}$.

Let $\tilde{p} = [p_1, \ldots, p_k, \tilde{p}_{k+1}, \ldots, \tilde{p}_n]^T$ denotes the updated permutation vector $p$ and $\tilde{P}_n = [e_{p_1}, \ldots, e_{p_k}, e_{\tilde{p}_{k+1}}, \ldots, e_{\tilde{p}_n}]^T$ denotes the updated permutation matrix $P_n$ obtained after $m$ steps of the modified Hessenberg process with pivoting, then $\tilde{L}_{s+1} = \tilde{P}_n L_{s+1}$ is a lower trapezoidal matrix. We mention that the first column of trailing $n \times (m+1)$ submatrix $L_{s+1}(; k+1 : s+1)$ of $L_{s+1}$ is $l_{k+1}$ defined by (7) and remaining columns of $L_{s+1}(; k+1 : s+1)$ are determined by the $m$ steps of the modified Hessenberg process. The leading principal $k \times k$ submatrix of the upper Hessenberg matrix $\bar{H}_s \in \mathbb{R}^{(s+1) \times s}$ is the upper triangular matrix $R_k$ in the LU factorization (6). The entries of the trailing $m$ columns of $\bar{H}_s$ are determined by the modified Hessenberg process. In addition, by defining the vector $f_{s+1} = [\gamma_1, \ldots, \gamma_k, 0, \ldots, 0]^T \in \mathbb{R}^{s+1}$, where $[\gamma_1, \ldots, \gamma_k]^T = C_n r_0$ and $\gamma_{k+1} = (I - L_k C_n)r_0\tilde{p}_{k+1}$, we can easily show that $r_0 = L_{s+1} f_{s+1}$.

The $s$th iterate of augmented CMRH is defined by

$$x_s = x_0 + W_s d,$$

(9)

where $d \in \mathbb{R}^s$. From the relation (8) and the definition of vector $f_{s+1}$, we have

$$r_s = b - A x_s = r_0 - AW_s d = L_{s+1} (f_{s+1} - \bar{H}_s d).$$

Since $L_{s+1}$ is not orthogonal, as CMRH method, we obtain $d$ such that $\| f_{s+1} - \bar{H}_s d \|$ is minimized. Hence (9) can be written as

$$x_s = x_0 + W_s \bar{H}_s^T f_{s+1}.$$

To find approximate eigenvectors of $A$, using the subspace spanned by the columns of $W_s$, the relation (5), and $AW_s = L_{s+1} \bar{H}_s$, we solve the small generalized eigenvalue problem

$$\bar{H}_s^T L_{s+1} L_{s+1} \bar{H}_s g_i = \theta_i \bar{H}_s^T L_{s+1} W_s g_i.$$  

(10)
The $g_i$’s associated with the $k$ smallest harmonic Ritz values (in magnitude) $\tilde{\theta}_i$ are needed and the corresponding harmonic Ritz vectors $\tilde{y}_i = W_s g_i$, $i = 1, \ldots, k$, will be used for adding to the next Krylov subspace. If $\tilde{y}_i$ is complex, the real and imaginary parts are used separately.

We mention that, the implementation is a little different for the first cycle ($i = 0$). Standard CMRH(s) is used and $W_s = L_s$ is produced with the Hessenberg process by the initial vector $r_0$. The algorithm is given just for the second and subsequent runs.

Now we can summarize one restart cycle $i$ of augmented-CMRH algorithm as shown in Algorithm 2.

**Algorithm 2. One restart cycle $i$ of the augmented-CMRH($m,k$)**

Let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k$ be given harmonic Ritz vectors.

Compute $r_i = b - A x_i$, and $s = m + k$. Set $w_j = \tilde{y}_j$, for $j = 1, \ldots, k$ and $p = [1, 2, \ldots, n]^T$.

Perform the $LU$ factorization of $AW_s(:,1:k)$ as $AW_s(:,1:k) = L_k U_k$ with the permutation matrix $P_k^T = [e_{p_1}, e_{p_2}, \ldots, e_{p_n}]$.

Set $L_{s+1}(:,1:n) = L_k$ and $H(:,1:k) = U_k$.

Set $u = r_i$.

For $j = 1, \ldots, k$,

$$\gamma_j = (u)_{p_j}, \quad u = u - \gamma_j l_j$$

Determine $i_0 \in \{k+1, \ldots, n\}$ such that $\| (u)_{p_{i_0}} \| = \| (u)_{p_{k+1:p_n}} \|$, set $\gamma_{k+1} = (u)_{p_{i_0}}$ and $p_{k+1} \leftrightarrow p_{i_0}$.

Set $l_{k+1} = u/\gamma_{k+1}$.

For $j = k+1, \ldots, s$,

$$u = A l_j,$$

For $t = 1, \ldots, j$

$$h_{t,j} = (u)_{p_t}, \quad u = u - h_{t,j} l_t,$$

end

If ($u \neq 0$) then

Determine $i_0 \in \{j+1, \ldots, n\}$ such that $\| (u)_{p_{i_0}} \| = \| (u)_{p_{j+1:p_n}} \|$, $h_{j+1,j} = (u)_{p_{i_0}}$, $l_{j+1} = u/h_{j+1,j}$,

$$p_{j+1} \leftrightarrow p_{i_0},$$

else

$$h_{j+1,j} = 0, \quad \text{Stop}.$$ end

Set $w_j = l_j$.

end

Set $L_{s+1} = [l_1, l_2, \ldots, l_{s+1}]$, $W_s = [w_1, w_2, \ldots, w_s]$, $H_s = \{h_{i,j}\}_{1 \leq i \leq j+1, 1 \leq j \leq s}$.

Set $f_{s+1} = [\gamma_1, \gamma_2, \gamma_{k+1}, 0, \ldots, 0]^T \in \mathbb{R}^{s+1}$.

Compute $x_{i+1} = x_i + W_s d$, where $d$ minimizes $\| f_{s+1} - H_s d \|$, $d \in \mathbb{R}^s$.

Solve the generalized eigen problem $H_s^T L_{s+1}^T L_{s+1} H_s g_j = \tilde{\theta}_j H_s^T L_{s+1}^T W_s g_j$ for the appropriate $g_j$ and form $\tilde{y}_j = W_s g_j$ for $j = 1, 2, \ldots, k$.

In the implementation of second algorithm (called CMRH-E), as GMRES-E [29], we first generate the basis of the Krylov subspace $K_m$, then we add the approximate eigenvectors to it. Suppose that $[l_1, l_2, \ldots, l_m]$ have been produced by Hessenberg process (with pivoting) with initial vector $r_0$, and $k$ harmonic Ritz vectors have been derived from the previous cycle. Let $W_s = [l_1, l_2, \ldots, l_m, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k]$, a slight modification in the Hessenberg procedure is used to
deduce a trapezoidal basis of $K$. It consists of defining $u$ in Algorithm 1 (line 5 of Algorithm 1) now as

$$u = Al_j \quad (1 \leq j \leq m) \quad \text{and} \quad u = A\bar{y}_{j-m} \quad (m + 1 \leq j \leq s).$$

With this definition, we obtain the modified Hessenberg relation $AW_s = L_{s+1}\bar{H}_s$, where $L_{s+1} = [l_1, l_2, \ldots, l_{s+1}]$ is an $n \times (s + 1)$ matrix and $\bar{H}_s$ is an $(s + 1) \times s$ upper-Hessenberg matrix. Let $P_{s+1} = [e_{p_1}, e_{p_2}, \ldots, e_{p_n}]^T$ denotes the permutation matrix obtained after $s$ steps of the modified Hessenberg process, then $\bar{L}_{s+1} = P_{s+1}L_{s+1}$ is a lower trapezoidal matrix.

The approximate solution of system (1) can be written as $x_s = x_0 + W_sd$, where $d \in \mathbb{R}^s$. So, we have

$$r_s = b - Ax_s = r_0 - AW_sd = L_{s+1}((r_0)_{p_1}e_1^{(s+1)} - \bar{H}_sd).$$

Since $L_{s+1}$ is not orthogonal, as CMRH, by defining $\tilde{d}$ as

$$\tilde{d} = \arg\min_{d \in \mathbb{R}^s} \| (r_0)_{p_1}e_1^{(s+1)} - \bar{H}_sd \|,$$

the approximate solution $x_s$ can be written as

$$x_s = x_0 + W_s\bar{H}_s^T(r_0)_{p_1}e_1^{(s+1)}.$$

In this method, as augmented CMRH algorithm, the needed approximate eigenvectors of $A$ can be found by using the subspace spanned by the columns of $W_s$ and solving the small generalized eigenvalue problem (10).

Putting these results together gives the Algorithm 3.

**Algorithm 3. One restarting cycle $i$ of the CMRH-E($m,k$)**

*Let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k$ be given harmonic Ritz vectors.*

*Compute $r_i = b - Ax_i$ and $s = m + k$. Set $p = [1,2,\ldots,n]^T$. Determine $i_0$ such that $|\langle r_i, e_1 \rangle| = \|r_i\|_\infty, \beta = \langle r_i, e_1 \rangle, l_1 = r_i/\beta, p_t \leftrightarrow p_{i_0}.$*

*For $j = 1, \ldots, s$

$$u = \begin{cases} Al_j & \text{if } j \leq m \\ A\tilde{y}_{j-m} & \text{otherwise.} \end{cases}$$

*For $t = 1, \ldots, j$

$$h_{t,j} = (u)_{p_t},$$

$$u = u - h_{t,j}l_t,$$

*end*

*If $u \neq 0$ then*

*Determine $i_0 \in \{j + 1, \ldots, n\}$ such that $|\langle u, e_{i_0} \rangle| = \|u\|_{p_{i_0}^{-1}:p_n}\|_\infty,$

$$h_{j+1,j} = (u)_{p_{i_0}}, l_{j+1} = u/h_{j+1,j}, p_{j+1} \leftrightarrow p_{i_0}\$$

*else*

$$h_{j+1,j} = 0, \text{ Stop.}$$

*end*

*Set $L_{s+1} = [l_1, l_2, \ldots, l_{s+1}], W_s = [l_1, \ldots, l_m, \tilde{y}_1, \ldots, \tilde{y}_k], \bar{H}_s = \{h_{i,j}\}_{1 \leq i \leq j+1; 1 \leq j \leq s}$*

*Compute $x_{i+1} = x_i + W_sd, \text{ where } d \text{ minimizes } \| \beta e_1^{(s+1)} - \bar{H}_sd \|, d \in \mathbb{R}^s,$

*Solve the generalized eigen problem $\bar{H}_s^T L_{s+1}^T L_{s+1} H_s g_j = \bar{\theta}_j \bar{H}_s^T L_{s+1}^T W_s g_j$ for the appropriate $g_j$ and form $\bar{y}_j = W_sg_j$, for $j = 1,2,\ldots,k.$*
4 CMRH with deflated restarting

The general idea of deflation is to split the approximation space into two complementary subspaces such that the linear system projected onto one of these subspaces, referred to as the deflated linear system, will be easier to solve iteratively than the original linear system (1). These subspaces can be chosen in different ways. The Krylov subspace method is then confined in one of this subspace, by projecting the initial residual into this space and by replacing A by its restriction to this space. If the projection operator is chosen properly, the deflated linear system will be easier to solve iteratively than the original linear system. We combine simultaneously deflation and augmentation in the CMRH method. In this case, the search space \(\mathcal{K}\) will be

\[
\mathcal{K} = \mathcal{K}_m(\hat{A}, \hat{r}_0) + \mathcal{U}_k,
\]

where \(\mathcal{U}_k\) is the augmentation space of dimension \(k\), \(\mathcal{K}_m(\hat{A}, \hat{r}_0)\) represents the deflated Krylov subspace, \(\hat{A}\) refers to the deflated operator, and \(\hat{r}_0\) refers to the deflated residual. Given any initial guess \(x_0\) and \(s = m + k\), we consider an approximation solution of the form

\[
x_s \in x_0 + \mathcal{K}_m(\hat{A}, \hat{r}_0) + \mathcal{U}_k.
\]

This implies the following relation for the residual

\[
r_s \in r_0 + A\mathcal{K}_m(\hat{A}, \hat{r}_0) + A\mathcal{U}_k.
\]

As in [31]; we select \(\mathcal{U}_k\) as an approximate invariant subspace and update this subspace at the end of each cycle. Let \(U_k\) be an \(n \times k\) matrix whose columns vectors form a basis of the approximate invariant subspace \(\mathcal{U}_k\). Assume that \(Z = AU_k\), then the matrix \(E := Z^T Z\) is nonsingular. We define the \(n \times n\) matrices

\[
Q := ZE^{-1}Z^T, \quad P := I_n - Q.
\]

We note that \(P^2 = P\), \(PAU_k = 0\), and \((AU_k)^T P = 0\), so, \(P\) is the projection on \((AU_k)^\perp\) along \(AU_k\).

Using matrices (11) and matrix \(U_k\), we set up the deflated system

\[
\hat{A}x = \hat{b},
\]

where \(\hat{A} := PA\) and \(\hat{b} := Pb\). We point out that \(\hat{A}\) is completely determined by \(A\) and the choice of the matrix \(U_k\). Also, the system (12) is consistent, since it results from a left-multiplication of the consistent matrix equation \(Ax = b\) with \(P\).

Since \(PAU_k = 0\), \(PA\) is singular, hence it is important to analyze the possibilities of a break down when solving the linear system (12). In the next subsection, we show that when CMRH is used to solve the deflated system (12), as GMRES, a break down can not occur if the condition \(\mathcal{N}(PA) \cap \mathcal{R}(PA) = \{0\}\) holds.

By using \(\hat{r}_0 = \hat{b} - \hat{A}x_0\) and the Hessenberg process (with pivoting) started with \(l_1 = \hat{r}_0/(\hat{r}_0)_{p1}\), where \(|(\hat{r}_0)_{p1}| = \|\hat{r}_0\|_{\infty}\), we can construct the basis \(L_m = [l_1, l_2, \ldots, l_m]\) for the Krylov subspace \(\mathcal{K}_m(\hat{A}, \hat{r}_0) = \text{span}\{\hat{r}_0, \hat{A}\hat{r}_0, \ldots, \hat{A}^{m-1}\hat{r}_0\}\). We have also the Hessenberg relation

\[
\hat{A}L_m = L_{m+1}\tilde{H}_m,
\]
where \( \tilde{H}_m \) is an \((m+1) \times m\) upper Hessenberg matrix. Hence the approximate solution \( x_s \) from the affine space \( x_0 + \mathcal{K}_m(\hat{A}, \hat{r}_0) + U_k \) can be written as

\[
x_s = x_0 + L_mk_s + Ukc_s,
\]

with coordinate vectors \( k_s \in \mathbb{R}^m \) and \( c_s \in \mathbb{R}^k \). So, we have

\[
r_s = b - Ax_s = r_0 - AL_mk_s - AU_kc_s.
\]

Let \( \beta = (\hat{r}_0)p_1 \). From (13), \( Z = AU_k \), and \( \hat{r}_0 = Pr_0 = (I_n - Q)r_0 \), we have

\[
r_s = \beta l_1 + qr_0 - (\hat{A} + QA)L_mk_s - Zc_s
\]

\[
= \beta l_1 + ZE^{-1}Z^Tr_0 - L_{m+1}\tilde{H}_mk_s - ZE^{-1}Z^TAL_mk_s - Zc_s
\]

\[
= [Z \quad L_{m+1}]B_s,
\]

where

\[
B_s = \begin{bmatrix} E^{-1}Z^Tr_0 \\ \beta e_1^{(m+1)} \end{bmatrix} - \begin{bmatrix} I_k & E^{-1}C_m \\ 0 & \tilde{H}_m \end{bmatrix} \begin{bmatrix} c_s \\ k_s \end{bmatrix},
\]

with \( C_m = Z^TAL_m \in \mathbb{R}^{k \times m} \). Since \([Z \quad L_{m+1}]\) has not orthonormal columns, for computing \( c_s \) and \( k_s \), we impose the following minimizing condition on the residual vectors \( r_s \)

\[
\min \| B_s \| = \min_{k_s \in \mathbb{R}^m, c_s \in \mathbb{R}^k} \| E^{-1}Z^Tr_0 - \begin{bmatrix} I_k & E^{-1}C_m \\ 0 & \tilde{H}_m \end{bmatrix} \begin{bmatrix} c_s \\ k_s \end{bmatrix} \|.
\]

This problem decouples into an \((m+1) \times m\) least squares problem for \( k_s \) and an explicit formula for \( c_s \):

\[
\min_{k_s \in \mathbb{R}^m} \| \beta e_1^{(m+1)} - \tilde{H}_mk_s \|, \quad c_s := E^{-1}Z^Tr_0 - E^{-1}C_mk_s.
\]

We observe that the explicit inclusion of \( U_k \) can be omitted when instead we first construct the iterate \( \hat{x}_m = x_0 + L_mk_s \in x_0 + \mathcal{K}_m(\hat{A}, \hat{r}_0) \) by using the quasi minimal residual norm and then apply the correction \( x_s = \hat{x}_m + U_kc_s \). We can easily show that the projected residuals are equal to the original ones.

To define the subspace \( U_k \) for the next cycle, first we use the subspace spanned by the columns of \( W_s = [U_k \quad L_m] \) which satisfies the relation

\[
AW_s = \hat{W}_{s+1}\tilde{G}_s,
\]

where

\[
\hat{W}_{s+1} = [Z \quad L_{m+1}], \quad \tilde{G}_s = \begin{bmatrix} I_k & E^{-1}C_m \\ 0 & \tilde{H}_m \end{bmatrix}.
\]
By using (5) and (15), we compute the required harmonic Ritz pairs \((\tilde{\theta}_j, \tilde{y}_j), j = 1, \ldots, s\), by solving the small generalized eigenvalue problem
\[
\tilde{G}_s^T \tilde{W}_s^T \tilde{W}_{s+1} \tilde{G}_s g_j = \tilde{\theta}_j \tilde{G}_s^T \tilde{W}_s^T W_s g_j, \quad \tilde{y}_j = W_s g_j. \tag{17}
\]
Next, we store the \(k\) eigenvectors \(g_j\) associated with the smallest eigenvalues (in magnitude) \(\tilde{\theta}_j\) in \(G_k\) and define \(\hat{Y}_k = W_s G_k\). By performing the LU factorization (with pivoting) of \(G_s G_k\), as \(G_s G_k = \hat{L}_k \hat{U}_k\), (with the permutation matrix \(\hat{P}_k = [e_{p_1}^{(s+1)}, e_{p_2}^{(s+1)}, \ldots, e_{p_\ell}^{(s+1)}]^T\), we define
\[
U_k^{\text{new}} = \hat{Y}_k \hat{U}_k^{-1} \quad \text{and} \quad Z^{\text{new}} = \tilde{W}_{s+1} \hat{L}_k.
\]
These relations and (15) together imply that \(AU_k^{\text{new}} = Z^{\text{new}}\).

Putting these results together gives the following algorithm.

**Algorithm 4. One restart cycle \(i\) of the CMRH-DR**

Let \(U_k\) and \(Z = AU_k\) be given matrices.

1. Compute \(r_i = b - Ax_i\) and \(s = m + k\). Set \(p = [1, 2, \ldots, n]^T\).
2. Compute \(E = Z^T Z, Q = Z E^{-1} Z^T, P = I_n - Q\).
3. Compute \(\hat{r}_i = Pr_i\) and \(\hat{A} = PA\).
4. Determine \(i_0\) such that \(||\hat{r}_i||_\infty = ||\hat{r}_{i_0}||_\infty, \beta = (\hat{r}_{i_0})_{i_0}, p_1 \leftrightarrow p_{i_0}, c = \beta e_1\).
5. Perform \(m\) Hessenberg steps with \(\hat{A}\), letting \(l_1 = \hat{r}_{i_0}/\beta\) and generating \(L_{m+1}, \hat{H}_m\), and \(C_m = Z^T A L_m\).
6. Set \(W_s = [U_k \quad L_m], \tilde{W}_{s+1} = [Z \quad L_{m+1}]\) and \(\tilde{G}_s = \begin{bmatrix} I_k & E^{-1} C_m \\ 0 & \hat{H}_m \end{bmatrix}\).
7. Solve \(\min \| c - H_m k_s \|\) for \(k_s\).
8. Compute \(c_s = E^{-1} Z^T r_0 - E^{-1} C_m k_s\).
9. Compute \(x_{i+1} = x_i + L_m k_s + U_k c_s, r_{i+1} = b - Ax_{i+1}\).
10. Compute the \(k\) eigenvectors \(g_j\) associated with the smallest eigenvalues (in magnitude) \(\tilde{\theta}_j\) of \(\tilde{G}_s^T \tilde{W}_s^T \tilde{W}_{s+1} \tilde{G}_s g_j = \tilde{\theta}_j \tilde{G}_s^T \tilde{W}_s^T W_s g_j\) and store in \(G_k\).
11. Compute \(\hat{Y}_k = W_s G_k\)
12. Perform the LU factorization of \(G_s G_k\) as \(\tilde{G}_s G_k = \hat{L}_k \hat{U}_k\) and \(\hat{P}_k = [e_{p_1}^{(s+1)}, e_{p_2}^{(s+1)}, \ldots, e_{p_\ell}^{(s+1)}]^T\)
13. Compute \(Z = \tilde{W}_{s+1} \hat{L}_k, U_k = \hat{Y}_k \hat{U}_k^{-1}\). (Then we have \(Z = AU_k\))

For the first run, standard CMRH(s) is used and \(W_s = L_s, \tilde{W}_{s+1} = L_{s+1},\) and \(\tilde{G}_s = \hat{H}_s\) are produced with the Hessenberg process by the initial vector \(r_0\).

### 4.1 CMRH for a singular systems

A deflated matrix \(\hat{A}\) is singular if \(U_k \neq 0\), and we have to discuss whether the application of CMRH to the deflated system yields a well defined sequence of iterates that terminates with a solution. The application of GMRES to such systems has been analyzed in \[7,17,22\]. As in [7]; we state the following lemma for the properties of CMRH applied to singular consistent systems.

**Lemma 1.** Apply CMRH to (1) and suppose that \(\dim K_k = k\) for some \(k \geq 0\). Then exactly one of the following holds:

(i) \(\dim A(K_k) = k - 1\) and \(A(x_0 + z) \neq b\) for every \(z \in K_k\);
(ii) \(\dim A(K_k) = k, \dim K_{k+1} = k, x_k\) is uniquely defined, and \(Ax_k = b\);
(iii) \(\dim A(K_k) = k, \dim K_{k+1} = k + 1, x_k\) is uniquely defined, and \(Ax_k \neq b\).
Proof. As shown in Lemma 2.1 in [7], we have $k - 1 \leq \dim A(K_k) \leq k$ for all $k \geq 0$. We have also $r_0 \notin A(K_{k-1})$ if $k > 0$. If $\dim A(K_k) = k - 1$, then conclusions (ii) and (iii) can not hold. In this case, as in [7], we can show that for every $z \in \mathcal{K}_k$, we have $A(x_0 + z) \neq b$, and (only) conclusion (i) holds.

If $\dim A(K_k) = k$, from $A(K_k) \subseteq \mathcal{K}_{k+1}$, we have $k = \dim A(K_k) \leq \dim \mathcal{K}_{k+1} \leq k + 1$. If $\dim \mathcal{K}_{k+1} = k$, then we must have $A(K_k) = \mathcal{K}_{k+1}$ and, hence $r_0 \in A(K_k)$ and $A L_k \in \mathcal{K}_k$, then $\lambda_{k+1} = 0$. In this case, from (3), we have $A L_k = L_k H_k$, \(\text{rank}(H_k) = k\), and consequently
\[
(\hat{H}_k)^\dagger = ((H_k)^{-1}, 0).
\] (18)
So, the iterate $x_k$, defined by (4), can be written as follows
\[
x_k = x_0 + L_k (H_k)^{-1}(r_0)_{p_1} e_1^{(k)}.
\]
Using $A L_k = L_k H_k$ and the last equality, we have
\[
r_k = b - A x_k = r_0 - (r_0)_{p_1} A L_k (H_k)^{-1} e_1^{(k)}
\]
\[
= r_0 - (r_0)_{p_1} L_k H_k (H_k)^{-1} e_1^{(k)} = r_0 - (r_0)_{p_1} l_1 = 0.
\]
Thus $A x_k = b$, and (only) conclusion (ii) holds. If $\dim \mathcal{K}_{k+1} = k + 1$, then $\lambda_{k+1} \neq 0$ in $A L_k = L_{k+1} H_k + h_{k+1} k_{k+1} e_1^{(k)}$. It follows that the decomposition $A L_k = L_{k+1} \hat{H}_k$ exists, the columns of the matrix $L_{k+1}$ form a basis of $\mathcal{K}_{k+1}$, and the matrix $\hat{H}_k$ is of full rank and the iterate $x_k$ can be uniquely defined by (4). In this case, we have $r_0 \notin A(K_k)$, $r_k \neq 0$, $A x_k \neq b$ and (only) conclusion (iii) holds.

Using this lemma, the next two theorems give condition under which the CMRH iterates converge safely to a solution of the system. The proof is similar to the ones of Theorem 2.2 in [7] and Theorem 4.1 in [16], so we omit them here.

Theorem 1. If the CMRH method is applied to (1), then, at some step, either
(a) CMRH breaks down through rank deficiency of the least-squares problem
\[
\min_{d \in \mathbb{R}^k} \| \hat{H}_k d - (r_0)_{p_1} e_1^{(k+1)} \|
\]
without determining a solution $x_k = x_0 + L_k d_k$ or
(b) CMRH determines a solution without breakdown and then breaks down at the next step through degeneracy of the Krylov subspace.

Theorem 2. Consider an arbitrary matrix $\hat{A} \in \mathbb{C}^{n \times n}$ and a vector $\hat{b} \in \mathcal{R}(\hat{A})$ (i.e., the linear system $\hat{A} x = \hat{b}$ is consistent). Then the following conditions are equivalent:
1. For each initial guess $x_0 \in \mathbb{R}^n$, the CMRH method applied to the linear equation $\hat{A} x = \hat{b}$ is well defined at each iteration step $k$ and it terminates with a solution of the system.
2. $\mathcal{N}(\hat{A}) \cap \mathcal{R}(\hat{A}) = \{0\}$.

Finally, by using the above results, we can show that, just as for deflated GMRES [16], for each initial guess $x_0 \in \mathbb{R}^n$, the CMRH method applied to the singular consistent equation (12) is well defined at each iteration step and it terminates with a solution of the system when $\mathcal{U}_k$ is an invariant subspace, i.e., when $A \mathcal{U}_k = \mathcal{U}_k$. 
5 Numerical expriments

In this section, we present some results of solving linear systems of the form $Ax = b$ to illustrate the performance of the proposed algorithms. The codes are written in the programming package MATLAB and tested on a Workstation Intel Corei3, 2.40GHz. For all problems, the initial vector is the zero vector. For all matrices, the right-hand side was taken to be a vector with entries having random values between 0 and 1. The Jacobi (or diagonal) preconditioner has been used for all the test problems. The stopping criterion $\|r_i\| < 10^{-8}\|r_0\|$ was used. The maximum allowed number of cycles is $kmax = 3000$. We compare CMRH($m + k$) with augmented-CMRH($m,k$), CMRH-E($m,k$), and CMRH-DR($m,k$) methods.

For the first set of examples, we used the matrices

\[
A_1(i,j) = \begin{cases} 
\epsilon, & \text{if } i = j, \\
\frac{2 \min(i,j)-1}{n-i+j}, & \text{if } i \neq j,
\end{cases}
\]

with $n = 100$, $\epsilon = 0.1$, and $\epsilon = 0.0001$, and

\[
A_2 = \begin{pmatrix}
\epsilon & 1 \\
-1 & \epsilon & 1 \\
& \ddots & \ddots & \ddots \\
& & -1 & \epsilon & 1 \\
& & & -1 & \epsilon
\end{pmatrix},
\]

with $n = 100$, $\epsilon = 0.01$, and $\epsilon = 0.0001$ [6].

For the second set of experiments, we used some matrices from Matrix Market collection\* for the matrix $A$. These matrices with their generic properties are given in Table 1. In Table

<table>
<thead>
<tr>
<th>Matrix</th>
<th>property</th>
<th>order</th>
<th>sym.</th>
<th>nnz</th>
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<td>Yes</td>
<td>8402</td>
<td></td>
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<tr>
<td>cdd1</td>
<td>961</td>
<td>No</td>
<td>4681</td>
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<td>2375</td>
<td></td>
</tr>
<tr>
<td>Sherman4</td>
<td>1104</td>
<td>No</td>
<td>3786</td>
<td></td>
</tr>
<tr>
<td>Sherman5</td>
<td>3312</td>
<td>No</td>
<td>20793</td>
<td></td>
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<td>No</td>
<td>22316</td>
<td></td>
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<td>13514</td>
<td>No</td>
<td>352702</td>
<td></td>
</tr>
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</table>

2, we give the number of cycles (Cycle), and the matrix-vector products (Mvp) required for convergence. The notation “×” means that the relative residual norms have not reached the accuracy $10^{-8}$ after 3000 cycles. The results presented in Table 2 indicate that the augmented-CMRH($m,k$), CMRH-E($m,k$), and CMRH-DR($m,k$) are effective for these problems and they are much better than standard CMRH($m + k$). As we observe, these examples get better results with CMRH-DR($m,k$) in terms of matrix-vector products (except for Sherman4 which has better results with augmented-CMRH(18,2)). In addition, using four approximate eigenvectors

Table 2: Cycles and matrix-vector products required for convergence.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>m</th>
<th>k</th>
<th>CMRH($m + k$)</th>
<th>augmented-CMRH($m, k$)</th>
<th>CMRH-E($m, k$)</th>
<th>CMRH-DR($m, k$)</th>
</tr>
</thead>
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<tr>
<td></td>
<td>Cycle</td>
<td>Mvp</td>
<td>Cycle</td>
<td>Mvp</td>
<td>Cycle</td>
<td>Mvp</td>
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<td>$A_1$ with 20</td>
<td>0</td>
<td>Cycle</td>
<td>Mvp</td>
<td>688</td>
<td>13760</td>
<td>125</td>
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<tr>
<td>$A_1$ with 18</td>
<td>2</td>
<td>Cycle</td>
<td>Mvp</td>
<td>16</td>
<td>51</td>
<td>1020</td>
</tr>
<tr>
<td>$A_1$ with 16</td>
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<td>Cycle</td>
<td>Mvp</td>
<td>16</td>
<td>51</td>
<td>1020</td>
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<td>Mvp</td>
<td>63</td>
<td>1260</td>
<td>17</td>
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<tr>
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<td>76</td>
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<td>Mvp</td>
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<td>Cycle</td>
<td>Mvp</td>
<td>23</td>
<td>460</td>
<td>2233</td>
</tr>
<tr>
<td>Saylr4 with 18</td>
<td>2</td>
<td>Cycle</td>
<td>Mvp</td>
<td>2710</td>
<td>54200</td>
<td>2233</td>
</tr>
<tr>
<td>Saylr4 with 16</td>
<td>4</td>
<td>Cycle</td>
<td>Mvp</td>
<td>2710</td>
<td>54200</td>
<td>2233</td>
</tr>
</tbody>
</table>

(m = 16 and k = 4) gives the lowest number of cycles and matrix-vector products for all test matrices (except for matrix Saylr4 which have better results with CMRH-E(18,2)).

For the matrices Sherman1, Sherman4, Sherman5, and Saylr4, the relative residual norms ($\|r_i\|/\|r_0\|$) are plotted against the number of matrix-vector products. See figures 1 and 2 for the graph of convergence of the methods. These figures show that augmented-CMRH(17,3), CMRH-E(17,3), and CMRH-DR(17,3) compete well for these examples and are much better than CMRH(20). CMRH-DR is also better than augmented-CMRH and CMRH-E in terms of matrix-vector products. The augmented-CMRH(17,3) and CMRH-E(17,3) have similar convergence. Finally, we mention that, for Saylr4 (Fig. 2 (right)), CMRH(20) is not convergence in 3000 cycles and reaches the relative residual norm of 0.0023, while augmented-CMRH(17,3), CMRH-E(17,3), and CMRH-DR(17,3) need 25820, 23240, and 14810 Mvps and have the relative residual norms of 9.8022e-09, 9.4538e-09, and 9.8836e-09, respectively.
Augmented and deflated CMRH method for solving nonsymmetric linear systems

6 Conclusions

In this paper, we have described three methods that accelerate the convergence of CMRH(m). The techniques are straightforward and easy to implement. Numerical experiments show that the new methods can shrink the slow convergence phase and thus considerably accelerate the convergence of CMRH. The methods are not really needed for easy problems where few restarts are used. The experiments show that the results of CMRH-DR algorithm are often better than those of augmented-CMRH and CMRH-E algorithms.

References


Augmented and deflated CMRH method for solving nonsymmetric linear systems


